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Random Walks on Trees with Finitely Many Cone Types

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This paper is devoted to the study of random walks on infinite trees with finitely many cone types (also called periodic trees). We consider nearest neighbour random walks with probabilities adapted to the cone structure of the tree, which include in particular the well studied classes of simple and homesick random walks. We give a simple criterion for transience or recurrence of the random walk and prove that the spectral radius is equal to 1 if and only if the random walk is recurrent. Furthermore, we study the asymptotic behaviour of return probabilitites and prove a local limit theorem. In the transient case, we also prove a law of large numbers and compute the rate of escape of the random walk to infinity, as well as prove a central limit theorem. Finally, we describe the structure of the boundary process and explain its connection with the random walk.

KEY WORDS: Tree; random walk; transience; rate of escape.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let T be a locally finite, infinite tree with root o. The root induces an orientation, where $x \xrightarrow{*} y$ for two vertices x, y if x lies on the geodesic from o to y. Given x, the cone C_x of T is the subtree rooted at x that is spanned by all y with $x \xrightarrow{*} y$. We say that T has *finitely many cone types* when the number of isomorphism classes of the rooted trees C_x, $x \in T$, is finite. A similar definition can be given for general rooted (connected) graphs, in particular Cayley graphs of groups, and in the latter context the notion of cone types was first introduced by Cannon (1984). This has led to the very

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popular theme of *automatic groups,* see the book by Epstein with coauthors (1992). Probability on trees with finitely many cone types has been studied in some detail by Lyons (1990) and Takacs (1997). In the present note, we pursue the investigation of random walks on trees with finitely many cone types that was taken up in the Ph.D. thesis of the first of us, Nagnibeda (1997). That is, we study nearest neighbour random walks that are adapted to the cone structure.

More precisely, we assign transition probabilities $p(x,y)>0$ to all pairs of neighbours $x, y \in T$ such that $\sum_y p(x,y)=1$ for every x. (If x, y are not neighbours then we set $p(x,y)=0$.) They give rise to a T-valued Markov chain $(Z_n)_{n\geqslant 0}$, where Z_n is the random position of a random walker at time n, who obeys the "rules" $\mathbb{P}[Z_{n+1}=y \mid Z_n=x]=p(x,y)$. The most typical example is the *simple random walk,* where $p(x,y)=1/\deg(x)$, when x and y are neighbours (and $\deg(x)$ is the number of neighbours of x). The class of examples studied by Lyons (1990) and Takacs (1997) are the *homesick* random walks: if x^- denotes the neighbour of x in T closer to o, then $p(x,x^-)=\lambda/(\lambda+\deg(x)-1)$, while $p(x,y)=1/(\lambda+\deg(x)-1)$ for all "forward" neighbours y of x, where $\lambda>0$ (the homesickness parameter). Here we start with arbitrary transition probabilities along the edges of T.

We now consider each cone C_x as a rooted graph that is labelled with these probabilities. Let $\mathscr{I}$ be the set of isomorphism classes of all labelled cones C_x, where $x \neq o$. Then we require $\mathscr{I}$ to be finite, and say that *the pair* (T,P) *has finitely many cone types.* For $x \in T\setminus\{o\}$, let $\imath(x) \in \mathscr{I}$ be the type of the cone C_x. If $\imath(x)=i$ and $y \in C_x$ is a neighbour of x with $\imath(y)=j$, then $p(x,y)=\mathsf{p}(i,j)$ depends only on i and j. Let $\mathsf{d}(i,j)$ be the number of neighbours of x in C_x that are of type j. Then the "backward probability"

$$p(x,x^-)=1-\sum_j \mathsf{d}(i,j)\,\mathsf{p}(i,j)=:\mathsf{p}(-i)$$

depends only on i and is determined by the "forward probabilities." We can encode this information in a digraph with vertex set $\mathscr{I}$ where vertices i and j are connected by $\mathsf{d}(i,j)$ oriented edges, each one carrying the label $\mathsf{p}(i,j)$. We also write $\mathscr{I}$ for this graph, and write $i \to j$ when $\mathsf{d}(i,j)>0$. If we add to $\mathscr{I}$ the vertex o with $\mathsf{d}(o,i)$ edges from o to each $i \in \mathscr{I}$ (where $\mathsf{d}(o,i)$ is the number of neighbours of o with type i), then we can label each of these edges with a probability $p(o,x)$, where $\imath(x)=i$. We write $\mathscr{I}^*$ for this enlarged labelled digraph. The original tree with its transition probabilities can be recovered from these data as the *directed cover* of $\mathscr{I}^*$. It consists of all oriented paths in $\mathscr{I}^*$ that start in o (and o itself corresponds to the path of length 0), and two paths are neighbours in the tree if one

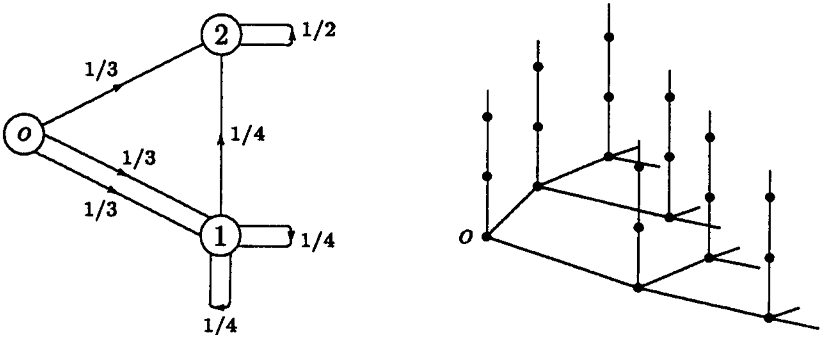


Fig. 1. A tree with two cone types.

extends the other by one step (edge), the label (forward transition probability) being the one of this last edge of $\mathscr{I}^*$.

(1.1) Example. A tree with two cone types. (See Fig. 1.)

The left hand side shows the augmented graph $\mathscr{I}^* = \{o\} \cup \mathscr{I}$, where $\mathscr{I} = \{1, 2\}$. The right hand side shows the resulting tree. The transition probabilities that are displayed along the edges of $\mathscr{I}^*$ correspond to the simple random walk on T. □

For $x, y \in T$ we write $\operatorname{dist}(x, y)$ for their distance (number of edges on the geodesic) in T, and $\operatorname{dist}_i(x, y)$ for the number of points of type i on the geodesic, where $i \in \mathscr{I}$. The *length* of x is $|x| = \operatorname{dist}(x, o)$, and $|x|_i = \operatorname{dist}_i(x, o)$.

Coming back to our original random walk on T, the questions that we shall consider here are the following:

(A) A simple criterion for *recurrence* or *transience* of the random walk.

Recall that transience means that the probability that the random walker will eventually return to the starting point is less than 1, while recurrence means that this probability is equal to one. Recurrence is further subdivided into positive recurrence and null recurrence (when the expected time until the first return is finite or infinite, resepectively)

(B) The question whether the *spectral radius* $\rho(P) = \lim_{n\to\infty} p^{(2n)}(o, o)^{1/2n}$ is equal to 1 or less than 1.

Here, as usual, $p^{(n)}(x, y) = \mathbb{P}[Z_n = y \mid Z_0 = x]$ is the probability that the random walk starting at x is at y after n steps. (Note that for our type of random walks, $p^{(n)}(x,x)=0$ when n is odd.) Thus, $\rho(P)<1$ means that the n-step transition probabilities decay exponentially.

(C) The asymptotic behaviour of $p^{(2n)}(o,o)$ as $n\to\infty$.

(D) Existence and computation of the *rate of escape*, that is, the almost sure limits of $|Z_n|/n$ and, more generally, of $\lim |Z_n|_i/n$, where $i\in\mathscr{I}$.

(E) A central limit theorem for $|Z_n|$.

Questions D and E regard specifically the transient case, when $|Z_n|\to\infty$ almost surely. In this case, the random walk converges almost surely to a random variable Z_∞ that takes its values in the *boundary* ∂T of T. (See Section 4 later for a brief description of the boundary. More details can be found in many sources, e.g., Cartier (1972), Woess (2000), Section 6.B.) For $x\in T$ and $\xi\in\widehat{T}=T\cup\partial T$, we denote by $\overline{x\xi}$ the geodesic from x to ξ. Suppose that $Z_0=o$, and let W_k be the element of length k on the random infinite geodesic $\overline{oZ_\infty}$. Then we shall be interested in the following.

(F) The structure of the *boundary process* $(W_k)_{k\geqslant 0}$ and how it is linked with the random walk.

We shall mostly assume that *the cone types are irreducible,* by which we mean that the graph $\mathscr{I}$ is strongly connected (for each pair of $i,j\in\mathscr{I}$, there is an oriented path from i to j). Equivalently, this means that the matrix

$$A=(\mathsf{a}(i,j))_{i,j\in\mathscr{I}},\qquad \text{where}\quad \mathsf{a}(i,j)=\frac{\mathsf{d}(i,j)\,\mathsf{p}(i,j)}{\mathsf{p}(-i)} \tag{1.2}$$

is irreducible in the sense of the Perron–Frobenius theory of non-negative matrices, see Seneta (1981). We write $\lambda(A)$ for the largest positive eigenvalue of A. In the irreducible case, this is the Perron–Frobenius eigenvalue. Even when A is not irreducible, it exists and is an eigenvalue with maximal modulus [Seneta (1981), Exercise 1.12]. Our first result does not require irreducibility. It extends the corresponding result of Lyons [(1990), Section 5] regarding homesick random walk.

(1.3) Theorem A. The random walk is

$$\begin{aligned}\text{positive recurrent} &\Leftrightarrow \lambda(A)<1\\ \text{null recurrent} &\Leftrightarrow \lambda(A)=1\\ \text{transient} &\Leftrightarrow \lambda(A)>1\end{aligned}$$

(1.4) Theorem B. If the cone types are irreducible, then $\rho(P) < 1$ if and only if the random walk is transient.

The "only if" is of course clear. The interesting part is that $\lambda(A) > 1 \Rightarrow \rho(P) < 1$ when A is irreducible. Example (1.1) shows that Theorem B fails when A is not irreducible, as in this case $\lambda(A) = 2$ and $\rho(P) = 1$.

(1.5) Theorem C. Suppose that the cone types are irreducible.

(a) If P is positive recurrent, then there is a constant $C_1 > 0$ such that

$$p^{(2n)}(o, o) \sim C_1 \qquad \text{as} \quad n \to \infty$$

(b) If P is null recurrent, then there is a constant $C_2 > 0$ such that

$$p^{(2n)}(o, o) \sim C_2 n^{-1/2} \qquad \text{as} \quad n \to \infty$$

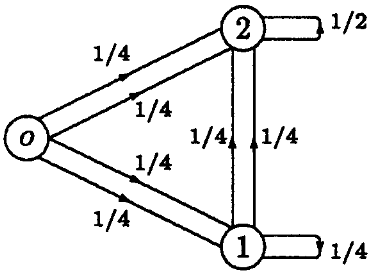
(c) If P is transient, then one of the following cases occurs:

$$\begin{aligned} p^{(2n)}(o, o) &\sim C_1 \rho(P)^{2n} && \text{as} \quad n \to \infty, \\ p^{(2n)}(o, o) &\sim C_2 \rho(P)^{2n}\, n^{-1/2} && \text{as} \quad n \to \infty, \\ p^{(2n)}(o, o) &\sim C_3 \rho(P)^{2n}\, n^{-3/2} && \text{as} \quad n \to \infty \end{aligned}$$

In the first two sub-cases of case (c), the random walk is ρ-recurrent, since then $\sum_n p^{(n)}(o, o)\, \rho(P)^n = \infty$. In the third sub-case, the random walk is ρ-transient. When the cone types are not irreducible, statement (a) will of course remain valid (since this is a property of general positive recurrent Markov chains), but (b) and (c) do not remain valid in the sense that different exponents may occur in the place of $-1/2$ or $-3/2$.

(1.6) Example. The two-dimensional *comb lattice. (See Fig. 2.)* The weights correspond to the simple random walk on T. In this example, the cone types are not irreducible, and $p^{(2n)}(o, o) \sim C\, n^{-3/4}$ as $n \to \infty$, see Gerl (1986) or Cassi and Regina (1992). □

We shall present examples for all cases of Theorem C, see in particular Section 4, Example (4.8) for the three sub-cases of case (c). It is however noteworthy that under additional assumptions, some of the cases will never occur, as the following example shows.



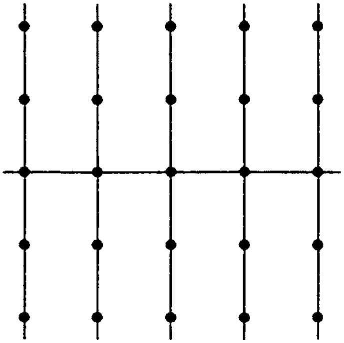


Fig. 2. The comb lattice.

(1.7) Example. Let AUT(T, P) be the group of all bijections $g: T \to T$ such that $p(gx, gy) = p(x, y)$ for all $x, y \in T$. Suppose that AUT(T, P) acts transitively on T. In this case (T, P) clearly has finitely many cone types, but they are not necessarily irreducible. Then there are the following possibilities:

(1) The tree T has linear growth, AUT(T, P) has an infinite cyclic subgroup that acts quasi-transitively, and

$$p^{(2n)}(o, o) \sim C_2 \rho(P)^{2n} n^{-1/2}$$

For the simple random walk, $\rho(P) = 1$. More generally, $\rho(P) = 1$ for random walks with *drift* (rate of escape) 0, see Krámli and Szász (1983).

(2) The tree has exponential growth, and

$$p^{(2n)}(o, o) \sim C_2 \rho(P)^{2n} n^{-3/2}$$

For the simple random walk, $\rho(P) < 1$. The case $\rho(P) = 1$ may occur when AUT(T, P) is an amenable (and non-unimodular) group, see Saloff-Coste and Woess (1996, 1997).

We give two specific examples where AUT(T, P) acts transitively.

(1.7.a) Example. Set $\mathcal{I} = \{1, \dots, N\}$ and choose $\mathsf{p}_1, \dots, \mathsf{p}_N > 0$ with $\sum_i \mathsf{p}_i = 1$. Set $\mathsf{p}(i, j) = \mathsf{p}_j$ and $\mathsf{d}(i, j) = 1$ when $j \neq i$, and $\mathsf{p}(i, i) = 0$. Then we have irreducible cone types. Furthermore, at the root o, let $\mathsf{p}(o, i) = \mathsf{p}_i$ for every $i \in \mathcal{I}$. The resulting Markov chain is a well-studied random walk on the group which is the free product of N copies of the two-element-group, or, equivalently, on the homogeneous tree with degree N (which is a

Cayley graph of that group). When $N=2$ we are in case (1), and when $N \geqslant 3$, we are in case (2) with $\rho(P)<1$. See, e.g., Gerl (1981).

(1.7.b) Example. Set $\mathscr{I}=\{1,2\}$ and choose parameters $\alpha \in (0,1)$ and $N \in \mathbb{N}$. Set

$$\mathsf{d}(1,1)=1, \quad \mathsf{d}(1,2)=N-1, \quad \mathsf{d}(2,2)=N, \quad \text{and}$$

$$\mathsf{p}(1,1)=\alpha, \quad \mathsf{p}(1,2)=\mathsf{p}(2,2)=(1-\alpha)/N, \quad \mathsf{p}(2,1)=0$$

The two cone types are not irreducible. To define $\mathscr{I}^*$, draw one edge from o to 1 with weight $\mathsf{p}(o,1)=\alpha$ and N edges from o to 2, each with weight $\mathsf{p}(o,2)=(1-\alpha)/N$. The resulting random walk takes place on the homogenous tree with degree $N+1$. When $N=1$ we obtain the random walk on $\mathbb{Z}$ with $p(k,k+1)=\alpha$ and $p(k,k-1)=1-\alpha$, and we have case (1). When $N \geqslant 2$, the group $\mathrm{AUT}(T,P)$ is amenable and non-unimodular, and we are in case (2). In both cases, $\rho(P)=1$ precisely when $\alpha=1/2$.

In order to see that one has indeed only the alternative between cases (1) and (2) as stated above, one can use the method applied by Gerl and Woess (1986). One first derives an equation for the function $G(z)=G(x,x\,|\,z)=\sum_n p^{(n)}(x,x)\,z^n$ $(z \in \mathbb{C})$, which is independent of $x \in T$ by transitivity:

$$G(z)=1+\tfrac{1}{2}\sum_{y}\left(\sqrt{1+4p(x,y)\,p(y,x)\,z^2\,G(z)^2}-1\right)$$

This is well known for free groups, see e.g., Levit and Molchanov (1971), Aomoto (1984) or Gerl and Woess (1986); in the general transitive case it is derived in the same way. An analysis of the principal singularity of $G(z)$ then leads to the result precisely as in Gerl and Woess (1986). In case (1) the tree T is a two-way-infinite path, in case (2) it is homogeneous with degree $\geqslant 3$. One can show that the same alternative holds more generally when the group acts with finitely many orbits.

(1.8) Example. When the graph $\mathscr{I}$ is a circle (without multiple edges) and in $\mathscr{I}^*$ the root is connected to precisely one vertex of $\mathscr{I}$, then T is a ray (half-line), and the random walk is a birth-and-death chain with periodic transition probabilities. Such chains were studied by Woess (1985), who showed that in the transient case, i.e., case (c) of Theorem C, the third subcase must occur with $\rho(P)<1$.

We now turn to questions D, E and F.

(1.9) Theorem D. Suppose that the cone types are irreducible, and that the random walk is transient. Then the almost sure limits

$$\ell(P) = \lim_{n\to\infty} \frac{|Z_n|}{n} \qquad \text{and} \qquad \ell_i(P) = \lim_{n\to\infty} \frac{|Z_n|_i}{n} \quad (i \in \mathscr{I})$$

exist and are all positive.

In Section 5 we shall deduce explicit formulas [(5.7) and (5.8)] for these limits in terms of the matrix A and the finite set of algebraic equations which is given in Proposition (2.5) below and which is the basis of almost all of the results presented here. These formulas extend results of Takacs (1997) regarding the rate of escape of homesick (or better "non-homesick" because concerning the transient regime) random walk on trees with finitely many cone types (and our proof is more direct and perhaps somewhat simpler).

(1.10) Theorem E. Suppose that the cone types are irreducible, and that the random walk is transient. Then we have convergence in law:

$$\frac{|Z_n| - n\ell(P)}{\sqrt{n}} \to N(0, \sigma^2)$$

the normal distribution with variance $\sigma^2 = \sigma^2(P) > 0$.

We postpone examples with concrete computations of $\ell(P)$, $\ell_i(P)$ and $\sigma^2(P)$ to Section 6, after deriving the relevant formulas. In particular, we emphasize that $\sigma^2 > 0$ strictly, thus generalizing the corresponding result of Sawyer and Steger (1987, Section 7) for free groups, whose proof is based on a sophisticated argument that is specific to that case. For bounded range random walks on free groups, Lalley (1993) has a clever method, via subshifts of finite type, to prove (among other things) that $\sigma^2 > 0$. We adapt Lalley's method to our situation. Note that in general, no reasonably large group acts on (T, P) in our context.

The following links the random walk with the boundary process.

(1.11) Theorem F. Suppose that the cone types are irreducible, and that the random walk is transient. Then

$$\lim_{n\to\infty} \frac{\operatorname{dist}(Z_n, \overline{oZ_\infty})}{\log n} = 0 \qquad \text{almost surely}$$

This is valid in greater generality than for nearest neighbour random walks which are transient in a "uniform" way. Under suitable conditions it will also hold for random walks on hyperbolic graphs. The proof is extrapolated from Ledrappier (1992), who applies a slightly more complicated argument to random walks on free groups.

The final result regards question F. We call it a "Theorem" because of its usefulness, although it turns out to be very simple. As a matter of fact, we shall prove it before Theorems D–F. It holds for arbitrary transient nearest neighbour random walks on trees and was already used by Takacs (1997, 1998) in the context of (non-)homesick random walks. This fact has also been observed by Sawyer and Steger (1987) and Lalley (1993) in the context of random walks on free groups.

(1.12) Theorem F′. In the transient case, the boundary process is a Markov chain. Its transition probabilities $\mathbb{P}[W_{k+1}=y \mid W_k=x]$ depend only on the cone types of x and y.

The paper is organized as follows. Section 2 contains several useful facts about the Green function of the random walk and its singularities. We also introduce a set of auxiliary generating functions $\{F_{-i}(z)\}_{i \in \mathscr{I}}$ depending on the cone types. Proposition (2.5) which says that they satisfy a finite system of algebraic equations is basic for most of our results. Theorem C is proved at the end of Section 2. Sections 3 and 4 are devoted to the proofs of Theorems A and B, respectively. In Section 5 we introduce the exit times for the random walk, which are used to prove Theorem F′ as well as some more facts about the boundary process on the tree. We then prove Theorem D and deduce explicit formulas (5.7)–(5.8) for the rate of escape $\ell(P)$. Theorem E is proved in Section 6. As a byproduct, we get yet another proof of Theorem D and formula (6.5) for $\ell(P)$. Finally, we prove Theorem F in Section 7. In conclusion, Section 8 contains some comments and ideas for future work.

2. GENERATING FUNCTIONS OF TRANSITION PROBABILITIES AND THEIR SINGULARITIES

Basic tools for our study will be the generating functions of n-step transition probabilities. We denote by $\mathbb{P}_x$ the probability measure on the trajectory space $\Omega = T^{\mathbb{N}_0}$ that governs the random walk starting at x. Besides $p^{(n)}(x,y)=\mathbb{P}_x[Z_n=y]$, where $x, y \in T$, we also define

$$f^{(n)}(x,y)=\mathbb{P}_x[Z_n=y,\, Z_k \neq y\,(k=1,\dots,n-1)]=\mathbb{P}_x[\mathbf{t}^y=n]$$

where $\mathbf{t}^y = \inf\{n \geqslant 1 : Z_n = y\}$ is the stopping time of the first visit to y (after the start). In particular, $f^{(0)}(x, y) = 0$. For complex z, let

$$G(x, y \mid z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)\, z^n \qquad \text{and} \qquad F(x, y \mid z) = \sum_{n=0}^{\infty} f^{(n)}(x, y)\, z^n$$

The radius of convergence $r = r(P) = 1/\rho(P)$ of the power series $G(x, y \mid z)$ is independent of x and y. If $s(x, y)$ denotes the radius of convergence of $F(x, y \mid z)$, then $s(x, y) \geqslant r(P)$. We have the following well-known and basic relations in the respective disks of convergence, see e.g., Woess (2000).

(2.1) Lemma.

(a) if $x, y, v \in T$ are distinct and y lies on the geodesic from x to v, then $F(x, v \mid z) = F(x, y \mid z)\, F(y, v \mid z)$.

(b) $G(x, x \mid z) = 1/(1 - F(x, x \mid z))$ and $G(x, y \mid z) = F(x, y \mid z)\, G(y, y \mid z)$, if $x \neq y$.

(c) $F(x, y \mid z) = p(x, y)\, z + \sum_{w \neq y} p(x, w)\, z F(w, y \mid z)$ for all x, y.

Only the first of the three statements depends on the tree structure. Now, *recurrence* means that $F(x, x \mid 1) = 1$ for some (equivalently, all) $x \in T$, or equivalently, that $G(x, y \mid 1) = \infty$ for some (equivalently, all) $x, y \in T$. In this case, *positive recurrence* means that $F'(x, x \mid 1) < \infty$, while *null recurrence* means that $F'(x, x \mid 1) = \infty$, and this is again independent of x. (More precisely, we mean $F'(x, x \mid 1-)$, the limit of the derivative from the left.) Analogously, the ρ-recurrent case is characterized by $F(x, x \mid r) = 1$ or equivalently $G(x, y \mid r) = \infty$ for some (all) x, y. One subdivides into ρ-positive-recurrence, when $F'(x, x \mid r) < \infty$, and ρ-null-recurrence, when $F'(x, x \mid r) = \infty$.

(2.2) Remark. In the case of (ordinary) recurrence, $F(x, y \mid 1) = 1$ for all x, y. However, in the case of ρ-recurrence with $\rho(P) < 1$, it is *not* true that $F(x, y \mid r) = 1$ for all x, y. As a matter of fact, in the ρ-recurrent case, it follows from Lemma (2.1.c) that the function

$$h(x) = \begin{cases} 1, & \text{if} \quad x = o, \\ F(x, o \mid r), & \text{otherwise} \end{cases}$$

satisfies $Ph = \rho h$, where $Ph(x) = \sum_y p(x, y)\, h(y)$. Therefore it cannot be constant when $\rho < 1$.

By Pringsheim's theorem ("for a power series with non-negative coefficients, the radius of convergence is a singularity"), $z = r$ is a singularity of each $G(x, y \mid z)$, and $z = s(x, y)$ is a singularity of $F(x, y \mid z)$. We shall write $s_o = s(o, o)$. The following does not depend on finiteness of the number of cone types.

(2.3) Lemma. With exception of the trivial case where $C_x = \{x\}$ and $p(x, x^-) = 1$, the radius of convergence $s(x, x^-)$ is always finite ($x \neq o$.)

Proof. Using (a) of Lemma (2.1), statement (c) can be rewritten as

$$F(x, x^- \mid z) = \frac{p(x, x^-)\, z}{1 - \tilde{F}(x \mid z)}, \qquad \text{where} \quad \tilde{F}(x \mid z) = \sum_{y:\, y^- = x} p(x, y)\, z F(y, x \mid z)$$

When $C_x \neq \{x\}$, the function $\tilde{F}(x \mid z)$ is non-constant and strictly increasing for positive $z \leqslant \tilde{s} = \min\{s(y, x) : y^- = x\}$, which is its smallest positive singularity. If $\tilde{F}(x \mid \tilde{s}) \leqslant 1$ then $\tilde{s} < \infty$, and $F(x, x^- | z)$ is analytic for $z \in [0, \tilde{s})$, but not beyond $\tilde{s}$. Thus, $s(x, x^-) = \tilde{s} < \infty$. If $\tilde{F}(x \mid \tilde{s}) > 1$ then $z = s(x, x^-)$ must be the unique solution in $(0, \tilde{s})$ of the equation $\tilde{F}(x \mid z) = 1$, and is again finite. □

Similar arguments are used in the following.

(2.4) Lemma.

(1) If $F(o, o \mid s_o) < 1$ then $r(P) = s_o$ and the random walk is ρ-transient (and consequently transient in the usual sense).

(2) If $F(o, o \mid s_o) = 1$ then $r(P) = s_o$ and the random walk is ρ-recurrent (and consequently transient in the usual sense when $s_o > 1$, recurrent when $s_o = 1$).

(3) If $F(o, o \mid s_o) > 1$ then $r(P)$ is the unique solution of the equation $F(o, o \mid r) = 1$ in the interval $[1, s_o)$, and the random walk is ρ-positive-recurrent (and consequently transient in the usual sense when $r(P) > 1$, recurrent when $r(P) = 1$).

Proof. This is immediate from Lemma (2.1.c), since $F(o, o \mid z)$ and $G(o, o \mid z)$ are monotone increasing for positive z, and s_o and $r(P)$ are their smallest positive singularities, respectively. □

Since the numbers $f^{(n)}(x, x^-)$ depend only on the transition probabilities within the cone C_x, the functions $F(x, x^- \mid z)$ coincide for all x with the same cone type $\iota(x) = i$, and we write $F_{-i}(z) = F(x, x^- \mid z)$. Also, we denote

$s_i = s(x, x^-)$ if $\imath(x) = i$. (On the other hand, it is not true in general that $F(x^-, x \mid z)$ depends only on z and the cone types of x^- and x.) Lemma (2.1) translates into a finite system of simple algebraic equations for the functions $F_{-i}(z)$, $i \in \mathscr{I}$, that will be fundamental for all that follows.

(2.5) Proposition. For each $i \in \mathscr{I}$, we have $F_{-i}(z) = \mathscr{Q}_i(z; F_{-j}(z), j \in \mathscr{I})$ where

$$\mathscr{Q}_i(z; w_j, j \in \mathscr{I}) = \mathsf{p}(-i)\, z + \sum_j \mathsf{d}(i, j)\, \mathsf{p}(i, j)\, z w_j w_i$$

Proof. Lemma (2.1.c + a) implies that on an arbitrary tree,

$$F(x, x^- | z) = p(x, x^-)\, z + \sum_{y:\, y^- = x} p(x, y)\, z F(y, x \mid z)\, F(x, x^- \mid z) \tag{2.6}$$

Rewriting this in terms of the respective cone types, we get the claim. □

From this, we can deduce first information about the critical singularities s_i of the functions $F_{-i}(z)$.

(2.7) Proposition. If the cone types are irreducible, then all s_i coincide. Denoting by $s = s(P)$ their common value, we have $s < \infty$ and $F_{-i}(s) < \infty$ for all i.

Proof. The equations of Proposition (2.5) imply that

$$s_i \leqslant \min\{s_i, s_j : j \in \mathscr{I}, i \to j\}$$

By irreducibility, $s_i \leqslant s_j$ for all $i, j \in \mathscr{I}$, and all s_i must coincide. Also by irreducibility, no cone is trivial, whence $s < \infty$. Suppose that $F_{-j}(s) = \infty$ for some j. Then $F_{-i}(s) = \infty$ for all i with $i \to j$. Again by irreducibility, we get that $F_{-i}(s) = \infty$ for all i. Let $z \in (0, s)$ and divide by $F_{-i}(z)$ in the equation of Proposition (2.5):

$$1 = \frac{\mathsf{p}(-i)\, z}{F_{-i}(z)} + \sum_j \mathsf{d}(i, j)\, \mathsf{p}(i, j)\, z\, F_{-j}(z)$$

Letting $z \to s$ from the left, we find $1 = \infty$, a contradiction. □

Next, we want to derive information about the behaviour of $F_{-i}(z)$ near the singularity $z = s$. Given the particularly simple form of the equations of Proposition (2.5), "elimination theory" of algebraic geometry implies that each $w = F_{-i}(z)$ is algebraic, that is, it satisfies a polynomial equation in the two variables z and w. See e.g., van der Waerden (1939), Section 2, or—in the context of formal power series—Kuich and Salomaa (1985), Section 16, for the algorithm to find the resultant polynomial. Therefore $F_{-i}(z)$ has an expansion as a *Puiseux series*

$$F_{-i}(z) = \sum_{n=0}^{\infty} a_i(n)(s-z)^{\alpha_i(n)}$$

valid in a neighbourhood of $z = s$ with the exception of the interval (s, ∞), where $\alpha_i(0) < \alpha_i(1) < \cdots$ is a discrete sequence of rational numbers, see e.g., Dimca (1977). As $F_{-i}(s) < \infty$, we must have $\alpha_i(0) = 0$ and $a_i(0) = F_{-i}(s)$. As $F_{-i}(z)$ is monotone increasing in $[0, s]$, we must have $a_i(1) < 0$. Thus the expansion starts with

$$F_{-i}(z) = a_i(0) + a_i(1)(s-z)^{\alpha_i(1)} + \text{h.o.t.}$$

where h.o.t. stands for "higher order terms."

(2.8) Lemma. If the cone types are irreducible, then there is $\alpha = \alpha(P) \in (0, 1]$ such that $\alpha_i(1) = \alpha$ for every $i \in \mathscr{I}$.

Proof. We insert all Puiseux expansions in the equation for $F_{-i}(z)$ of Proposition (2.5). Using the fact that $a_i(0) = \mathsf{p}(-i)\, s + \sum_j \mathsf{d}(i, j)\, \mathsf{p}(i, j) \times s a_j(0)\, a_i(0)$, which is the same equation at $z = s$, we find

$$\begin{aligned} &a_i(1)(s-z)^{\alpha_i(1)} + \text{h.o.t.} \\ &\quad = -\frac{a_i(0)}{s}(s-z) \\ &\qquad + \sum_j \mathsf{d}(i, j)\, \mathsf{p}(i, j)\, s\big(a_j(0)\, a_i(1)(s-z)^{\alpha_i(1)} + a_i(0)\, a_j(1)(s-z)^{\alpha_j(1)} + \text{h.o.t.}\big) \end{aligned}$$

Therefore

$$\alpha_i(1) = \min\{1, \alpha_i(1), \alpha_j(1) : j \in \mathscr{I}, i \to j\}$$

Consequently $\alpha_i(1) \leqslant 1$ and $\alpha_i(1) \leqslant \alpha_j(1)$ for all j with $i \to j$. By irreducibility, all $\alpha_i(1)$ have a common value α, where $0 < \alpha \leqslant 1$. □

Consider the Jacobian matrix of the system of equations of Proposition (2.5):

$$\mathfrak{J}(z) = \left(\frac{\partial \mathscr{Q}_i(z; F_{-k}(z), k \in \mathscr{I})}{\partial w_j}\right)_{i, j \in \mathscr{I}}$$

Its (i, j)-element is

$$\begin{cases} \mathsf{d}(i,i)\,\mathsf{p}(i,i)\,zF_{-i}(z) + \sum_{k\in\mathcal{I}} \mathsf{d}(i,k)\,\mathsf{p}(i,k)\,zF_{-k}(z), & \text{if} \quad j = i, \\ \mathsf{d}(i,j)\,\mathsf{p}(i,j)\,zF_{-i}(z), & \text{if} \quad j \neq i \end{cases}$$

For real $z \in (0, s]$, this is a non-negative, irreducible matrix. By the Perron–Frobenius theorem, its largest positive eigenvalue $\lambda(\mathfrak{J}(z))$ is a simple root of the characteristic equation, and the corresponding eigenvector has all entries > 0. The function $z \mapsto \lambda(\mathfrak{J}(z))$ is continuous and increasing.

We now apply a method that we have learnt from the paper of Lalley (1993) and which is exposed in detail in Woess (2000), Section 19. It has also been used in other branches of mathematics, see e.g., Hersonsky and Hubbard (1997), Drmota (1997). Following the argument as presented in Woess (2000), pp. 210–211, word by word (only the $\mathtt{r}$ there has to be replaced with our s), we obtain the following.

(2.9) Proposition. If the cone types are irreducible then $\alpha(P) = 1/2$ and $s(P) = \min\{z > 0 : \lambda(\mathfrak{J}(z)) = 1\}$.

(2.10) Proposition. If the cone types are irreducible, then the asymptotic behaviour of the first return probabilities to the root is

$$f^{(2n)}(o, o) \sim C_0 s(P)^{-2n}\, n^{-3/2}, \qquad \text{as} \quad n \to \infty$$

Proof. By Lemma (2.1.c), $F(o, o \mid z) = \sum_x p(o, x)\, zF(x, o \mid z)$. If $\imath(x) = i$ then $F(x, o \mid z) = F_{-i}(z)$. Therefore the Puiseux expansion of $F(o, o \mid z)$ at $z = s$ is of the form

$$F(o, o \mid z) = a(0) + a(1)(s - z)^{1/2} + \text{h.o.t.} \tag{2.11}$$

Since $f^{(n)}(o, o) = 0$ for odd n, also $z = -s$ is a singularity, and the Puiseux expansion of $F(o, o \mid z)$ at $z = -s$ is of the same form, with $(s + z)^{1/2}$ in the place of $(s - z)^{1/2}$. We claim that $F(o, o \mid z)$ has no further singularities on the circle of convergence $|z| = s$. The result is then an immediate application of Darboux's method; see Szegö (1939), Section 8.4.

Now, P defines a *reversible* Markov chain: if we define a measure m on T recursively by

$$m(o) = 1 \qquad \text{and} \qquad m(x) = m(x^-)\,\frac{p(x^-, x)}{p(x, x^-)}, \qquad \text{if} \quad x \neq o \tag{2.12}$$

(it is well defined because T is a tree), then P acts via $f \mapsto Pf$ as a self-adjoint operator on the space $\ell^2(T, m)$ of all functions $f: T \to \mathbb{R}$ with

$\|f\|_2^2 = \sum_x f(x)^2\, m(x) < \infty$. Therefore its resolvent is analytic in $\mathbb{C} \setminus \mathbb{R}$. The matrix elements of the resolvent are the functions $G(x, y \mid 1/z)$, where $x, y \in T$. This yields that each $G(x, y \mid z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$. Now $F(x, o \mid z) = G(x, o \mid z)/G(o, o \mid z)$ by Lemma (2.1.b), when $x \neq o$, and a non-real singularity of $F(x, o \mid z)$ on the circle of convergence $|z| = s$ could only be a pole. But this is impossible, since $|F(x, o \mid z)| \leqslant F(x, o \mid s) < \infty$. ☐

Proof of Theorem C. If $F(o, o \mid s) > 1$ then by Lemma (2.4), $r(P) = 1/\rho(P)$ is the unique solution in $[1, s)$ of the equation $F(o, o \mid z) = 1$. By the strict monotonicity of $F(o, o \mid \cdot)$ in $[0, s]$, it must be a simple pole of $G(o, o \mid z) = 1/(1 - F(o, o \mid z))$. The same holds for $z = -r(P)$, because $G(o, o \mid -z) = G(o, o \mid z)$. We know that there are no further singularities on the circle of convergence, see the proof of Proposition (2.10). Therefore

$$p^{(2n)}(o, o) \sim C_1 \rho(P)^{2n}$$

If $F(o, o \mid s) = 1$ then $r(P) = 1/\rho(P) = s$, and (2.11) yields that the Puiseux expansion of $G(o, o \mid z)$ at $z = r(P)$ is of the form

$$G(o, o \mid z) = b(1)(r - z)^{-1/2} + \text{h.o.t.}$$

and analogously at $z = -r(P)$. An application of Darboux's method as above leads to

$$p^{(2n)}(o, o) \sim C_2 \rho(P)^{2n}\, n^{-1/2}$$

Finally, if $F(o, o \mid s) < 1$ then again $r(P) = 1/\rho(P) = s$, but this time (2.11) yields that the Puiseux expansion of $G(o, o \mid z)$ at $z = r(P)$ is of the form

$$G(o, o \mid z) = b(0) + b(1)(r - z)^{1/2} + \text{h.o.t.}$$

Another application of Darboux's method yields

$$p^{(2n)}(o, o) \sim C_3 \rho(P)^{2n}\, n^{-3/2}$$

This implies the general statement of Theorem C. In the recurrent case, $\rho(P) = 1$, thus proving (a) and (b). ☐

3. RECURRENCE AND TRANSIENCE

Our next goal is to prove Theorem A. Recall that here we only assume finiteness of the number of cone types, but not their irreducibility. Note that it is implicit in our definition that T does possess cones of each of the types $i \in \mathcal{I}$.

We start with positive recurrence. It is well known that a time-homogeneous Markov chain is positive recurrent if and only if it has a stationary probability measure μ (that is, $\mu(y) = \sum_x \mu(x)\, p(x, y)$ for all y), and that in this case every stationary measure is a constant multiple of μ.

Now, our random walk on T is reversible: the measure m defined in (2.12) satisfies $m(x)\, p(x, y) = m(y)\, p(y, x)$ for all x, y. Therefore m is stationary. Thus, the random walk is positive recurrent if and only if $m(T) = \sum_x m(x) < \infty$.

For each $i \in \mathscr{I}$, we form a tree T_i with root o by taking a cone C_x of T, where $\imath(x) = i$, and connecting x with o by a single edge. On T_i, we consider the random walk whose transition probabilities are the original ones within C_x, the probability of going from x to o is $\mathsf{p}(-i)$, and the probability of going from o to x is 1. This random walk is again reversible with respect to the measure m_i that is defined as in (2.12), but for the random walk on T_i. The following is obvious.

(3.1) Lemma. (1) For every $x \in T \setminus \{o\}$ with $\imath(x) = i$,

$$m(C_x) = \frac{p(o, x_1)}{p(x_1, o)} \cdots \frac{p(x_{k-2}, x_{k-1})}{p(x_{k-1}, x_{k-2})}\, p(x_{k-1}, x_k)(m_i(T_i) - 1)$$

where $[o, x_1, \ldots, x_{k-1}, x]$ is the geodesic from o to x in T.

(2)
$$m(T) = \sum_x p(o, x)\, m_{\imath(x)}(T_{\imath(x)})$$

Thus $m(T) < \infty$ if and only if $m_i(T_i) < \infty$ for all $i \in \mathscr{I}$.

We now define T_i^n as the finite subtree of T_i spanned by all its vertices at distance $\leqslant n$ from the root, where $n \geqslant 0$. The sequence $(m_i(T_i^n))_n$ is increasing with limit $m_i(T_i)$. From the definition of m_i we get $m_i(T_i^0) = 1$ and

$$m_i(T_i^n) - 1 = \frac{1}{\mathsf{p}(-i)} \left(1 + \sum_{j:\, i \to j} \mathsf{d}(i, j)\, \mathsf{p}(i, j)(m_j(T_j^{n-1}) - 1) \right)$$

for each i. Consider the column vectors $\mathbf{m}_n = (m_i(T_i^n) - 1)_{i \in \mathscr{I}}$ and $\mathbf{p} = (1/\mathsf{p}(-i))_{i \in \mathscr{I}}$. Then the last equation can be rewritten as $\mathbf{m}_n = \mathbf{p} + A\mathbf{m}_{n-1}$, where A is the matrix defined in (1.2). We have obtained

$$\mathbf{m}_n = \mathbf{p} + A\mathbf{p} + A^2\mathbf{p} + \cdots + A^{n-1}\mathbf{p}$$

Thus, the sequence $(\mathbf{m}_n)_n$ will converge to a limit that is finite in each component, if and only if $\sum_{n \geqslant 0} A^n$ converges in each component. But this holds if and only if $\lambda(A) < 1$. We have proved the first part of Theorem A:

(3.2) Proposition. The random walk is positive recurrent if and only if $\lambda(A) < 1$.

Next, we study null recurrence. From the system of equations of Proposition (2.5) it is clear that the values $F_{-i}(1)$ depend continuously on the numbers $\mathsf{p}(-i)$ and $\mathsf{p}(i, j)$, $i, j \in \mathscr{I}$. Thus, it is quite easy to understand that $\lambda(A) = 1$ implies null recurrence. To be more precise, we use a slight extension of the "homesick" random walk of Lyons (1990). Let $\tau > 0$, the *homesickness parameter* (denoted λ by Lyons, who starts with the simple random walk). Define new transition probabilities

$$p_\tau(x, x^-) = \frac{p(x, x^-)\,\tau}{1 + p(x, x^-)(\tau - 1)}, \qquad p_\tau(x, y) = \frac{p(x, y)}{1 + p(x, x^-)(\tau - 1)}$$

where $x \in T \setminus \{o\}$ and y is any forward neighbour of x (i.e., $y^- = x$). The transition probabilities at o remain unchanged. When $\tau = 1$, this is the original random walk, and when τ increases, its tendency to move towards the origin increases.

When (T, P) has finitely many cone types then so has (T, P_τ). The matrix over $\mathscr{I}$ associated with P_τ as in (1.2) is

$$A_\tau = \frac{1}{\tau} A, \qquad \text{and} \qquad \lambda(A_\tau) = \frac{1}{\tau}\lambda(A)$$

(3.3) Proposition. If $\lambda(A) = 1$ then the random walk is null recurrent.

Proof. If $\lambda(A) = 1$ then $\lambda(A_\tau) < 1$ for all $\tau > 1$, and P_τ is positive recurrent. Therefore we have $F^{(\tau)}_{-i}(1) = 1$ for the associated hitting probabilities. By Proposition (2.5), these numbers depend continuously on τ, whence $F_{-i}(1) = 1$ for all $i \in \mathscr{I}$. Thus, P is recurrent. This must be null-recurrence, since we do not have $\lambda(A) < 1$. □

The converse of Proposition (3.3) will complete the proof of Theorem A.

(3.4) Proposition. If the random walk is null recurrent then $\lambda(A) = 1$.

Proof. We start with some observations concerning the Perron–Frobenius structure of the non-negative matrix A. Note that $i \to j$ in the graph $\mathscr{I}$ precisely when $\mathsf{a}(i, j) > 0$. For $i, j \in \mathscr{I}$ we write $i \xrightarrow{*} j$ if there is an oriented path in $\mathscr{I}$ from i to j. This includes paths of length 0, so that $i \xrightarrow{*} i$ for all i. Next, we write $i \leftrightarrow j$ if $i \xrightarrow{*} j$ and $j \xrightarrow{*} i$. This is an equivalence relation. We write $[\mathscr{I}]$ for the factor set and $[i]$ for the equivalence class of $i \in \mathcal{I}$, and denote by $A_{[i]}$ the restriction of A to $[i]$. The $A_{[i]}$, $[i] \in [\mathcal{I}]$, are precisely the irreducible blocks of the matrix A. Then

$$\lambda(A) = \max\{\lambda(A_{[i]}) : [i] \in [\mathcal{I}]\}$$

We shall show that null recurrence implies $\lambda(A_{[i]}) \leqslant 1$ for all i, whence $\lambda(A) \leqslant 1$. In view of Proposition (3.2), we then must have $\lambda(A) = 1$.

As already mentioned, null recurrence is characterized by

$$F(o, o \mid 1) = 1 \qquad \text{and} \qquad F'(o, o \mid 1-) = \infty$$

and in this case, $F(x, y \mid 1) = 1$ for all x, y. In particular, $z = 1$ is a singularity of $F(o, o \mid \cdot)$.

Even if the cone types are not irreducible, the equations of Proposition (2.5) imply that all functions F_{-i}, $i \in \mathcal{I}$, are algebraic. Each function F_{-i} is either analytic at $z = 1$, or it has a singularity there. In both cases, there will be a Puiseux expansion near $z = 1$, except for real $z > 1$, of the form

$$F_{-i}(z) = 1 - f_i(1-z)^{\gamma_i} + \text{h.o.t.}$$

where γ_i is a positive rational. When F_{-i} is analytic at $z = 1$ then $\gamma_i = 1$. Monotonicity of $F_{-i}(z)$ for $z \in [0, 1]$ implies that $f_i > 0$. Proceeding as in the proof of Lemma (2.8), we find

$$\gamma_i = \min\{1, \gamma_i, \gamma_j : j \in \mathcal{I}, i \to j\}$$

In particular, $\gamma_i \leqslant 1$, and $\gamma_i \leqslant \gamma_j$ for all $j \in \mathcal{I}_i = \{j \in \mathcal{I} : i \xrightarrow{*} j\}$. Note that $\mathcal{I}_i \cup \{o\}$, where the origin o is connected to i, is the augmented graph of cone types that encodes the tree T_i used on the way of proving Proposition (3.2).

Let A_i be the matrix associated with T_i according to (1.2). Then A_i is the restriction of A to $\mathcal{I}_i$, and $A_{[i]}$ is an irreducible block of A_i.

First suppose that $\gamma_i = 1$. Note that the generating function of the first return probabilities to o in T_i is $F_i(o, o \mid z) = zF_{-i}(z)$. We find that $F_i(o, o \mid 1) = 1$ and $F_i'(o, o \mid z) < \infty$, so that the random walk on T_i is positive recurrent. Proposition (3.2) yields $\lambda(A_i) < 1$, and therefore also $\lambda(A_{[i]}) < 1$.

Now suppose that $\gamma_i < 1$. For $j \in \mathcal{I}_i$, let

$$f_j^* = \begin{cases} f_j, & \text{if} \quad \gamma_j = \gamma_i, \\ 0, & \text{if} \quad \gamma_j > \gamma_i \end{cases}$$

By differentiating the equations of Proposition (2.5), we get for $j \in \mathscr{I}_i$

$$F'_{-j}(z) = p(-j) + \sum_{k \in \mathscr{I}_i} \mathsf{d}(j,k)\, \mathsf{p}(j,k)$$

$$\times (F_{-k}(z)\, F_{-j}(z) + zF'_{-k}(z)\, F_{-j}(z) + zF_{-k}(z)\, F'_{-j}(z))$$

We divide by $\gamma_i(1-z)^{\gamma_i - 1}$ and take the limit when $z \to 1$ from the left. Since $\gamma_j \geqslant \gamma_i$ for all $j \in \mathscr{I}_i$,

$$\lim_{z \to 1-} \frac{F_{-j}(z)'}{\gamma_i(1-z)^{\gamma_i - 1}} = f^*_j$$

and we obtain

$$f^*_j = \sum_{k \in \mathscr{I}_i} \mathsf{d}(j,k)\, \mathsf{p}(j,k)(f^*_k + f^*_j)$$

Denoting $\mathbf{f}^*_i = (f^*_j)_{j \in \mathscr{I}_i}$ (column vector), and observing that $\sum_k \mathsf{d}(j,k) \times \mathsf{p}(j,k) = \mathsf{p}(-j)$, we can rewrite this as $A_i \mathbf{f}^*_i = \mathbf{f}^*_i$.

Now let $\mathbf{f}_{[i]}$ be the restriction of $\mathbf{f}^*_i$ from $\mathscr{I}_i$ to $[i]$. This vector is strictly positive in each entry, and $A_{[i]}\mathbf{f}_{[i]} \leqslant \mathbf{f}_{[i]}$ elementwise. The subinvariance theorem of the Perron–Frobenius theory implies $\lambda(A_{[i]}) \leqslant 1$, see Seneta (1981), Thm. 1.6. □

4. THE SPECTRAL RADIUS IN THE TRANSIENT CASE

In this section we shall prove Theorem B. Starting point is the following formula

$$\sum_{y : y^- = x} p(x,y)(1 - F(y, x \mid 1)) = p(x, x^-)\left(\frac{1}{F(x, x^- \mid 1)} - 1\right) \tag{4.1}$$

when $x \neq o$, which follows from Lemma (2.1) and holds for arbitrary nearest neighbour random walks on arbitrary trees.

In the case of finitely many, irreducible cone types, we see from Proposition (2.5) that transience implies $F_{-i}(1) < 1$ for all $i \in \mathscr{I}$. Therefore we can define the matrix $Q = (\mathsf{q}(i,j))_{i,j \in \mathscr{I}}$ with

$$\mathsf{q}(i,j) = \frac{1 - F_{-j}(1)}{1 - F_{-i}(1)}\, F_{-i}(1)\, \mathsf{a}(i,j) \tag{4.2}$$

with $\mathsf{a}(i, j)$ as defined in (1.2). We introduce the diagonal matrix

$$D(z) = \operatorname{diag}(F_{-i}(z))_{i \in \mathcal{I}} \tag{4.3}$$

Then

$$Q = (I - D(1))^{-1}\, D(1)\, A\, (I - D(1))$$

is an irreducible and—by (4.1)—stochastic matrix, whence $\lambda(Q) = 1$. Therefore we also have

$$\lambda(D(1)\, A) = 1 \tag{4.4}$$

Next, let $\mathbf{f}(z) = (F_{-i}(z))_{i \in \mathcal{I}}$ and (as in Section 3) $\mathbf{p} = (1/\mathsf{p}(-i))_{i \in \mathcal{I}}$, two column vectors. By differentiating the equations of Proposition (2.5) as at the end of Section 3, we get in vector notation

$$\mathbf{f}'(z) = \frac{1}{z^2}\, D(z)^2\, \mathbf{p} + D(z)^2\, A \mathbf{f}'(z)$$

Recall from Proposition (2.7) that all $F_{-i}(z)$ have a common radius of convergence $s(P)$. For $0 < z < s(P)$, both $D(z)$ and $\mathbf{f}'(z)$ are finite, analytic and positive. For small positive z, we have $\lambda(D(z)^2\, A) < 1$ and

$$\mathbf{f}'(z) = \frac{1}{z^2}\, (I - D(z)^2\, A)^{-1}\, D(z)^2\, \mathbf{p} \tag{4.5}$$

By analycity, we must have $\lambda(D(z)^2\, A) < 1$ for all positive $z < s(P)$. For $z = s(P)$, the matrix $D(z)$ is finite by Proposition (2.7), and continuity yields $\lambda(D(z)^2\, A) \leqslant 1$. But each component of $\mathbf{f}'(s(P))$ is infinite by Proposition (2.9), whence

$$\lambda(D(s(P))^2\, A) = 1$$

In particular, we have obtained the following in the case of irreducible cone types.

(4.6) Lemma. $s(P) = \sup\{z > 0 : \lambda(D(z)^2\, A) < 1\}$.

Proof of Theorem B. We know that because of transience and irreducibility of cone types $F_{-i}(1) < 1$ for all $i \in \mathcal{I}$. Therefore we have

$D(1)^2 A < D(1) A$ strictly in each positive matrix element, and $\lambda(D(1) A) = 1$ by (4.4). The monotonicity of the Perron–Frobenius eigenvalue yields $\lambda(D(1)^2 A) < 1$. Consequently $s(P) > 1$.

By Lemma (2.1.c), the radius of convergence of $F(o, o \mid z)$ is also $s(P)$, and $F(o, o \mid s(P)) < 1$. By Lemma (2.4), $\rho(P) = 1/s(P)$. □

(4.7) Remark. Thus, in the case of irreducible cone types, we always have $s(P) \geqslant 1$, and $s(P) = 1$ if and only if the random walk is null recurrent. □

(4.8) Example. Let $\mathscr{I} = \{1, 2\}$ and set $\mathsf{d}(i, j) = 1$ for all i and j. Set $\mathsf{p}(1, 1) = \mathsf{p}(1, 2) = (1 - \mathsf{p})/2$ and $\mathsf{p}(2, 1) = \mathsf{p}(2, 2) = (1 - \mathsf{q})/2$, where $0 < \mathsf{p}, \mathsf{q} < 1$. One computes $\lambda(A) = \frac{1}{2\mathsf{p}} + \frac{1}{2\mathsf{q}} - 1$. In particular, when $\mathsf{p} \leqslant 1/4$, the random walk is transient for any value of q. The equations (2.5) become

$$F_{-1}(z) = \mathsf{p}\, z + \frac{1-\mathsf{p}}{2}\, z(F_{-1}(z)^2 + F_{-1}(z)\, F_{-2}(z)) \qquad \text{and}$$

$$F_{-2}(z) = \mathsf{q}\, z + \frac{1-\mathsf{q}}{2}\, z(F_{-2}(z)^2 + F_{-1}(z)\, F_{-2}(z))$$

Eliminating F_{-2}, we find the following equation for F_{-1}.

$$(\mathsf{p}-\mathsf{q})(1-\mathsf{p})\, zF_{-1}(z)^3 - (2(\mathsf{p}-\mathsf{q}) + (1-\mathsf{p})(\mathsf{p}+\mathsf{q}-2\mathsf{p}\mathsf{q})\, z^2)\, F_{-1}(z)^2$$
$$+ 2\mathsf{p}(1+\mathsf{p}-2\mathsf{q})\, zF_{-1}(z) - 2\mathsf{p}^2(1-\mathsf{q})\, z^2 = 0$$

(By exchanging p and q, we get the analogous equation for F_{-2}.) Using this equation and Maple we computed $s(P)$ and $s(P)\, F_{-i}(s(P))$ for a few values of p and q.

(p, q)	$(0.1, 0.7)$	$(0.1, 0.5)$	$(0.1, 0.393)$	$(0.1, 0.3)$
$s(P)$	1.119	1.164	1.111	1.182
$s(P)\, F_{-1}(s(P))$	0.463	0.414	0.393	0.393
$s(P)\, F_{-2}(s(P))$	1.158	1.081	1.000	0.901

(The values are rounded at the 3rd digit.) Now take $T = T_2$, that is, the root o is only connected to a vertex of type 2. Then the resulting random walk is ρ-recurrent with $1/s(P) < \rho(P) < 1$ when $\mathsf{p} = 0.1$ and $\mathsf{q} > 0.393$, and we are in case (c.1) of Theorem C. If $\mathsf{p} = 0.1$ and $\mathsf{q} = 0.393$ then the random walk is still ρ-recurrent, with $\rho(P) = 1/s(P)$, and we are in case 2 of

Theorem 3(c). Finally, if $\mathsf{p} = 0.1$ and $\mathsf{q} < 0.393$ then the random walk is ρ-transient with $\rho(P) = 1/s(P)$, and we are in case (c.3) of Theorem C. □

5. EXIT TIMES AND THE ASYMPTOTIC FREQUENCY OF CONE TYPES

Suppose that the transition matrix P gives rise to a transient nearest neighbour random walk $(Z_n)_{n \geqslant 0}$ on the infinite tree T with root o. We do not yet suppose to have finitely many cone types, and nothing of what we are going to do next depends on the choice of the root. The boundary ∂T of T consists of all equivalence classes of one-way-infinite geodesics, where two geodesics are equivalent if they differ only by finite initial pieces. For $x \in T$, we write ∂C_x for the set of boundary points that have a representative geodesic in the cone C_x, and $\widehat{C}_x = C_x \cup \partial C_x$. The topology on $\widehat{T} = T \cup \partial T$ is discrete on T. A neighbourhood base of $\xi \in \partial T$ is given by all sets $\widehat{C}_x$, where x lies on the geodesic $\overline{o\xi}$ from o to ξ.

Transience implies that $|Z_n| \to \infty$, and that Z_n converges in the topology of $\widehat{T}$ to a random point $Z_\infty \in \partial T$. This is well known and easy to see, compare e.g. with Woess (2000), Thm. 21.15. We assume that $Z_0 = o$. Then we can define the *exit times* $\mathbf{e}_k$, $k \geqslant 0$, by

$$\mathbf{e}_k = \sup\{n \geqslant 0 : |Z_n| = k\}$$

We have $\mathbf{e}_k < \mathbf{e}_{k+1} < \infty$ almost surely. We define

$$W_k = Z_{\mathbf{e}_k}$$

Then $W_0 = o$. If $W_k = x$, where $x \in T$ has length $|x| = k$, then (Z_n) has to move to a forward neighbour y of x at time $\mathbf{e}_k + 1$, and it has to stay in C_y from this time onwards (since otherwise, by the nearest neighbour property, it would have to return to x). Thus, $(W_k)_{k \geqslant 0}$ is indeed the boundary process defined in Section 1, and

$$\mathbb{P}_o[Z_\infty \in \widehat{C}_x] = \mathbb{P}_o[W_k = x]$$

In the sequel, we shall write $G(x, y) = G(x, y \mid 1)$ and $F(x, y) = F(x, y \mid 1)$. Again, we shall use the formula (4.1) to get the following.

(5.1) Lemma. If $|x| = k \geqslant 1$ then

$$\mathbb{P}_o[W_k = x] = G(o, x)\, p(x, x^-)\left(\frac{1}{F(x, x^-)} - 1\right)$$

Proof. We decompose according to the value of $\mathbf{e}_k$.

$$\begin{aligned}\mathbb{P}_o[W_k = x] &= \sum_l \mathbb{P}_o[\mathbf{e}_k = l, Z_l = x]\\ &= \sum_l \mathbb{P}_o[Z_l = x, Z_n \in C_x \setminus \{x\}\ \forall n > l]\\ &= \sum_l \mathbb{P}_o[Z_l = x]\ \mathbb{P}_x[Z_n \in C_x \setminus \{x\}\ \forall n \geqslant 1]\\ &= \sum_l p^{(l)}(o, x) \sum_{y:\, y^- = x} p(x, y)(1 - F(y, x))\end{aligned}$$

The rest follows from (4.1). □

We now consider the subtree T^o spanned by all $x \in T$ that are attained by (W_k) with positive probability. To make this clearer, observe that transience implies that $F(x, o) < 1$ for some forward neighbour of o, and if $x \in T \setminus \{o\}$ is such that $F(x, x^-) < 1$ then there must be some forward neighbour y of x with $F(y, x) < 1$. Conversely, if $F(x, x^-) = 1$ then $F(y, x) = 1$ for all forward neighbours of x, and consequently $F(y, y^-) = 1$ for all $y \in C_x$. Therefore $\{o\} \cup \{x \neq o : F(x, x^-) < 1\}$ is (the vertex set of) an infinite subtree, and this is T^o, since for $|x| = k > 0$ we have $\mathbb{P}_o[W_k = x] > 0$ precisely when $F(x, x^-) < 1$. In particular, $Z_\infty \in \partial T^o$ almost surely.

Proof of Theorem F′. Let $x, y \in T^o$ with $|x| = k$ and $y^- = x$. We have already observed that $[W_{k+1} = y] \subset [W_k = x]$. Therefore Lemma (5.1) implies that (W_k) is a Markov chain, and for $k > 0$, its transition probabilities are

$$q(x, y) = \mathbb{P}_o[W_{k+1} = y \mid W_k = x] = \frac{1 - F(y, x)}{1 - F(x, x^-)} \frac{p(x, y)}{p(x, x^-)} F(x, x^-) \tag{5.2}$$

In this computation, we have used reversibility:

$$\frac{G(o, y)}{G(o, x)} = \frac{m(y)\, G(y, o)}{m(x)\, G(x, o)} = \frac{p(x, y)}{p(y, x)} F(y, x)$$

Clearly, the transition probabilities in (5.2) depend only on the cone types of x and y. □

Next, consider the increments of the exit times: $\mathbf{i}_0 = \mathbf{e}_0$ and $\mathbf{i}_k = \mathbf{e}_k - \mathbf{e}_{k-1}$ for $k \geqslant 1$. Note that $\mathbf{i}_0$ takes every even value and $\mathbf{i}_k$ $(k \geqslant 1)$

every odd value with positive probability. We shall now strengthen Theorem F′ by the following.

(5.3) Proposition. Let $x, y \in T^o$ with $|x| = k \geqslant 1$ and $y^- = x$. Then for all odd m, n,

$$\begin{aligned}&\mathbb{P}_o[W_{k+1} = y, \mathbf{i}_{k+1} = n \mid W_k = x, \mathbf{i}_k = m]\\ &\quad = \frac{F(x, x^-)}{1 - F(x, x^-)} \frac{1 - F(y, x)}{F(y, x)} \frac{p(x, y)}{p(x, x^-)} f^{(n)}(y, x)\end{aligned}$$

If y is a neighbour of o, then for all even m and odd n

$$\mathbb{P}_o[W_1 = y, \mathbf{i}_1 = n \mid \mathbf{i}_0 = m] = G(o, y) \frac{1 - F(y, x)}{F(y, x)} p(o, y) f^{(n)}(y, o)$$

Proof. Recall that $W_k = x$ implies $W_{k-1} = x^-$ and $Z_n = x$ for $n = \mathbf{e}_{k-1} + 1$. In the same way as in Lemma (5.1), we compute

$$\begin{aligned}&\mathbb{P}_o[W_k = x, \mathbf{i}_k = m]\\ &= \sum_l \mathbb{P}_o[\mathbf{e}_{k-1} = l, Z_l = x^-, \mathbf{e}_k = l + m, Z_{l+m} = x]\\ &= G(o, x^-)\, \mathbb{P}_{x^-}[Z_r \in C_x\, (r = 1, \ldots, m-1), Z_m = x]\, \mathbb{P}_x[Z_r \in C_x \setminus \{x\}\ \forall r \geqslant 1]\end{aligned}$$

Now note that by reversibility,

$$\begin{aligned}&\mathbb{P}_{x^-}[Z_r \in C_x\, (r = 1, \ldots, m-1), Z_m = x]\\ &\quad = \frac{m(x)}{m(x^-)} \mathbb{P}_x[Z_{m-r} \in C_x\, (r = 1, \ldots, m-1), Z_m = x^-]\\ &\quad = \frac{p(x^-, x)}{p(x, x^-)} f^{(m)}(x, x^-)\end{aligned}$$

Using this and (4.1), we conclude that

$$\begin{aligned}&\mathbb{P}_o[W_k = x, \mathbf{i}_k = m]\\ &\quad = G(o, x^-) \frac{p(x^-, x)}{p(x, x^-)} f^{(m)}(x, x^-) \sum_{y:\, y^- = x} p(x, y)(1 - F(y, x))\\ &\quad = G(o, x^-)\, p(x^-, x) \left(\frac{1}{F(x, x^-)} - 1 \right) f^{(m)}(x, x^-)\end{aligned}$$

In the same way,

$$\begin{aligned}
&\mathbb{P}_o[W_{k+1}=y,\, \mathbf{i}_{k+1}=n,\, W_k=x,\, \mathbf{i}_k=m]\\
&\quad = G(o,x^-)\,\mathbb{P}_{x^-}\left[\begin{matrix} Z_r\in C_x\,(r=1,\dots,m-1),\, Z_m=x,\, Z_{m+s}\in C_y \\ (s=1,\dots,n-1),\, Z_{m+n}=y,\, Z_{m+q}\in C_y\setminus\{y\}\ \forall q>m+n \end{matrix}\right]\\
&\quad = G(o,x^-)\,\frac{p(x^-,x)}{p(x,x^-)}\, f^{(m)}(x,x^-)\,\frac{p(x,y)}{p(y,x)}\, f^{(n)}(y,x)\, p(y,x)\left(\frac{1}{F(y,x)}-1\right)
\end{aligned}$$

Taking quotients, the first of the two proposed formulas follows. The second one is slightly different, because o has no predecessor. We leave the (analogous) details to the reader. □

Note in particular that the conditional probabilities of Proposition (5.3) do not depend on m. Combining this with the fact that $(W_k)_{k\geqslant 0}$ is a Markov chain with state space T^o, we conclude that $(W_k, \mathbf{i}_k)_{k\geqslant 0}$ is a Markov chain with state space $T^o\times\mathbb{N}$. Its transition probabilities $\tilde{q}((x,m),(y,n))$ computed in Proposition (5.3) do not depend on m. Therefore this Markov chain factorizes over $\mathbb{N}$, and its image under the natural projection $(x,m)\mapsto x$ is the boundary process (W_k) on T.

We now apply these results to our situation of finitely many irreducible cone types, thereby proving Theorem D. We know that transience implies $F_{-i}(1)<1$ for all $i\in\mathscr{I}$. Therefore the transient part T^o of T coincides with the whole of T. By Theorem F′, the boundary process factorizes with respect to the cone types, that is, the sequence of $\mathscr{I}$-valued random variables $(\imath(W_k))_{k\geqslant 1}$ is an $\mathscr{I}$-valued Markov chain. (The time $k=0$ plays a special role because the origin has no predecessor.) By (5.2), its transition matrix is precisely the matrix $Q=(I-D(1))^{-1}\,D(1)\,A\,(I-D(1))$ defined in (4.2). Thus, $(\imath(W_k))_{k\geqslant 1}$ is positive recurrent. Let π be its stationary probability distribution on $\mathscr{I}$, that is, the solution of

$$\pi\, Q=\pi, \qquad \sum_i \pi(i)=1 \tag{5.4}$$

where $\pi=(\pi(i))_{i\in\mathscr{I}}$ is thought of as a row vector. Note that

$$|W_k|_i=\sharp\{m\leqslant k : \imath(W_m)=i\}$$

From the ergodic theorem for positive recurrent Markov chains we immediately obtain the following.

(5.5) Corollary. $\lim_k |W_k|_i/k=\pi(i)$ $\mathbb{P}_o$-almost surely for every $i\in\mathscr{I}$.

Next, we consider the Markov chain $(W_k, \mathbf{i}_k)_{k \geqslant 1}$ on $(T \setminus o) \times \mathbb{N}_{\text{odd}}$. (As above, the time $k = 0$ plays an extra role.) From Proposition (5.3) we see that its transition probabilities from (x, m) to (y, n) depend only on n and the cone types of x and y. Thus, we can factorize with respect to the cone types, and obtain the Markov chain $(\imath(W_k), \mathbf{i}_k)_{k \geqslant 1}$ on $\mathscr{I} \times \mathbb{N}_{\text{odd}}$. Its transition probabilities are

$$\tilde{\mathsf{q}}((i, m), (j, n)) = \mathsf{q}(i, j) \frac{\mathsf{f}^{(n)}(-j)}{F_{-j}(1)}$$

where $\mathsf{f}^{(n)}(-j) = f^{(n)}(y, y^-)$ for $y \in T \setminus \{o\}$ with $\imath(y) = j$, and $\mathsf{q}(i, j)$ is given by (4.2). This Markov chain is irreducible and factorizes over $\mathbb{N}_{\text{odd}}$, and the factor chain is $(\imath(W_k))_{k \geqslant 1}$. Thus, it is easy to find that $\tilde{Q}$ has the stationary distribution

$$\tilde{\pi}(j, n) = \sum_i \pi(i) \, \tilde{\mathsf{q}}((i, m), (j, n))$$

which is independent of the choice of m. In particular, $(\imath(W_k), \mathbf{i}_k)_{k \geqslant 1}$ is positive recurrent.

(5.6) Proposition. For the exit times, we have $\mathbb{P}_o$-almost surely

$$\lim_k \frac{\mathbf{e}_k}{k} = \sum_i \pi(i) \frac{F'_{-i}(1)}{F_{-i}(1)}$$

Note that $F'_{-j}(1)/F_{-j}(1) = \mathbb{E}_y(\mathbf{t}^x \mid \mathbf{t}^x < \infty)$, where $\mathbb{E}_y$ denotes the expectation with respect to the measure $\mathbb{P}_y$ on the trajectory space, $y \in T \setminus \{o\}$ has cone type $\imath(y) = j$, and $\mathbf{t}^x$ is the first arrival time at $x = y^-$.

Proof of Proposition (5.6). Consider the projection $g\colon \mathscr{I} \times \mathbb{N}_{\text{odd}} \to \mathbb{N}$, $(j, n) \mapsto n$. Then the right hand side of the proposed limit is $\int g \, d\tilde{\pi}$, which is finite, so that we may apply the ergodic theorem. Thus

$$\lim_k \frac{1}{k} \sum_{m=1}^{k} g(\imath(W_k), \mathbf{i}_k) = \int g \, d\tilde{\pi} \qquad \mathbb{P}_o\text{-almost surely}$$

The term in the limit on the left hand side is $(\mathbf{e}_k - \mathbf{e}_0)/k$. □

Proof of Theorem D. In the trajectory space Ω, let $\Omega_\infty = \{\omega\colon Z_n(\omega) \to Z_\infty(\omega)\}$. Then $\mathbb{P}_o(\Omega_\infty) = 1$ and for each $\omega \in \Omega_\infty$, all $\mathbf{e}_k(\omega)$ exist, are finite and tend to ∞. For each $n \in \mathbb{N}$, we define

$$k(n) = k(n, \omega) = \max\{k\colon \mathbf{e}_k(\omega) \leqslant n\}$$

Then $k(n) \to \infty$ $\mathbb{P}_o$-almost surely, $Z_{\mathbf{e}_{k(n)}} = W_{k(n)}$, and since our random walk is of nearest neighbour type,

$$|Z_n|_i - |W_{k(n)}|_i \leqslant \operatorname{dist}(Z_n, W_{k(n)}) \leqslant n - \mathbf{e}_{k(n)} < \mathbf{e}_{k(n)+1} - \mathbf{e}_{k(n)}$$

Now

$$0 < \frac{\mathbf{e}_{k(n)+1} - n}{n} \leqslant \frac{\mathbf{e}_{k(n)+1} - \mathbf{e}_{k(n)}}{n} \leqslant \frac{\mathbf{e}_{k(n)+1} - \mathbf{e}_{k(n)}}{\mathbf{e}_{k(n)}} \to 0$$

and consequently also

$$\frac{|Z_n|_i - |W_{k(n)}|_i}{n} \to 0 \qquad \text{and} \qquad \frac{\mathbf{e}_{k(n)}}{n} \to 1$$

$\mathbb{P}_o$-almost surely when $n \to \infty$. Therefore, using Corollary (5.5),

$$\frac{|Z_n|_i}{n} = \frac{|Z_n|_i - |W_{k(n)}|_i}{n} + \frac{|W_{k(n)}|_i}{k(n)} \frac{k(n)}{\mathbf{e}_{k(n)}} \frac{\mathbf{e}_{k(n)}}{n} \to \pi(i)\, \ell(P)$$

where

$$\ell(P) = 1 \Big/ \sum_i \pi(i) \frac{F'_{-i}(1)}{F_{-i}(1)} \tag{5.7}$$

is the reciprocal of the limit in Proposition (5.6). ☐

The formula for $\ell(P)$ can also be rewritten in the following way, which is simpler, because it does not involve derivatives.

(5.8) Lemma.

$$\ell(P) = 1 \Big/ \sum_i \pi(i) \frac{F_{-i}(1)}{\mathsf{p}(-i)((1 - F_{-i}(1))}$$

Proof. The row vector $\pi = (\pi(i))_{i \in \mathcal{I}}$ satisfies $\pi Q = Q$, and (4.2) implies

$$\pi(I - D(1))^{-1}\, D(1)\, A\, D(1) = \pi(I - D(1))^{-1}\, D(1)$$

or equivalently,

$$\pi(I - D(1))^{-1}\, (I - D(1)\, A\, D(1)) = \pi$$

Set $z = 1$ in (4.5) and rewrite (4.5) as

$$D(1)^{-1}\,\mathbf{f}'(1) = (I - D(1)\,A\,D(1))^{-1}\,D(1)\,\mathbf{p}$$

Therefore

$$\begin{aligned} 1/\ell(P) &= \pi D(1)^{-1}\,\mathbf{f}'(1) \\ &= \pi(I - D(1)\,A\,D(1))^{-1}\,D(1)\,\mathbf{p} = \pi(I - D(1))^{-1}\,D(1)\,\mathbf{p} \end{aligned}$$

This is the proposed formula. □

When specializing to the case of non-homesick random walk, the formula (5.8) coincides with the one given by Takacs (1997). In the next section we shall find yet another formula for $\ell(P)$ (see (6.5)).

6. THE CENTRAL LIMIT THEOREM

The tool that we shall use for proving Theorem E has been developed by Sawyer and Steger (1987), Theorem B.2 and Section 4 (in particular, see the last paragraph of Section 4). We restate it here for the convenience of the reader.

(6.1) Theorem [Sawyer and Steger]. Let (Y_n) be a sequence of real-valued random variables such that for some $\delta > 0$,

$$\mathbb{E}\left(\sum_{n=0}^{\infty} \exp(-rY_n - sn)\right) = \frac{C(r, s)}{g(r, s)} \qquad \text{for} \quad 0 < r, s < \delta$$

where $C(r, s)$ and $g(r, s)$ are analytic for $|r| < \delta$ and $|Re(s)| < \delta$ and $C(0, 0) \neq 0$. Then

$$\frac{Y_n}{n} \to \ell = \frac{g_r(0, 0)}{g_s(0, 0)} \quad \text{almost surely, and}$$

$$\frac{Y_n - n\ell}{\sqrt{n}} \to N(0, \sigma^2) \quad \text{in law, where}$$

$$\sigma^2 = \frac{-g_{rr}(0, 0) + 2\ell g_{rs}(0, 0) - \ell^2 g_{ss}(0, 0)}{g_s(0, 0)} \geqslant 0$$

(When $\sigma^2 = 0$ then the limit distribution is the point mass at 0.)

Proof of Theorem E. We set $Y_n = |Z_n|$ and substitute $e^{-r} = \zeta$, $e^{-s} = z$. Thus, we have to compute the double generating function

$$\mathscr{E}(\zeta, z) = \sum_{x \in T} \sum_{n=0}^{\infty} p^{(n)}(o, x)\, \zeta^{|x|}\, z^n$$

We can rewrite this as

$$\begin{aligned} \mathscr{E}(\zeta, z) &= \sum_{x \in T} G(o, x \mid z)\, \zeta^{|x|} \\ &= \left(1 + \sum_{x \in T \setminus \{o\}} \zeta^{|x|} m(x)\, F(x, o \mid z)\right) G(o, o \mid z) \end{aligned} \tag{6.2}$$

In the second step, we have used reversibility and Lemma (2.1.b). We now proceed similarly to proof of Proposition (2.5). Recall the definition of the tree T_i, its subtree (ball) T_i^n and the measure m_i on T_i (see Section 3). We consider

$$\mathscr{F}_i(\zeta, z) = \sum_{x \in T_i \setminus \{o\}} \zeta^{|x|}\, m_i(x)\, F(x, o \mid z) \qquad \text{and}$$

$$\mathscr{F}_i^{(n)}(\zeta, z) = \sum_{x \in T_i^n \setminus \{o\}} \zeta^{|x|} m_i(x)\, F(x, o \mid z)$$

In these definitions, $|x|$ denotes the distance between x and o in T_i. Let $[o = x_0, x_1, \ldots, x_k = x]$ be the geodesic path from o to x in T_i, and let $i = i_1, \ldots, i_k$ be the types of $x_1, \ldots, x_k$. Then we know that $F(x, o \mid z) = F_{-i_1}(z) \cdots F_{-i_k}(z)$. Therefore we obtain the recursion

$$\mathscr{F}_i^{(n)}(\zeta, z) = \frac{\zeta F_{-i}(z)}{\mathsf{p}(-i)} \left(1 + \sum_j \mathsf{d}(i, j)\, \mathsf{p}(i, j)\, \mathscr{F}_j^{(n-1)}(\zeta, z)\right)$$

Now consider the column vectors $\mathbf{F}(\zeta, z) = (\mathscr{F}_i(\zeta, z))_{i \in \mathscr{I}}$ and $\mathbf{F}^{(n)}(\zeta, z) = (\mathscr{F}_i^{(n)}(\zeta, z))_{i \in \mathscr{I}}$. Then

$$\begin{aligned} \mathbf{F}^{(n)}(\zeta, z) &= \zeta \cdot D(z)\, \mathbf{p} + \zeta \cdot D(z)\, A \mathbf{F}^{(n-1)}(\zeta, z) \\ &= \sum_{k=0}^{n-1} (\zeta \cdot D(z)\, A)^k\, \zeta \cdot D(z)\, \mathbf{p} \end{aligned}$$

If $\zeta > 0$ and $0 < z < s(P)$ then this sum converges if and only if $\zeta < 1/\lambda(z)$, where

$$\lambda(z) = \lambda(D(z)\, A) \tag{6.3}$$

Since $|\mathscr{F}_i^{(n)}(\zeta, z)| \leqslant \mathscr{F}_i^{(n)}(|\zeta|, |z|)$, we get convergence in $\{(\zeta, z) : |z| < s(P), |\zeta| < 1/\lambda(z)\}$. In this domain, the limit $\mathbf{F}(\zeta, z)$ is analytic and can be computed as

$$\mathbf{F}(\zeta, z) = (I - \zeta \cdot D(z)\, A)^{-1}\, \zeta \cdot D(z)\, \mathbf{p} \tag{6.4}$$

From this we infer that one can write

$$\mathscr{F}_i(\zeta, z) = \frac{B_i(\zeta, z)}{h(\zeta, z)}$$

where $B_i(\zeta, z)$ and $h(\zeta, z) = \det\,(I - \zeta \cdot D(z)\, A)$ are analytic in $\{(\zeta, z) : \zeta \in \mathbb{C}, |z| < s(P)\}$. Consequently

$$\mathscr{E}(\zeta, z) = \frac{B(\zeta, z)}{h(\zeta, z)}, \qquad \text{where}$$

$$B(\zeta, z) = \left(h(\zeta, z) + \sum_{x \in T : o \to x} p(o, x)\, B_{\iota(x)}(\zeta, z) \right) G(o, o \mid z)$$

The last function is analytic in $\{(\zeta, z) : \zeta \in \mathbb{C}, |z| < r(P)\}$, and we know from Theorem B that $1 < r(P) = 1/\rho(P) \leqslant s(P)$. From (4.4), we have that $\lambda(1) = 1$, and by the Perron–Frobenius theorem, this is a simple root of $h(1, z)$. On the other hand, $\mathscr{E}(1, z) = 1/(1 - z)$. Therefore $B(1, z) = h(1, z)/(1 - z)$ does not vanish at $z = 1$. Thus, if we set $C(r, s) = B(e^{-r}, e^{-s})$ and $g(r, s) = h(e^{-r}, e^{-s})$, then all hypotheses of the theorem of Sawyer and Steger are satisfied. We now want to compute $\ell(P)$ and in particular $\sigma^2(P)$ by this method.

If we want to solve $h(\zeta, z) = 0$ for positive real $z < s(P)$, then $\zeta(z) = 1/\lambda(z)$ is the unique solution by the Perron–Frobenius theorem. Thus, in a real neighbourhood of the point $(0, 0)$, the implicit function theorem allows us to factorize

$$g(r, s) = g_1(r, s)(e^r - \lambda(e^{-s}))$$

where $g_1(r, s)$ is real-analytic and $g_1(0, 0) \neq 0$. A straightforward computation now leads to

$$\ell(P) = \frac{1}{\lambda'(1)} \qquad \text{and} \qquad \sigma^2(P) = \frac{\lambda''(1) + \lambda'(1) - \lambda'(1)^2}{\lambda'(1)^3} \tag{6.5}$$

It remains to show that $\sigma^2 > 0$. Let $\kappa(s) = \log \lambda(e^s)$, where $-\infty < s < \log s(P)$. Then, as $\lambda(1) = 1$, we compute $\kappa''(0) = \lambda''(1) + \lambda'(1) - \lambda'(1)^2$. In the next lemma, we shall prove that $\kappa''(s) > 0$ for all s, thus concluding the proof of Theorem E. □

The following lemma shows that under suitable assumptions, the Perron–Frobenius eigenvalue is a log-convex function. The short and efficient proof given below was comunicated to us by Steve Lalley. As a matter of fact, Lalley (1993) indicates a more general result, valid in the more general context of subshifts of finite type.

(6.6) Lemma. Let $s \mapsto M_s = (\mathsf{m}_s(i, j))_{i, j \in \mathscr{I}}$ be a function from some open interval (a, b) to the set of irreducible, non-negative matrices over $\mathscr{I}$, such that all M_s are positive in the same entries. Assume that $s \mapsto \mathsf{m}_s(i, j)$ is twice continuously differentiable and log-convex for all i, j for which $\mathsf{m}_s(i, j) > 0$.

Then also $s \mapsto \lambda(M_s)$ is log-convex, i.e.,

$$\kappa''(s) \geqslant 0 \quad \text{for all } s \in (a, b), \quad \text{where } \kappa(s) = \log \lambda(M_s)$$

Furthermore, if for some s and i, j, the second derivative of $\log \mathsf{m}_s(i, j)$ is strictly positive, then also $\kappa''(s) > 0$.

Proof. We write v and h for measures (row vectors) and functions (column vectors) on $\mathscr{I}$, respectively. Let

$$\mathscr{S} = \left\{ (v, h) : v(i) > 0, h(i) > 0, \sum_i v(i) = \sum_i h(i)\, v(i) = 1 \right\} \tag{6.7}$$

Then we have the following "variational principle" for the Perron–Frobenius eigenvalue, see Seneta (1981).

$$\lambda(M_s) = \max\{v M_s h : (v, h) \in \mathscr{S}\}$$

Since the sum of two positive log-convex functions is again log-convex, each function $s \mapsto v\, M_s\, h$, where $(v, h) \in \mathscr{S}$, is log-convex. Consequently also $s \mapsto \lambda(M_s)$ is log-convex.

To see that $\kappa''(s^*) > 0$ when the second derivative of $\log \mathsf{m}_s(i, j)$ is strictly positive in s^* for some $i, j \in \mathscr{I}$, first observe that $s \mapsto \log v M_s h$ has positive second derivative in s^* for all $(v, h) \in \mathscr{S}$. Now let $(v_s, h_s) \in S$ be the left and right Perron–Frobenius eigenvectors of M_s. The Implicit Function Theorem implies that $\lambda(M_s)$, v_s and h_s are in $C^2(a, b)$. We then have

$\lambda(M_s) \geqslant v_{s*} M_s h_{s*}$ for all $s \in (a, b)$, and these two functions of s coincide when $s = s^*$. Therefore

$$\kappa''(s^*) \geqslant \frac{d^2}{ds^2} \, v_{s*} M_s h_{s*}|_{s=s^*} > 0 \qquad \square$$

Conclusion of the Proof of Theorem E. In our case, $M_s = D(e^s)\, A$, and $\mathsf{m}_s(i, j) = F_{-i}(e^s)\, \mathsf{a}(i, j)$. When $\mathsf{a}(i, j) \neq 0$, the second derivative of $\log \mathsf{m}_s(i, j)$ is strictly positive, as it is the variance of the distribution on $\mathbb{N}$ given by $p_{i,s}(n) = e^{ns}\, \mathsf{f}^{(n)}(-i)/F_{-i}(e^s)$, which is supported on all odd non-negative integers. $\square$

(6.8) Example. Let $\mathscr{I} = \{1, ..., N\}$ carry the structure of a cycle, that is, $\mathsf{a}(i, j) > 0$ precisely when $j = i + 1$ modulo N. We have transience precisely when $\mathsf{p}(1, 2) \cdots \mathsf{p}(N-1, N)\, \mathsf{p}(N, 1) > \mathsf{p}(-1) \cdots \mathsf{p}(N)$. We compute the Perron–Frobenius eigenvalue of $D(z)\, A$,

$$\lambda(z) = \left(\prod_{i=1}^{N} F_{-i}(z)\, \mathsf{a}(i, i+1) \right)^{1/N}$$

We find

$$\frac{1}{\ell(P)} = \frac{1}{N} \sum_i \frac{F'_{-i}(1)}{F_{-i}(1)}, \qquad \sigma^2(P) = \frac{1}{N} \sum_i \left(\frac{F''_{-i}(1)}{F_{-i}(1)} + \frac{F'_{-i}(1)}{F_{-i}(1)} - \frac{F'_{-i}(1)^2}{F_{-i}(1)^2} \right) \ell(P)^3$$

The functions $F_{-i}(z)$ and their derivatives can be computed via continued fractions, see Woess (1985). $\square$

(6.9) Example. We continue Example 1.7.a (a nearest neighbour random walk on a homogeneous tree) in order to compare our results with the corresponding results of Sawyer and Steger (1987), who computed $\ell(P)$ and $\sigma(P)^2$ in this case. The function $G(z) = G(x, x \mid z)$ is independent of $x \in T$ by group invariance and is determined by the functional equation

$$G(z) = 1 + \frac{1}{2} \sum_{i=1}^{N} \left(\sqrt{1 + 4\mathsf{p}_i^2 z^2 G(z)^2} - 1 \right)$$

see Woess (1984), Sawyer and Steger (1987), or various other references. Furthermore,

$$F_{-i}(z) = \frac{1}{2\mathsf{p}_i z G(z)} \left(\sqrt{1 + 4\mathsf{p}_i^2 z^2 G(z)^2} - 1 \right)$$

The matrix $D(z)\,A$ has (i,j)-entry $F_{-i}(z)\,\mathsf{p}_j/\mathsf{p}_i$ when $j\neq i$, and 0, when $j=i$. Let v_z and h_z be the left and right Perron–Frobenius (row and column) eigenvectors of $D(z)\,A$, respectively, such that $(v_z, h_z)\in\mathcal{S}$ as in (6.7). With the auxiliary function $N(z)=\sum_i v_z(i)\,F_{-i}(z)/\mathsf{p}_i$, the equation $v_z D(z)\,A=\lambda(z)\cdot v_z$ yields $v_z(i)=\mathsf{p}_i N(z)/(\lambda(z)+F_{-i}(z))$. Reinserting this into the definition of $N(z)$, we obtain an implicit equation for $\lambda(z)$:

$$\sum_{i=1}^{N}\frac{F_{-i}(z)}{\lambda(z)+F_{-i}(z)}=1 \tag{6.10}$$

Analogously, we compute $\mathsf{p}_i\,h_z(i)=H(z)\,F_{-i}(z)/(\lambda(z)+F_{-i}(z))$, where $H(z)=\sum_i \mathsf{p}_i\,h_z(i)$.

Since the stochastic matrix Q of (4.2) is the conjugate of $D(1)\,A$ by $I-D(1)$, we find that the stationary probability π of the boundary process is $\pi(i)=h_1(i)\,v_1(i)$, whence

$$\pi(i)=\frac{F_{-i}(1)}{(1+F_{-i}(1))^2}\Bigg/\sum_{j=1}^{N}\frac{F_{-j}(1)}{(1+F_{-j}(1))^2}$$

We can now use formula (6.9) to compute $\ell(P)$. In the present example, we have $\mathsf{p}_i(1-F_{-i}(1)^2)=F_{-i}(1)/G(1)$, so that (6.9) becomes

$$\ell(P)=\frac{1}{G(1)}\sum_{i=1}^{N}\frac{F_{-i}(1)}{(1+F_{-i}(1))^2},\qquad\text{and}\qquad \ell_i(P)=\frac{1}{G(1)}\frac{F_{-i}(1)}{(1+F_{-i}(1))^2}$$

In order to compute $\sigma^2(P)$, it is better to rewrite (6.10) as $\sum_i 1/(e^{\kappa(s)-\varphi_i(s)}+1)^2=1$, where $\kappa(s)=\log\lambda(e^s)$ and $\varphi_i(s)=\log F_{-i}(e^s)$. We derive twice with respect to s to find $\kappa''(0)$, and, as $\kappa'(0)=\lambda'(1)=1/\ell(P)$ and $\kappa''(0)=\lambda''(1)+\lambda'(1)-\lambda'(1)^2$, (6.5) yields

$$\sigma^2(P)=\sum_{i=1}^{N}\left(\varphi_i''(0)+\frac{1-F_{-i}(1)}{1+F_{-i}(1)}\,(1-\ell(P)\,\varphi_i'(0))^2\right)\ell_i(P)$$

Note that $\varphi_i'(0)$ and $\varphi_i''(0)$ are, respectively, the expectation and variance of the distribution q_i on $\mathbb{N}$ given by $q_i(n)=\mathsf{f}^{(n)}(-i)/F_{-i}(1)$. □

(6.11) Example. Continuation of Example (4.8). One computes, by definitions (6.3) and (4.3),

$$\lambda(z)=\frac{1}{2}\left(F_{-1}(z)\,\frac{1-\mathsf{p}}{\mathsf{p}}+F_{-2}(z)\,\frac{1-\mathsf{q}}{\mathsf{q}}\right)$$

The functions $F_{-1}(z)$ and $F_{-2}(z)$ can be computed by the equations given in (4.8). Using formulas (6.5) and Maple we get the following values for $\ell(P)$ and $\sigma^2(P)$ (the test values of p and q are the same as in (4.8)).

(p, q)	$(0.1, 0.7)$	$(0.1, 0.5)$	$(0.1, 0.393)$	$(0.1, 0.3)$
$\ell(P)$	0.330	0.442	0.525	0.606
$\sigma^2(P)$	0.542	0.658	0.659	0.609

7. THE DEVIATION FROM THE LIMITING GEODESIC

Finally, we want to prove Theorem F. We first explain the general result that stands behind this theorem. Let T be a locally finite tree with boundary ∂T, and suppose that $(Z_n)_{n\geqslant 0}$ is a transient random walk on T which *converges to the boundary*, that is, there is a ∂T-valued random variable Z_∞ such that

$$Z_n \to Z_\infty \qquad \mathbb{P}_x\text{-almost surely for every } x \in T$$

in the topology of the compactification $\hat{T} = T \cup \partial T$. We do not require (Z_n) to be of nearest neighbour type. For $x, y \in T$, let $\hat{C}_{x,y}$ denote the closure of the cone of y, when x plays the role of the root, i.e., the set of all $\xi \in \hat{T}$ for which y lies on the geodesic from x to ξ. Also, let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly decreasing function that satisfies

$$\lim_{t\to\infty} \Phi(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t\varepsilon)}{\Phi(t)} = c_\varepsilon > 0 \quad \text{for all} \quad \varepsilon > 0$$

(7.1) Theorem. If under the above assumptions

$$\mathbb{P}_x[Z_\infty \in \hat{C}_{x,y}] \leqslant \Phi(\operatorname{dist}(x, y)) \qquad \text{for all} \quad x, y \in T$$

then

$$\lim_{n\to\infty} \frac{\operatorname{dist}(Z_n, \overline{xZ_\infty})}{\Phi^{-1}(1/n^{1+\delta})} = 0$$

almost surely for every $\delta > 0$.

Proof. Let the starting point be $Z_0 = o$. We need the notion of the *confluent* $\xi \wedge \eta$ of two elements ξ, η of $\hat{T}$ with respect to o: this is the last

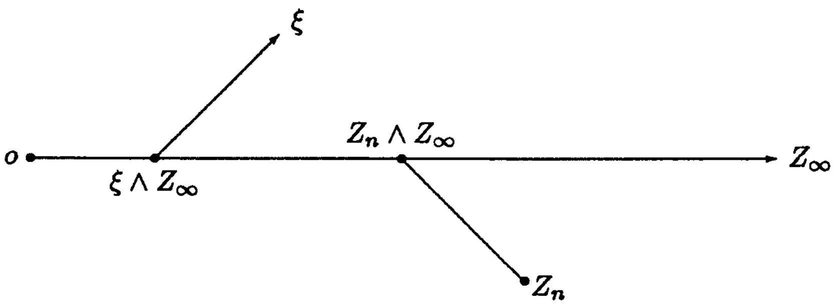


Fig. 3.

common element on the geodesics $\overline{o\xi}$ and $\overline{o\eta}$. It is a point of T, unless $\xi = \eta \in \partial T$.

Now choose and fix an element $\xi \in \partial T$. From Fig. 3 one sees the following: if $|Z_n \wedge Z_\infty| > |\xi \wedge Z_\infty|$ then $\operatorname{dist}(Z_n, \overline{oZ_\infty}) = \operatorname{dist}(Z_n, \overline{\xi Z_\infty})$. (The latter is a bi-infinite geodesic.) Therefore, if $r > 0$, then

$$\begin{aligned}\mathbb{P}_o[\operatorname{dist}(Z_n, \overline{oZ_\infty}) \geqslant r, |Z_n \wedge Z_\infty| > |\xi \wedge Z_\infty|] &\leqslant \mathbb{P}_o[\operatorname{dist}(Z_n, \overline{\xi Z_\infty}) \geqslant r] \\ &= \sum_{x \in T} \mathbb{P}_o[Z_n = x]\, \mathbb{P}_x[\operatorname{dist}(x, \overline{\xi Z_\infty}) \geqslant r] \\ &\leqslant \sum_{x \in T} \mathbb{P}_o[Z_n = x]\, \Phi(r) = \Phi(r)\end{aligned}$$

since $\operatorname{dist}(x, \overline{\xi Z_\infty}) \geqslant r$ means that $Z_\infty \in \hat{C}_{x,y}$, where y is the element on $\overline{x\xi}$ at distance $\lceil r \rceil$ from x.

Now choose $\delta > 0$. For $\varepsilon > 0$, let $r_n^\varepsilon = \varepsilon\, \Phi^{-1}(1/n^{1+\delta})$. Then $r_n^\varepsilon \to \infty$ and $n^{1+\delta}\Phi(r_n^\varepsilon) \to C_\varepsilon$. In particular, $\sum_n \Phi(r_n^\varepsilon) < \infty$. Consider the sequence of events

$$A_n^\varepsilon = [\operatorname{dist}(Z_n, \overline{oZ_\infty}) \geqslant r_n^\varepsilon, |Z_n \wedge Z_\infty| > |\xi \wedge Z_\infty|]$$

in the trajectory space. Then by the above, $\sum_n \mathbb{P}_o(A_n^\varepsilon) < \infty$ and, by Borel–Cantelli, $\mathbb{P}_o(\limsup_n A_n^\varepsilon) = 0$. But

$$\mathbb{P}_o(\limsup_n [|Z_n \wedge Z_\infty| > |\xi \wedge Z_\infty|]) \geqslant \mathbb{P}_o[Z_\infty \neq \xi] = 1 - \lim_{r \to \infty} \Phi(r) = 1$$

Therefore $\mathbb{P}_o(\limsup_n [\operatorname{dist}(Z_n, \overline{oZ_\infty}) \geqslant r_n^\varepsilon]) = 0$, which implies that

$$\limsup_n \frac{\operatorname{dist}(Z_n, \overline{xZ_\infty})}{\Phi^{-1}(1/n^{1+\delta})} \leqslant \varepsilon \qquad \mathbb{P}_o\text{-almost surely for every } \varepsilon > 0 \qquad \square$$

(7.2) Corollary. If (Z_n) is the nearest neighbour random walk on a tree T and there is $\theta < 1$ such that $F(x, x') \leqslant \theta$ for all but finitely many pairs of neighbours $x, x' \in T$, then

$$\lim_{n \to \infty} \frac{\operatorname{dist}(Z_n, \overline{xZ_\infty})}{\log n} = 0$$

almost surely.

Proof. We use the following well known formula for the $\mathbb{P}_x$-distribution of Z_∞, see e.g., Cartier (1972).

$$\mathbb{P}_x[Z_\infty \in \hat{C}_{x,y}] = F(x, y) \frac{1 - F(y, y')}{1 - F(y, y')\, F(y', y)} \tag{7.3}$$

where y' is the predecessor of y on the geodesic $\overline{xy}$. Using Lemma (2.1.a) and the assumption, we get $\mathbb{P}_x[Z_\infty \in C_{x,y}] \leqslant \text{const} \cdot \theta^{\operatorname{dist}(x,y)}$, where $0 < \theta < 1$, and Theorem (7.1) applies with $\Phi(t) = \text{const} \cdot \theta^t$. ☐

Proof of Theorem F. We apply Corollary (7.2).

Set $\theta_0 = \max\{F(x, x^-) : x \neq o\} < 1$. By our assumptions of transience and irreducibility of cone types, we have $F_{-i}(1) < 1$ for all $i \in \mathcal{I}$. Therefore $\theta_0 < 1$.

Suppose first that in the graph $\mathcal{I}$, every vertex i has outdegree equal to 1. Then each cone C_x, $x \neq o$, is a one-way-infinite geodesic, and T is a "star", where finitely many inifinte geodesics are connected at the root. In this case, transience implies that with probability one, $\operatorname{dist}(Z_n, \overline{oZ_\infty}) = 0$ for all but finitely many n.

Let us now assume that there are $i \in \mathcal{I}$ with outdegree at least 2, and write $\mathcal{I}^+$ for the set of these elements. Set

$$\theta_1 = \max\{1 - \mathsf{p}(i, j) + \mathsf{p}(i, j)\, \theta_0 : i \in \mathcal{I}^+, j \in \mathcal{I}, i \to j\}$$

Let $x \in T \setminus \{o\}$ with $\imath(x) \in \mathcal{I}^+$, and let y be a forward neighbour of x in T. Then, by Lemma (2.1),

$$F(x, y) = p(x, y) + \sum_{w \neq y} p(x, w)\, F(w, x)\, F(x, y) \leqslant \theta_1$$

since in the last sum there appears at least one forward neighbour $w \neq y$ of x, and $F(w, x) \leqslant \theta_0$.

Next, if $x \neq o$ has outdegree 1, but x^- has outdegree $\geqslant 2$, and y is the forward neighbour of x, then

$$F(x, y) = p(x, y) + p(x, x^-)\, F(x^-, x)\, F(x, y) \leqslant \theta_2$$

where for $k \geqslant 1$ we define θ_{k+1} inductively by

$$\theta_{k+1} = \max\{1 - \mathsf{p}(-i) + \mathsf{p}(-i)\, \theta_k : i \in \mathscr{I} \setminus \mathscr{I}^+\}$$

Also inductively, we obtain $F(x, y) \leqslant \theta_{k+1}$, when y is the (only) forward neighbour of x, and k is the minimal index such that the kth predecessor of x has outdegree $\geqslant 2$.

We have $\theta_k < 1$ for all k. Thus, the proof of Theorem F will be concluded when we show that there is K such that all $x \in T$ with $|x| > K$ have a kth predecessor with outdegree $\geqslant 2$, where $k \leqslant K$. We claim that we can choose $K = \sharp\mathscr{I} - 1$. Indeed, suppose that there is x with $|x| > K$ for which this claim fails. Let $[x_0, x_1, \ldots, x_K = x]$ be the final piece of length K on the geodesic from o to x. Then all x_m, $m = 0, \ldots, K$, have outdegree 1, and $[x_0, x_1, \ldots, x_K = x]$ projects to an oriented path $[\imath(x_0), \imath(x_1), \ldots, \imath(x_K)]$ in $\mathscr{I} \setminus \mathscr{I}^+$. Since $\mathscr{I}^+ \neq \varnothing$, this path must have at least one point repeated. This means that there is a closed path in $\mathscr{I}$ with all vertices of outdegree equal to 1. But this is impossible since $\mathscr{I}$ is strongly connected, so that there must be an "exit" from our path to $\mathscr{I}^+$. □

8. CONCLUSIVE REMARKS AND OUTLOOK

I. Note that we have found two different ways to derive the law of large numbers and the formula for the rate of escape. We think that the first one (Section 5) is more instructive, since it also sheds light on the structure of the boundary process.

II. One might want to get rid of the special status of the root o by "moving the root backwards to infinity." That is, a boundary point $\eta \in \partial T$ plays the role of the root, the predecessor of any $x \in T$ is the neighbour of x on the geodesic $\overline{x\eta}$, and the cone rooted at x is the subtree spanned by all y such that $x \in \overline{y\eta}$. We then require that there are only finitely many cone types, and write $\mathscr{I}$ for the corresponding finite graph. However, in this situation, reconstructing T from the augmented graph $\mathscr{I}^* = \mathscr{I} \cup \{o\}$ becomes not as immediate as before. In general, there are uncountably many trees with cone structure encoded by $\mathscr{I}$.

Let us suppose that $\mathsf{d}(i, j) \leqslant 1$ for all $i, j \in \mathscr{I}$. (This can always be achieved with an appropriate alteration of $\mathscr{I}$.) Then a unique tree with "cone graph" $\mathscr{I}$ is determined when we choose a backward sequence of types, i.e., a geodesic $\overline{x\eta} = [x = x_0, x_{-1}, x_{-2}, \ldots]$ such that $\imath(x_{k-1}) \to \imath(x_k)$ for all k; each x_k is the root of a cone of the corresponding type. This is in some sense a generalization of the tree of Example 1.7.b.

We have not (yet) studied in detail random walks on these trees rooted at infinity with finitely many cone types. However, there is a rather simple observation regarding the rate of escape for irreducible cone types: if $\lambda(A)>1$ for the matrix defined in (1.2), then the rate of escape is always the number $\ell(P)$ of (5.7) and (5.8). This can be seen as follows.

If $\lambda(A)>1$ then $F_{-i}(1)<1$ for all $i \in \mathscr{I}$, see Proposition (2.5). Therefore (7.3) implies that $\mathbb{P}_x[Z_\infty \neq \eta]=1$ for all x. Thus, if $Z_0=x$ and $\overline{x\eta}=[x=x_0, x_{-1}, x_{-2},\ldots]$, then with probability one there are random indices N and L such that $Z_N=x_{-L}$ and $Z_n \in C_{x_{-L}}$ for all $n \geqslant N$. This implies that the boundary process evolves inside of the latter cone, and that there is a random K such that the process $(\imath(W_k), \mathbf{i}_k)_{k>K}$ has the transition probabilities used in Proposition (5.6). Thus, the only change in the proof of Proposition (5.6) is that we have to consider the summation $\sum_{m=K+1}^{k}$, that is, $\lim_k(\mathbf{e}_k-\mathbf{e}_K)/k$.

This sheds some light on the derivation of the rate of escape by Takacs (1997). In order to study the (non)-homesick random walk on a rooted tree with finitely many cone types, Takacs constructs a random environment, i.e., a probability space consisting of all possible trees rooted at infinity with the same cone type graph, and proves that on this random tree, the rate of escape is almost surely $\ell(P)$. The above reasoning shows that the rate of escape on this random tree is $\ell(P)$ deterministically.

III. A next step is to study random walks with bounded range (instead of nearest neighbour random walks) on trees with finitely many cone types, in a spirit similar to Lalley's study of bounded range random walks on free groups, compare with the very recent paper of Lalley (2000) (and see the note below, at the end). Also, it seems promising to study nearest neighbour random walks on graphs that are "context free" in the sense of Muller and Schupp (1985), which means that they are bi-Lipschitz perturbations of trees and have finitely many cone types in a slightly different sense.

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NOTE

Upon submitting this paper to J. Theoretical Probability, we were informed about the very recent paper by Lalley (2000), whose main result overlaps substantially with our Theorem C (in spite of the different appearance at first glance). It is amusing to realize how we had been unaware of this while at the same time exchanging email with Steve Lalley on Lemma 6.6.

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