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The Dirichlet problem at infinity for random walks on graphs with a strong isoperimetric inequality

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Summary. We study the spatial behaviour of random walks on infinite graphs which are not necessarily invariant under some transitive group action and whose transition probabilities may have infinite range. We assume that the underlying graph G satisfies a strong isoperimetric inequality and that the transition operator P is strongly reversible, uniformly irreducible and satisfies a uniform first moment condition. We prove that under these hypotheses the random walk converges almost surely to a random end of G and that the Dirichlet problem for P-harmonic functions is solvable with respect to the end compactification. If in addition the graph as a metric space is hyperbolic in the sense of Gromov, then the same conclusions also hold for the hyperbolic compactification in the place of the end compactification. The main tool is the exponential decay of the transition probabilities implied by the strong isoperimetric inequality. Finally, it is shown how the same technique can be applied to Brownian motion to obtain analogous results for Riemannian manifolds satisfying Cheeger's isoperimetric inequality. In particular, in this general context new (and simpler) proofs of well known results on the Dirichlet problem for negatively curved manifolds are obtained.

1 Introduction

Let G be an infinite, connected, locally finite graph. Consider a random walk (time-homogeneous Markov chain) X_n, $n=0,1,2,\ldots$, with state space G such that the one-step transition operator P is in some way adapted to the underlying graph structure. Also, consider a compactification $\bar{G}$ of G which is "natural" under a geometrical viewpoint. If (X_n) is transient, then the question arises whether it converges almost surely in the topology of $\bar{G}$ to some random variable which takes its values in the boundary $bG=\bar{G}\setminus G$. In other words, we ask whether the boundary bG can serve as a model for the points attained at infinity

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by our random walk. If the answer is positive, then for every starting point $x \in G$ we have the hitting distribution (*harmonic measure*) v_x on bG. Assuming irreducibility of P, all the harmonic measures v_x are mutually absolutely continuous; we say that they constitute the *harmonic measure class* v on the boundary. Now, for every bounded measurable function h^* on bG its *Poisson integral* $h(x) = \langle h^*, v_x \rangle$ is a bounded function on G harmonic with respect to the transition operator P of the chain. For different functions $h^* \in L^\infty(bG, v)$ their Poisson integrals h are also different, i.e. we have an imbedding of the space $L^\infty(bG, v)$ into the space $H^\infty(G, P)$ of all bounded harmonic functions on G with the sup-norm. This imbedding is norm preserving.

Now one can ask three different questions about the boundary bG, the harmonic measures v_x on it, and the corresponding Poisson integrals:

(i) Does the *Dirichlet problem* admit solution: given a continuous function on the boundary, does there exist a continuous extension to $\bar{G}$ which is harmonic on G? In other words, is it true that for every continuous function h^* its Poisson integral is continuous up to the boundary bG? One can easily see that this is true if and only if for any sequence (x_n) in G convergent to a point $\zeta \in bG$ the corresponding harmonic measures v_{x_n} on bG converge weakly to the point measure δ_ζ.
(ii) Is the boundary bG large enough so that *every* bounded harmonic function h on G can be obtained as the Poisson integral of some bounded measurable function h^* on bG, i.e., is the embedding $L^\infty(bG, v) \to H^\infty(G, P)$ an isomorphism? If this is the case, then the boundary bG coincides as a measure space with the *Poisson boundary* of the random walk.
(iii) Is the compactification $\bar{G} = G \cup bG$ homeomorphic to the *Martin compactification* of the graph G corresponding to the transition operator P?

The problem (ii) is a measure theoretic version of problem (iii), so that a positive answer to (iii) implies a positive answer to (ii). We emphasize that in general the Dirichlet problem for the Martin compactification does not necessarily admit solution (even in the case when the Martin boundary coincides with its active part, i.e. the support of the harmonic measure class in the whole Martin boundary – we give an example at the end of Sect. 6): a positive answer to (ii) or (iii) *does not necessarily imply* a positive answer to (i). On the other hand the Dirichlet problem always admits (trivial) solution for the one-point compactification. Note also that if the Dirichlet problem with respect to a nontrivial boundary bG admits solution, then the support of the harmonic measure class coincides with bG, hence there exist nonconstant bounded harmonic functions and the Poisson boundary of the random walk is nontrivial.

In the present paper, we study convergence to the boundary and solvability of the Dirichlet problem 1) for the *end compactification* (as introduced by Freudenthal [Fr]) for arbitrary G (Sect. 5) and 2) for the *hyperbolic compactification*, if G – as a metric space – is hyperbolic in the sense of Gromov [Gr] (Sect. 6).

We prove convergence to the boundary and give positive answers to question (i) under the following assumptions: G satisfies a strong isoperimetric inequality, and the random walk is strongly reversible, uniformly irreducible and satisfies a uniform first moment condition with respect to the distance in G; see Sect. 2 for details concerning notation and preliminaries.

We point out that *no group invariance* is assumed, and that we are particularly interested in a setting where (X_n) does *not* have bounded range (bounded range means that one-step transitions can occur only at bounded distances). In this setting, our main tools are the exponential decay of the transition probabilities (a corollary of the strong isoperimetric inequality) and the uniform first moment condition, which is a strengthening of tightness of the distance (step length) distributions. In Sect. 3 we show that under the latter assumption the distance increments of (X_n) can be controlled by i.i.d. random variables on the nonnegative integers which have finite first moment. This in combination with the exponential decay of the transition probabilities is used in Sect. 4 to derive distance estimates for (X_n) which are uniform with respect to the starting point. As a corollary, we obtain a weak form of the law of large numbers for the distance between X_n and a reference vertex. Observe that the uniform first moment condition gives a natural intermediate class between bounded range random walks and arbitrary walks. For example, it is the finiteness of the first moment that permits one to describe the Poisson boundary in intrinsic terms for a wide class of random walks on groups using a condition close to hyperbolicity, see Kaimanovich [K 1].

The main results, convergence to the boundary and the solution of the Dirichlet problem, are given in Sect. 5 for the end compactification and in Sect. 6 for hyperbolic graphs. At the end of Sect. 6, we give a brief review of results related to ours and present a simple example where the answer to question (i) is negative, even though the harmonic measure class is supported by the whole of bG.

One can also ask questions (i)–(iii) for a Markov process with a continuous state space equipped with a certain compactification. For example, for Brownian motion on negatively curved simply connected manifolds with the natural visibility compactification, convergence to the visibility boundary sphere was proved by Prat [Pr], and the questions formulated above were answered positively by Anderson [An] (i), (ii), Sullivan [Su] (i), Anderson-Schoen [A-S] and Ancona [A 1] (iii), respectively.

In Sect. 7, due to first author only, it is shown how the results obtained here for graphs can be reformulated for Brownian motion on Riemannian manifolds – the natural continous counterpart to reversible random walks on graphs. The main assumption is Cheeger's isoperimetric inequality – indeed, the strong isoperimetric inequality for graphs was introduced by Dodziuk [Do] in analogy with Cheeger's inequality. The latter is well known to be equivalent to the fact that $\lambda(M)$ (the top of the spectrum of Laplacian on M) is nonzero, i.e., to exponential decay of the heat kernel. This fact and boundedness of geometry alone are used to derive estimates for the rate of escape of the Brownian motion. Finally, convergence to the boundary and solvability of the Dirichlet problem are obtained for the end and hyperbolic compactifications of Riemannian manifolds satisfying Cheeger's inequality.

2 Notation, preliminaries

Throughout Sects. 2–6, G denotes an infinite, connected, locally finite graph. Writing $x \in G$ we mean that x is a vertex of G. The edges are unoriented, loops are permitted, but no multiple edges. By $d(x, y)$ we denote the natural *distance* (number of edges in a shortest path) between the vertices x and y. We select, once for all, a reference vertex o and write $|x|=d(x,o)$. The *boundary* of a set $U\subset G$ is

$$\partial U=\{x\in U\,|\,x \text{ has a neighbour in } G\setminus U\}.$$

We shall always assume that G satisfies the following structure property.

(IS) *Strong isoperimetric inequality:* there is a constant $\kappa>0$ such that

$$|\partial U|\geqq \kappa\cdot|U| \qquad \text{for every finite } U\subset G.$$

See e.g. Dodziuk [Do], Gerl [Ge] and Ancona [A2] for properties of graphs satisfying (IS).

On G, we consider a *random walk* with Markov transition operator $P=(p(x,y))_{x,y\in G}$, where $p(x,y)$ is the probability of passing from $x\in G$ to $y\in G$ in one step. As usual, we describe the position of the random walk at time n by the nth projection X_n of the *trajectory space* $\Xi=G^{\mathbb{N}}$ onto G ($\mathbb{N}$ is the set of nonnegative integers): if $\xi=(x_n)_{n\geqq 0}\in\Xi$ then $X_n(\xi)=x_n$. On Ξ, equipped with the σ-algebra induced by the X_n, we consider the family of probability measures $\Pr_x$, $x\in G$, given by P and the initial point x. Thus

$$\Pr_x[X_0=x]=1, \qquad \Pr_x[X_1=y]=p(x,y),$$

and

$$\Pr_x[X_n=y]=p^{(n)}(x,y),$$

the (x,y)-entry of the nth power P^n. The *distance* (*step length*) *distribution* at vertex x is given by

$$\mu_x(k)=\Pr_x[d(X_0,X_1)=k]=\sum_{y:d(x,y)=k} p(x,y), \qquad k\in\mathbb{N}.$$

In order to study the (spatial) asymptotic behaviour of (X_n) on G, we need some assumptions which relate P with the structure of G.

(SR) *Strong reversibility* (compare with [Ge]): P is *reversible*, i.e., there exists a positive measure ($\equiv$function) m on G such that

$$m(x)\,p(x,y)=m(y)\,p(y,x) \qquad \text{for all } x,y\in G,$$

and there are constants $M_1>0$, $M_2<\infty$ such that

$$M_1\leqq m(x)\leqq M_2 \qquad \text{for every } x\in G.$$

(UI) *Uniform irreducibility* (compare with Picardello and Woess [P-W] and Ancona [A2]): there are $K\in\mathbb{N}$ and $c>0$ such that $d(x,y)=1$ implies $p^{(k)}(x,y)\geqq c$ for some $k\leqq K$.

(UM) *Uniform first moment condition:* we assume

$$\sum_{n=1}^{\infty}\phi(n)<\infty, \qquad \text{where } \phi(n)=\sup_{x\in G}\mu_x([n,\infty)).$$

Note that for an irreducible random walk on (a Cayley graph of) a finitely generated group, (UI) is automatically true, while (UM) becomes finiteness of the first moment of the underlying probability measure on the group. If the measure on the group is symmetric, then the random walk is strongly reversible with respect to the Haar (≡counting) measure on the group. In our general setting, it is easy to see that uniform irreducibility implies that G has bounded vertex degrees:

$$|\{y \mid d(x,y)=1\}| \leqq K/c \qquad \text{for every } x \in G.$$

Hence there exists a constant $D>0$ such that

$$|\{y \mid d(x,y) \leqq n\}| \leqq D^n \qquad \text{for every } x \in G \quad \text{and} \quad n \geqq 0. \tag{2.1}$$

The transition operator P acts on functions $h: G \to \mathbb{R}$ by

$$Ph(x) = \sum_y p(x,y)\, h(y),$$

whenever this series converges for all $x \in G$. A *harmonic function* is a function satisfying $Ph=h$.

We shall frequently use the following generalization of the results of [Ge] to random walks with infinite range.

Theorem A (Kaimanovich [K2]) *Under conditions* (SR) *and* (UI), *the strong isoperimetric inequality implies that* $\|P\|<1$, *where* $\|P\|$ *is the norm of* P *as an operator on* $L^2(G,m)$, *and there is a constant* $M>0$ *such that*

$$p^{(n)}(x,y) \leqq M\,\|P\|^n \qquad \textit{for all } x, y \in G, n \geqq 0.$$

3 The uniform first moment condition and polymorphisms of the trajectory space

In this section we study in detail the meaning of the uniform first moment condition (UM).

Definition 1 A family $(\mu_i)_{i \in I}$ of probability distributions on $\mathbb{N}$ is called *tight* (see e.g. Billingsley [Bi]) if

$$\phi(n) = \sup_{i \in I} \mu_i([n,\infty)) \to 0 \qquad \text{as } n \to \infty,$$

and it satisfies the *uniform first moment condition* if in addition

$$\sum_{n=1}^{\infty} \phi(n) < \infty.$$

Definition 2 Given two probability distributions μ and ν on $\mathbb{N}$, we say that μ *dominates* ν (notation $\mu \succeq \nu$) if

$$\mu([n,\infty)) \geqq \nu([n,\infty)) \qquad \text{for every } n \geqq 0.$$

Equivalently, this says that $F_\mu \leqq F_\nu$ for the corresponding distribution functions. One can easily see that this property means that ν can be obtained from μ

by redistributing the mass of μ from larger values $n\in\mathbb{N}$ to smaller ones, i.e. $\mu\succeq\nu$ iff there exists a family $(\nu_n)_{n\in I}$ of probability measures on $\mathbb{N}$ such that $\operatorname{supp}\nu_n\subset[0,n]$ and $\nu=\sum_n\mu(n)\,\nu_n$.

Lemma 1 *A family $(\mu_i)_{i\in I}$ of probability distributions on $\mathbb{N}$ is tight if and only if there exists a probability distribution μ on $\mathbb{N}$ which dominates all of the μ_i. In addition, $(\mu_i)_{i\in I}$ satisfies the uniform first moment condition if and only if μ can be chosen to have a finite first moment.*

Proof. If $\mu_i\preceq\mu$ for all $i\in I$ then

$$\phi(n)=\sup_{i\in I}\mu_i([n,\infty))\leqq\mu([n,\infty))\to 0 \qquad \text{as } n\to\infty.$$

If in addition μ has a finite first moment then

$$\sum_{n=1}^{\infty}\phi(n)\leqq\sum_{n=1}^{\infty}\mu([n,\infty))=\sum_{k=1}^{\infty}k\mu(k)<\infty.$$

Conversely, assume tightness and let $\phi(n)$ be as in Definition 1. Then $\phi(0)=1$, and $\phi(n)\geqq 0$ is decreasing. Thus, setting $\mu(n)=\phi(n)-\phi(n+1)$ defines a probability distribution on $\mathbb{N}$, and $\mu\succeq\mu_i$ for all $i\in I$. The first moment of μ is

$$\sum_{k=1}^{\infty}k\mu(k)=\sum_{n=1}^{\infty}\phi(n). \quad \square$$

Definition 3 (Vershik [Ve]) A *polymorphism* of two probability spaces $(Y,\Pr)$ and $(Y',\Pr')$ is a triple consisting of a probability space $(\Lambda,\mathbb{P})$ and two measurable, measure preserving mappings

$$\pi:\Lambda\to Y \quad\text{and}\quad \pi':\Lambda\to Y'.$$

We illustrate the situation with the following diagram:

$$(Y,\Pr)\xleftarrow{\ \pi\ }(\Lambda,\mathbb{P})\xrightarrow{\ \pi'\ }(Y',\Pr').$$

Thus $(\Lambda,\mathbb{P})$ subsumes the information contained in both of the given probability spaces. (In our notation, we omit the respective σ-algebras).

Let μ be a probability distribution on $\mathbb{N}$, the set of nonnegative integers. By $(\mathbb{N}^\infty,\mu^\infty)$ we denote the product probability space consisting of all sequences $(k_n)_{n\geqq 1}$ in $\mathbb{N}$ equipped with the usual σ-algebra generated by the projections $Z_n:\mathbb{N}^\infty\to\mathbb{N}$ and carrying the infinite product measure arising from μ. In particular, the Z_n, $n\geqq 1$, are i.i.d. with distribution μ.

Proposition 1 *The distance distributions μ_x, $x\in G$, of the random walk on G given by P are tight if and only if there exist a probability space $(\Lambda,\mathbb{P})$ and a probability distribution μ on $\mathbb{N}$ with the following property: for every $x\in G$ there exists a polymorphism*

$$(\Xi,\Pr_x)\xleftarrow{\ \pi_x\ }(\Lambda,\mathbb{P})\xrightarrow{\ \pi\ }(\mathbb{N}^\infty,\mu^\infty)$$

such that for $\mathbb{P}$*-almost every* $\lambda \in \Lambda$

$$d(X_{n-1}^x, X_n^x) \leqq Z_n^\sim, \qquad n = 1, 2, 3, \ldots,$$

where $X_n^x(\lambda) = X_n(\pi_x(\lambda))$ *and* $Z_n^\sim(\lambda) = Z_n(\pi(\lambda))$*. In addition, the random walk satisfies the uniform first moment condition if and only if* μ *can be chosen to have a finite first moment.*

Proof. Suppose that the distance distributions μ_x, $x \in G$, are tight. By Lemma 1 there exists a probability distribution μ on $\mathbb{N}$ which dominates all of the μ_x. We equip $\mathbb{N}^\infty$ with the corresponding product measure μ^∞.

In order to construct the required polymorphism, we define $\Lambda = [0, 1)^\infty$, the space of all sequences $\lambda = (\lambda_n)_{n \geqq 1}$ in $[0, 1)$ equipped with the usual Borel product σ-algebra. For $\mathbb{P}$ we choose the infinite product of the Lebesgue measure on $[0, 1)$. Thus, the projections of Λ onto $[0, 1)$ are independent, uniformly distributed random variables. We subdivide $[0, 1)$ into consecutive intervals

$$I_0 = [0 = a_0, a_1), \qquad I_1 = [a_1, a_2), \ldots,$$

of lengths

$$a_{i+1} - a_i = \mu(i), \qquad i \geqq 0.$$

If $\lambda = (\lambda_n)_{n \geqq 1} \in \Lambda$ then we define $\pi(\lambda) = (k_n)_{n \geqq 1}$, where

$$k_n = i \Leftrightarrow \lambda_n \in I_i, \qquad i \geqq 0. \tag{3.1}$$

Thus, $\pi\colon (\Lambda, \mathbb{P}) \to (\mathbb{N}^\infty, \mu^\infty)$ is measurable and measure preserving.

Now choose $x \in G$ and choose an enumeration, depending on x, of the vertices of G: $G = \{x_l \mid l \in \mathbb{N}\}$, in such a way that $x_0 = x$ and $l' \geqq l$ implies $d(x_{l'}, x) \geqq d(x_l, x)$. In analogy with the above construction, subdivide $[0, 1)$ into successive intervals, denoted J_{x, x_l}, of length $p(x, x_l)$, $l \geqq 0$. Do this for every $x \in G$. Thus, for $n \geqq 0$ the set $\bigcup_{y: d(x,y) \leqq n} J_{x,y}$ is an interval $[0, b_{x,n})$ of length $b_{x,n} = \mu_x([0, n]) \geqq \mu([0, n])$, i.e.

$$\bigcup_{y: d(x,y) \leqq n} J_{x,y} \supset \bigcup_{i=0}^{n} I_i.$$

In other words,

$$d(x, y) \geqq n \quad \text{implies} \quad J_{x,y} \subset \bigcup_{i=n}^{\infty} I_i. \tag{3.2}$$

If $\lambda = (\lambda_n)_{n \geqq 1} \in \Lambda$, then we define $\pi_x(\lambda) = \xi = (x_n)_{n \geqq 0}$, where

$$x_0 = x \quad \text{and} \quad x_n = y \Leftrightarrow \lambda_n \in J_{x_{n-1}, y}, \qquad n \geqq 1. \tag{3.3}$$

Thus, $\pi_x\colon (\Lambda, \mathbb{P}) \to (\Xi, \Pr_x)$ is measurable and measure preserving.

Finally, let $\lambda=(\lambda_n)_{n\geqq 1}\in\Lambda$, $(x_n)_{n\geqq 0}=\pi_x(\lambda)$ and $(k_n)_{n\geqq 1}=\pi(\lambda)$. If $d(x_{n-1},x_n)=l$ then $J_{x_{n-1},x_n}\subset\bigcup_{i=l}^{\infty} I_i$ by (3.2). By (3.3) we have $\lambda_n\in J_{x_{n-1},x_n}$. Hence (3.1) yields $k_n\geqq l$. Thus

$$d(X_{n-1}^x, X_n^x)\leqq Z_n^{\sim}$$

as proposed.

Conversely, it is obvious that the existence of a polymorphism with the stated properties yields $\mu_x\precsim\mu$ for every $x\in G$. □

The above proposition allows us to majorize the distance increments of (X_n) with i.i.d. random variables Z_n on $\mathbb{N}$ independently of the starting point $x\in G$.

4 Distance estimates

In this Section we derive several estimates for the random walk on G resulting from the assumptions (IS), (SR), (UI), (UM) made in Sect. 2, which will be used below for studying convergence to the boundary and the Dirichlet problem.

Lemma 2 *Suppose that the random walk on G satisfies* (UM). *Then we have the following*:

(a) $$\lim_{n\to\infty}\frac{1}{n}d(X_{n-1},X_n)=0$$

$\Pr_x$-*almost surely for every* $x\in G$, *and convergence in probability is uniform in* x:

$$\lim_{n\to\infty}\Pr_x\left[\sup_{k\geqq n}\frac{1}{k}d(X_{k-1},X_k)>\varepsilon\right]=0$$

uniformly in $x\in G$ *for every* $\varepsilon>0$;

(b) $$\lim_{n\to\infty}\Pr_x\left[\frac{1}{n}\sup_{k\leqq n}d(X_{k-1},X_k)>\varepsilon\right]=0$$

uniformly in $x\in G$ *for every* $\varepsilon>0$;

(c) $$\limsup_{n\to\infty}\frac{1}{n}\sup_{k\leqq n}d(X_0,X_k)\leqq\bar{\mu}=\sum_{n=1}^{\infty}\phi(n)$$

$\Pr_x$-*almost surely for every* $x\in G$, *and*

$$\lim_{n\to\infty}\Pr_x\left[\frac{1}{n}\sup_{k\leqq n}d(X_0,X_k)\geqq C_1\right]=0$$

uniformly in $x\in G$ *for every* $C_1>\bar{\mu}$.

Proof. Consider the polymorphism constructed in Proposition 1. Then

$$(4.1)\qquad d(X^x_{n-1}, X^x_n) \leqq Z^\sim_n \qquad \mathbb{P}\text{-a.s.} \qquad \text{for every } x \in G.$$

By construction, the $Z^\sim_n$ are i.i.d. with common distribution μ having finite first moment $\bar{\mu}$. By the law of large numbers,

$$(4.2)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Z^\sim_k = \bar{\mu} \qquad \mathbb{P}\text{-a.s.}$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} Z^\sim_n = 0 \qquad \mathbb{P}\text{-a.s.}$$

This and (4.1) together with the fact that π_x is measure preserving yield (a). Relation (4.1) also yields

$$\frac{1}{n} \sup_{k\leqq n} d(X^x_0, X^x_k) \leqq \frac{1}{n} \sum_{k=1}^{n} Z^\sim_k \qquad \mathbb{P}\text{-a.s.}$$

In combination with (4.2), we obtain (c). Finally,

$$\mathbb{P}\left[\frac{1}{n} \sup_{k\leqq n} d(X^x_{k-1}, X^x_k) > \varepsilon\right] \leqq \sum_{k=1}^{n} \mathbb{P}[Z^\sim_k > \varepsilon n] \leqq n\phi(\varepsilon n).$$

As ϕ is decreasing and $\sum_{n=1}^{\infty} \phi(n) < \infty$, we have $\lim_{n\to\infty} n\phi(\varepsilon n) = 0$ for every $\varepsilon > 0$. This yields (b). $\square$

Lemma 3 *Suppose that the graph G and the random walk satisfy assumpons* (IS), (SR) *and* (UI). *Then there exists a constant* $C_0 > 0$ *such that*

$$\liminf_{n\to\infty} \frac{1}{n} \inf_{k\geqq n} d(X_0, X_k) > C_0$$

Pr_x*-almost surely for every* $x \in G$, *and*

$$\lim_{n\to\infty} \mathrm{Pr}_x\left[\inf_{k\geqq n} \frac{1}{n} d(X_0, X_k) \leqq C_0\right] = 0$$

uniformly in $x \in G$.

Proof. Let

$$A_n = [\inf_{k\geqq n} d(X_0, X_k) \leqq C_0\, n].$$

From Theorem A and (2.1) we get

$$\begin{aligned}\Pr_x(A_n) &\leqq \sum_{k=n}^{\infty} \Pr_x[d(X_0, X_k) \leqq C_0\, n] = \sum_{k=n}^{\infty} \sum_{y: d(x,y) \leqq C_0 n} p^{(k)}(x, y)\\ &\leqq \sum_{k=n}^{\infty} M \cdot \|P\|^k \cdot |\{y \mid d(x, y) \leqq C_0\, n\}| \leqq \frac{M}{1-\|P\|} (D^{C_0} \|P\|)^n.\end{aligned}$$

Thus, if we choose $C_0 > 0$ small enough such that $D^{C_0} \|P\| < 1$, then $\sum_{n=1}^{\infty} \Pr_x(A_n)$ converges uniformly in $x \in G$. Now the Borel-Cantelli Lemma yields the result. □

Corollary 1 *Under assumptions* (IS), (SR), (UI) *and* (UM)

$$0 < C_0 < \liminf_{n\to\infty} \frac{1}{n} |X_n| \leqq \limsup_{n\to\infty} \frac{1}{n} |X_n| < C_1 < \infty$$

$\Pr_x$*-almost surely for every* $x \in G$.

This is a "law of large numbers" with respect to the distance in G. For an irreducible random walk with finite first moment on a finitely generated group the conditions (UI) and (UM) are satisfied with respect to the Cayley graph (≡word) metric(s) of the group. In this case, $|X_n|/n$ converges almost surely to a constant (the *rate of escape*) by Kingman's subadditive ergodic theorem (see e.g. Derriennic [D2]). This limit is always positive if the group is *nonamenable*, i.e. the Cayley graphs of the group satisfy (IS), see Kaimanovich and Vershik [K-V]. Corollary 1 generalizes these facts (in a weaker form) to a non group-invariant setting. Compare also with Varopoulos [Va].

Lemma 4. *Suppose that* (IS), (SR), (UI) *and* (UM) *hold. If* $\varepsilon > 0$ *is sufficiently small, then*

$$\lim_{|x|\to\infty} \Pr_x[|X_n| \leqq \varepsilon |x| \textit{ for some } n \geqq 0] = 0.$$

Proof. Let $\varepsilon < 1/2$ and $\alpha > 0$. Then

$$\begin{aligned}&\Pr_x[\exists n: |X_n| \leqq \varepsilon |x|]\\ &\qquad \leqq \Pr_x[\exists n \leqq \alpha |x|: |X_n| \leqq \varepsilon |x|] + \Pr_x[\exists n \geqq \alpha |x|: |X_n| \leqq \varepsilon |x|],\end{aligned}$$

where

$$\begin{aligned}\Pr_x[\exists n \leqq \alpha |x|: |X_n| \leqq \varepsilon |x|] &\leqq \Pr_x[\exists n \leqq \alpha |x|: d(X_0, X_n) \geqq (1-\varepsilon)\, |x|]\\ &= \Pr_x\left[\frac{1}{\alpha |x|} \sup_{n \leqq \alpha |x|} d(X_0, X_n) \geqq (1-\varepsilon)/\alpha\right]\\ &\leqq \Pr_x\left[\frac{1}{\alpha |x|} \sup_{n \leqq \alpha |x|} d(X_0, X_n) \geqq 1/2\alpha\right]\end{aligned}$$

and

$$\Pr_x[\exists n \geqq \alpha|x| : |X_n| \leqq \varepsilon|x|] \leqq \sum_{n=\alpha|x|}^{\infty} \sum_{y:|y| \leqq \varepsilon|x|} p^{(n)}(x, y)$$
$$\leqq \frac{M}{1-\|P\|} (\|P\|^{\alpha} D^{\varepsilon})^{|x|}$$

as follows from Theorem A and (2.1). Now set $\alpha = 1/2C_1$ and choose $\varepsilon > 0$ small enough such that $\|P\|^{\alpha} D^{\varepsilon} < 1$. Then applying Lemma 2(c) we get the desired result. □

Combining Lemmas 2–4, we obtain the following technical corollary, which will be useful in the sequel.

Corollary 2 *Suppose that* (IS), (SR), (UI) *and* (UM) *hold. Let* C_0 *be as in Lemma* 3, *and let* α, $\varepsilon > 0$. *For* $x \in G$, *consider the event* $A_x = A_x(\alpha, \varepsilon)$ *in the trajectory space defined by the following five properties*:

(i) $X_0 = x$,
(ii) $d(X_n, X_{n+1}) \leqq \varepsilon|x|$ *for all* $n \leqq \alpha|x|$,
(iii) $d(X_n, X_{n+1}) \leqq \varepsilon n$ *for all* $n \geqq \alpha|x|$,
(iv) $d(X_0, X_n) > C_0\, n$ *for all* $n \geqq \alpha|x|$,
(v) $|X_n| > \varepsilon|x|$ *for all* $n \geqq 0$.

There exists a constant $\varepsilon_0 > 0$ (*independent of* α) *such that if* $\varepsilon \leqq \varepsilon_0$, *then*

$$\lim_{|x| \to \infty} \Pr_x(A_x) = 1.$$

5. Convergence to the boundary and Dirichlet problem in the end compactification

We recall the construction of the end compactification of a graph G, see Freudenthal [Fr], Jung [Ju], Cartwright, Soardi and Woess [C-S-W].

An *infinite path* is a one-sided infinite sequence p of successively adjacent vertices without repetitions. Two infinite phaths p, p' are *equivalent* if for every finite $U \subset G$, all but finitely many of the vertices of p and p' lie in the same connected component of $G \setminus U$. An *end* is an equivalence class under this relation. The set of all ends is denoted by Ω. If $U \subset G$ is finite, then we add to each component of $G \setminus U$ all ends having a representative path lying entirely in that component. It $z \in G \cup \Omega \setminus U$, then among the components augmented in this way there is precisely one which contains z. This is the component of z with respect to U, denoted by $C(U, z)$. Varying U (finite) and z, the family of all sets $C(U, z)$ becomes a basis of the *end topology*. In this way, $G \cup \Omega$ becomes a compact, totally disconnected Hausdorff space in which G ($\equiv$ the vertex set of G) is discrete, open and dense.

We remark that the space of ends can also be introduced as a projective limit: let U be a finite ($\equiv$ compact) subset of G. Denote by $\mathscr{C}_U$ the (finite, discrete) space of all infinite connected components of $G \setminus U$. For any two sets $U_1 \subset U_2$ there exists a natural projection $\mathscr{C}_{U_2} \to \mathscr{C}_{U_1}$. The projective limit of the system $\{\mathscr{C}_U \mid U \subset G\}$ is another model for Ω. See e.g. [C-S-W] for more details.

We begin with the following simple statement.

Lemma 5 *Every sequence* (x_n) *of vertices in* G *such that*

$$|x_n|+|x_{n+1}|-d(x_n, x_{n+1})\to\infty$$

converges to an end.

Proof. Take a finite subset $U\subset G$. Suppose two points x, $y\in G\backslash U$ belong to different connected components of the set $G\backslash U$. It means that all the paths (and, in particular, the shortest paths) connecting x and y pass through the set U. Hence there exists a point $z\in U$ such that

$$d(x,y)=d(x,z)+d(z,y)\geqq|x|+|y|-2|z|\geqq|x|+|y|-2R_U,$$

where

$$R_U=\max_{z\in U}|z|.$$

Thus the condition of the Lemma implies that for every finite U there exists an integer N and a set $C(U,\omega)$ such that $x_n\in C(U,\omega)$ for all $n\geqq N$. Moreover, $x_n\to\infty$. Since the sets $C(U,\omega)$ form a basis of the end topology, the Lemma is proven. □

Theorem 1 *Assume that the graph* G *and the random walk satisfy assumptions* (IS), (SR), (UI) *and* (UM). *Then there exists an* Ω*-valued random variable* X_∞ *such that for every* $x\in G$,

$$\lim_{n\to\infty} X_n=X_\infty \qquad \mathrm{Pr}_x\text{-almost surely}$$

in the end topology.

Proof. Choose C_0 as in Lemma 3. Consider the event E in the trajectory space defined by

$$(5.1)\qquad \lim_{n\to\infty}\frac{1}{n}d(X_n,X_{n+1})=0 \quad\text{and}\quad \liminf_{n\to\infty}\frac{1}{n}d(X_0,X_n)>C_0.$$

Then $\mathrm{Pr}_x(E)=1$ by Lemmas 2(a) and 3. But as follows from Lemma 5, every trajectory in E converges to an end of G. □

In view of Theorem 1, we may define the family of hitting distributions (harmonic measures) ν_x, $x\in G$, on Ω: for a Borel set $B\subset\Omega$,

$$(5.2)\qquad \nu_x(B)=\mathrm{Pr}_x[X_\infty\in B].$$

Observe that for every continuous function h^* on Ω, the Poisson integral

$$(5.3)\qquad h(x)=\int_\Omega h^*\,d\nu_x$$

defines a bounded harmonic function on G. We are now ready to formulate the solution of the Dirichlet problem with respect to the end compactification.

Theorem 2 *Under the assumptions of Theorem* 1, *every continuous function* h^* *on* Ω *admits a unique continuous extension to* $G\cup\Omega$ *which is harmonic on* G. *The extension is given by* (5.3).

Proof. Uniqueness follows from the maximum principle. In view of (5.2) and (5.3), what we have to show is that

$$\lim_{x\to\omega} \nu_x = \delta_\omega \qquad \text{weakly,}$$

whenever x converges in G to $\omega \in \Omega$ (here, δ_ω denotes the unit mass at ω). It is sufficient to show that

$$\lim_{x\to\omega} \nu_x(C(U,\omega)\cap\Omega) = 1$$

for every basic neighbourhood $C(U,\omega)$ of ω, where $U \subset G$ is finite.

To prove this, consider the set $A_x \subset \Xi$ defined in Corollary 2 (we shall specify the values of α and ε later) and take a trajectory $\xi \in A_x$. Then for $X_n = X_n(\xi)$

$$|X_n| + |X_{n+1}| - d(X_n, X_{n+1}) > \varepsilon |x|$$

for all $n \leqq \alpha |x|$, and

$$\begin{aligned} |X_n| + |X_{n+1}| - d(X_n, X_{n+1}) &> 2(C_0 n - |x|) - \varepsilon n \\ &= (2C_0 - \varepsilon)\, n - 2|x| \\ &\geqq ((2C_0 - \varepsilon)\,\alpha - 2)\,|x| \end{aligned}$$

for all $n \geqq \alpha |x|$. Take now $\varepsilon < \min(\varepsilon_0, 2C_0)$ and $\alpha = (\varepsilon_0 + 2)/(2C_0 - \varepsilon_0)$ with ε_0 as in Corollary 2. Thus we have that

$$|X_n| + |X_{n+1}| - d(X_n, X_{n+1}) > \varepsilon_0 |x|$$

for all $n \geqq 0$.

Now from the proof of Lemma 5 follows that if $x \in C(U,\omega)$ and $\varepsilon|x| > 2R_U$, then none of the trajectories in the set A_x ever leaves the set $C(U,\omega)$. Hence

$$\nu_x(C(U,\omega)\cap\Omega) \geqq \Pr_x(A_x) \to 1, \qquad \text{as } x \to \omega. \quad \square$$

6 Convergence to the boundary and Dirichlet problem on hyperbolic graphs

In this section we assume that G, viewed as a metric space with the natural discrete metric d, is a *hyperbolic space* in the sense of [Gr]; see also Ghys and de la Harpe [G-H], Short [Sh] or Coornaert, Delzant, Papadopoulos [C-D-P]. We review the basic features and point out the necessary adaptations.

With respect to the reference vertex o, define for x, $y \in G$ the *Gromov product*

$$(x\,|\,y) = \tfrac{1}{2}(|x| + |y| - d(x,y)).$$

The graph G is *hyperbolic* if there exists a $\delta > 0$ (possibly large) such that for all x, y, $z \in G$

$$(6.1) \qquad (x\,|\,z) \geqq \min\{(x\,|\,y), (y\,|\,z)\} - \delta.$$

On the basis of [G-H, Sect. 7.2], we describe the hyperbolic compactification. Assume that G is hyperbolic. (Observe that our definition of hyperbolicity involves only one base point, so that the δ of [G-H] has to be chosen equal to twice times the δ of (6.1), compare with [Gr].) Choose $a>0$ such that $a' = e^{2\delta a} - 1 < \sqrt{2} - 1$ and define for $x, y \in G$

$$\rho_a(x, y) = \begin{cases} 0, & \text{if } x = y; \\ e^{-a(x|y)}, & \text{otherwise.} \end{cases} \tag{6.2}$$

Then, precisely as in [G-H, Sect. 7.2], one verifies that ρ_a is symmetric, $\rho_a(x, y) = 0$ if and only if $x = y$, and

$$\rho_a(x_0, x_n) \leqq \frac{1}{1-2a'} \sum_{i=0}^{n} \rho_a(x_{i-1}, x_i) \tag{6.3}$$

for all $n>0$, $x_0, \ldots, x_n \in G$. Thus, ρ_a is "almost" a metric. If we define for $x, y \in G$

$$\theta_a(x, y) = \inf\left\{ \sum_{i=1}^{n} \rho_a(x_{i-1}, x_i) \,|\, n \geqq 1, x = x_0, x_1, \ldots, x_n = y \in G \right\},$$

then θ_a is a metric on G, and

$$(1-2a')\, \rho_a(x, y) \leqq \theta_a(x, y) \leqq \rho_a(x, y) \qquad \text{for all } x, y \in G. \tag{6.4}$$

This can be proved exactly as in [G-H, Sect. 7.2]. The completion $\hat{G}$ of G with respect to θ_a as a topological space does not depend on the choice of a. The induced topology on $\hat{G}$ is the *hyperbolic topology*. In this way, $\hat{G}$ becomes a compact Hausdorff space in which G is discrete, open and dense. The *hyperbolic boundary* is $\partial G = \hat{G} \setminus G$.

Thus, a sequence (x_n) of vertices of G converges to some hyperbolic boundary point if and only if

$$\lim_{m, n \to \infty} (x_m | x_n) = \infty,$$

and two such sequences (x_n) and (y_n) converge to the same boundary point if and only if

$$\lim_{m, n \to \infty} (x_m | y_n) = \infty.$$

(Because of (6.1) this is an equivalence relation, and the boundary can be introduced as the set of equivalence classes.)

We remark that it is easy to verify (cf. Lemma 5) that for hyperbolic G, the hyperbolic compactification is finer that the end compactification: the identity on G extends to a continuous surjection $\hat{G} \to G \cup \Omega$, which maps ∂G onto Ω. On the other hand, even a graph with a rich end structure need not be hyperbolic in general.

With an abuse of the notation introduced in Sect. 5, we shall write X_∞ and v_x, $x \in G$, for the limit random variable of (X_n) and their $\Pr_x$-distributions on ∂G, whose existence is guaranteed by the following theorem.

Theorem 3 *Assume that G is a hyperbolic graph satisfying the condition* (IS), *and that P gives rise to a random walk on G with the properties* (SR), (UI) *and*

(UM). *Then there exists a ∂G-valued random variable X_∞ such that for every $x \in G$*

$$\lim_{n\to\infty} X_n = X_\infty \qquad \Pr_x\text{-almost surely}$$

in the hyperbolic topology.

Proof. Consider the set E of trajectories defined by (5.1) in the proof of Theorem 1. Then $\Pr_x(E)=1$, and if $\xi \in E$ then for $X_n = X_n(\xi)$

$$\liminf_{n\to\infty} \frac{1}{n}(X_n|X_{n+1}) > C_0.$$

Consequently, for $m \geqq n$ we have by (6.3)

$$\rho_a(X_n, X_m) \leqq \frac{1}{1-2a'} \sum_{k=n}^{\infty} e^{-a(X_k|X_{k+1})},$$

which tends to zero as $n\to\infty$. Therefore (X_n) is a Cauchy sequence and converges in the hyperbolic topology. As $|X_n|\to\infty$, the limit must be a boundary point. □

We define the hitting probabilities of (X_n) on ∂G as mentioned above: if $x \in G$ and B is a Borel set in ∂G, then

$$\nu_x(B) = \Pr_x[X_\infty \in B]. \tag{6.5}$$

Once more, the Dirichlet problem admits solution.

Theorem 4 *Under the assumptions of Theorem* 3, *every continuous function h^* on ∂G has a unique continuous extension to $\hat{G}$ which is harmonic on G. The extension is given by the Poisson integral*

$$h(x) = \int_{\partial G} h^* \, d\nu_x, \qquad x \in G.$$

Proof. We proceed precisely as in the proof of Theorem 2. Let $\zeta \in \partial G$ and consider a basic neighbourhood

$$N(\zeta, s) = \{\eta \in \hat{G} \,|\, \theta_a(\eta, \zeta) < s\}, \qquad s > 0.$$

In order to show that

$$\lim_{x\in G,\, x\to\zeta} \nu_x(N(\zeta, s) \cap \partial G) = 1,$$

we consider the set $A_x \subset \Xi$ defined in Corollary 2 and choose

$$\varepsilon < \min\{\varepsilon_0, C_0/2\}, \qquad \alpha = 2/C_0. \tag{6.6}$$

Let $\xi \in A_x$, $X_n = X_n(\xi)$. We claim that if $|x|$ is large enough then $\theta_a(x, X_n) < s/2$ for all n. Thus, if in addition $x \in N(\zeta, s/2)$, then the limit points of all the trajectories in A_x lie in $N(\zeta, s)$, and

$$\nu_x(N(\zeta, s) \cap \partial G) \geqq \Pr_x(A_x) \to 1, \qquad \text{as } x \to \zeta.$$

To prove the claim, observe that for $n \geqq \frac{2}{C_0}|x|$,

$$|X_n| > C_0 n - |x| \geqq \frac{C_0}{2} n$$

by (iv) of Corollary 2, so that in combination with (iii) and (6.6) we get

$$(X_n | X_{n+1}) > \frac{1}{2}(C_0 n - \varepsilon n) > \frac{C_0}{4} n.$$

On the other hand, for $n \leqq \frac{2}{C_0}|x|$,

$$(X_n | X_{n+1}) \geqq \frac{1}{2}(2\varepsilon|x| - \varepsilon|x|) = \frac{\varepsilon}{2}|x|$$

by (v) and (ii). Now (6.3) yields

$$\begin{aligned}(1-2a')\,\rho_a(x, X_n) &\leqq \sum_{k=0}^{\infty} e^{-a(X_k|X_{k+1})} \\ &\leqq \frac{2}{C_0}|x|\, e^{-a\varepsilon|x|/2} + \sum_{k=2|x|/C_0}^{\infty} e^{-aC_0 k/4} = \beta(|x|),\end{aligned}$$

which tends to zero as $|x| \to \infty$. If $|x|$ is large enough such that $\beta(|x|) < (1-2a')\,s/2$, then by (6.4) we have $\theta_a(x, X_n) < s/2$ for all n, as proposed. □

Before concluding, we remark that in all results of this paper, (IS) and (SR) can be replaced by the following weaker hypothesis:

(E) There are $\lambda < 1$ and $M > 0$ such that

$$p^{(n)}(x, y) \leqq M\lambda^n \qquad \text{for all } x, y \in G, n \geqq 0.$$

Indeed, for Theorems 1 and 3 it is even enough to have (UI), (UM) and $\lambda(P) < 1$, where $\lambda(P) = \limsup p^{(n)}(x, y)^{1/n}$ (independent of x and y).

We now briefly review results related to ours concerning random walks on graphs and groups.

If (X_n) has *bounded range* (i.e., $\sup\{d(x,y)|p(x,y)>0\}<\infty$) and is transient, then it is easy to verify that it converges a.s. to a random end, without further hypotheses. For *nearest neighbour* random walks (i.e., $p(x,y)>0 \Leftrightarrow x$ and y are neighbours), the Dirichlet problem with respect to the end compactification admits solution if and only if the Green kernel vanishes at infinity, given that there are at least two ends, see Benjamini and Peres [B-P] for trees and Cartwright, Soardi and Woess [C-S-W] for even non locally finite graphs. The (similar) proofs in [B-P] and [C-S-W] carry over immediately to bounded range random walks on locally finite graphs.

In a group-invariant setting, the solution of the Dirichlet problem has been given much earlier by Derriennic [D 1] for finite range random walks on finitely generated free groups (≡ homogeneous trees) as a by-product of his identification of the space of infinite words (≡ space of ends) with the Martin boundary. Convergence to a random end for group invariant random walks on trees has been proved by Sawyer and Steger [S-S] assuming a moment condition, see also Sawyer [Sa] and Cartwright and Sawyer [C-Sa].

An earlier approach due to Furstenberg turned out to be very useful in proving that a.e. random walk trajectory converges to a (random) boundary point without any additional conditions imposed on the walk. The idea consists in verifying that the group action on the boundary satisfies a certain property. It was first applied for proving the convergence of random walks on Fuchsian groups to the boundary circle of the Poincaré disc [Fu, Theorem 1.3]. Margulis remarked that this method also works for the end compactification of finitely generated free groups (unpublished, cf. [K-V]). Independently, Cartwright and Soardi [C-So] used this approach for proving convergence to a random end for group invariant random walks on trees in general; a special case (amenability of the group) has to be excluded here. For irreducible random walks which are invariant under a nonamenable, vertex-transitive group acting on a graph with more than two (⇔ infinitely many) ends, the same method yields convergence to a random end and solvability of the Dirichlet problem without any moment condition, see Woess [Wo]. This applies in particular to all finitely generated groups with infinitely many ends. The same technique easily permits one to prove convergence to a random boundary point and solvability of the Dirichlet problem for the hyperbolic compactification of nonamenable hyperbolic groups as well.

For general (non group-invariant) hyperbolic graphs, the only known result is due to Ancona [A 2]: it applies to bounded range random walks satisfying (UI) and such that $\lambda(P)<1$. In this case, the Martin boundary is identified with the hyperbolic boundary, so that a.s. convergence to the hyperbolic boundary is verified; solvability of the Dirichlet problem is proved under the assumption that the Green kernel vanishes at infinity. This generalizes earlier results of Series [Se] for Fuchsian groups.

We now give the example, announced in the introduction, of a random walk on a graph (a tree), where the whole Martin boundary is active, but the Dirichlet problem does not admit solution. Other examples can also be found in [B-P].

by connecting a finite path of length n^3 to r_0. As the root of T_n, take the vertex r_n of T_n which has only one neighbour. Now build a tree T from all the T_n by drawing an edge between r_n and r_{n+1}, $n \geqq 0$: see the figure.

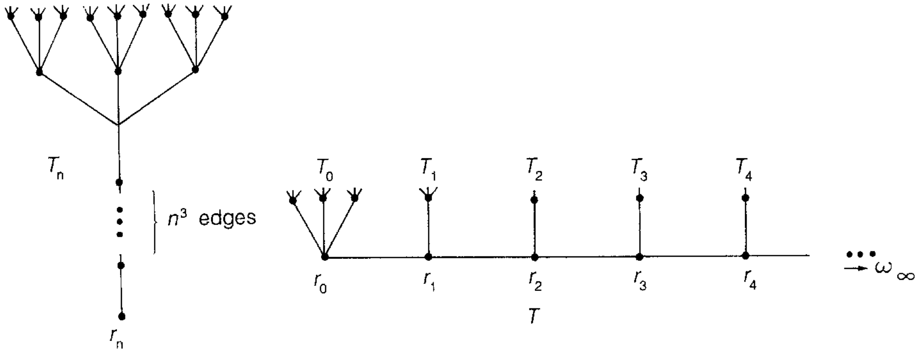

Fig. 1

Being a tree, T is a hyperbolic graph, and its end compactification coincides with the hyperbolic one. The space of ends of T is given by the union of the spaces of ends of all the T_n plus an additional one, denoted ω_∞, which is the limit of (r_n). Because of the presence of finite unramified paths (successive vertices with only two neighbours) of arbitrary length, T does *not* satisfy the strong isoperimetric inequality.

Let P be the transition operator of the *simple random walk* on T, which moves from a vertex of T to any of its neighbours with equal probability. It is easily seen to be transient.

By Cartier [Ca], the Martin compactification of (T, P) coincides with the end compactification.

Let $g(x, y)$ be the Green kernel of P, i.e. $g(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y)$. By [B-P] and [C-S-W], an end ω of T is *regular for the Dirichlet problem* (i.e., $v_x \to \delta_\omega$ weakly as $x \to \omega$) if and only if

$$\lim_{x \to \omega} g(x, r_0) = 0. \tag{6.7}$$

Using the well known behaviour of the simple random walk on the binary tree, one obtains that (6.7) holds for every $\omega \in \Omega \setminus \{\omega_\infty\}$. On the other hand, using for example the methods of Gerl and Woess [G-W], one can prove that

$$\lim_{n \to \infty} g(r_n, r_0) > 0$$

(we omit the details of the calculation).

Thus, the Dirichlet problem for P-harmonic functions is not solvable with respect to Ω. On the other hand, being regular, all points of $\Omega \setminus \{\omega_\infty\}$ lie in the support of the harmonic measure class v. But the closure of the latter set is Ω, so that the support of v is the whole boundary.

7 Convergence to the boundary and Dirichlet problem for Brownian motion on Riemannian manifolds

In this section (due to the first author only) it is shown how the results obtained above can be reformulated for Brownian motion on Riemannian manifolds.

Let M be a complete connected Riemannian manifold with the Riemannian metric d. Fix a reference point $x_0 \in M$ and denote by $|x|$ the distance $d(x, x_0)$ for all $x \in M$. We assume that M has *bounded geometry*, i.e. the sectional curvatures on M are uniformly bounded and the injectivity radius is bounded away from zero. The Laplace-Beltrami operator Δ of the Riemannian metric on M determines a diffusion process with state space M and generator Δ which is called the *Brownian motion* on M. Denote by $\Pr_x$ the probability measure in the space of sample paths $(X_t)_{t \geqq 0}$ of the Brownian motion starting from a point $x \in M$, and by $p_t(x, \cdot)$ the density of the one-dimensional distribution of the measure $\Pr_x$ at time t with respect to the Riemannian volume m, i.e. the fundamental solution of the heat equation on M (see e.g. Ikeda and Watanabe [I-W] or Pinski [Pi]).

Now consider the Riemannian analogues of the discrete conditions introduced in Sect. 2. It is well known that the densities p_t are symmetric: $p_t(x, y) = p_t(x, y)$ for all $x, y \in M$. In other words, the Brownian motion is reversible with respect to the Riemannian volume. This corresponds to the fact that the Laplace-Beltrami operator Δ is self-adjoint in the space $L^2(M, m)$. It is the natural analogue of condition (SR).

Boundedness of the geometry of M implies that the growth of the volume of Riemannian balls in M is uniformly bounded (see Cheeger and Ebin [C-E]): there exists a constant $D > 0$ such that

$$m\{y \in M : d(x, y) \leqq r\} \leqq D^r \quad \text{for every } x \in M \quad \text{and} \quad r \geqq 0.$$

This is an analogue of the inequality (2.1).

From the comparison theorems for Brownian motion and standard scaling estimates for the decay at infinity of the heat kernel follows [Pi] that there exists a measure μ on $\mathbb{R}$ with a finite first moment such that

$$\Pr_x[\sup_{0 \leqq \tau \leqq 1} d(X_0, X_\tau) \geqq r] \leqq \mu([r, \infty)) \quad \text{for all } x \in M, \quad r \geqq 0,$$

which is an analogue of condition (UM) (in fact, this measure μ can be taken to have Gaussian decay).

Finally, instead of conditions (IS) and (UI) we can directly assume that 0 does not belong to the spectrum of the Laplace-Beltrami operator Δ in the space $L^2(M, m)$, i.e. $\lambda(M) < 0$ for the top of the spectrum of Δ. This condition is well known to be equivalent to *Cheeger's isoperimetric inequality*:

$$\operatorname{area}(\partial U) \geqq \kappa \cdot m(U)$$

for a certain constant $\kappa=\kappa(M)>0$ and for every compact connected subdomain $U\subset M$ with smooth boundary ∂U; see Cheeger [Che] and Buser [Bu]. It implies that there exist constants $K>0$, $\theta<1$ such that

$$p_t(x,y)\leqq K\theta^t \quad \text{for all } x,y\in M, \quad t\geqq 0.$$

Now, reproducing the argument from Sect. 4 we get the following statements.

Lemma 6 *Let M be a complete connected Riemannian manifold with bounded geometry such that $\lambda(M)\neq 0$. Then there exist constants C_0, C_1 such that*

$$0<C_0<\liminf_{t\to\infty}\frac{1}{t}|X_t|\leqq\limsup_{t\to\infty}\frac{1}{t}|X_t|<C_1<\infty$$

and

$$\lim_{t\to\infty}\frac{1}{t}\sup_{0\leqq\tau\leqq 1} d(X_t,X_{t+\tau})\to 0$$

$\Pr_x$*-almost surely for every $x\in M$.*

Lemma 7 *Let M be a complete connected Riemannian manifold with bounded geometry such that $\lambda(M)<0$. For $x\in M$, consider the event $A_x=A_x(\alpha,\varepsilon)$ in the trajectory space of the Brownian motion on M defined by the following five properties:*

(i) $X_0=x$,
(ii) $d(X_t,X_{t+\tau})\leqq\varepsilon|x|$ *for all* $t\leqq\alpha|x|$, $0\leqq\tau\leqq 1$,
(iii) $d(X_t,X_{t+\tau})\leqq\varepsilon t$ *for all* $t\geqq\alpha|x|$, $0\leqq\tau\leqq 1$,
(iv) $d(X_0,X_t)>C_0\, t$ *for all* $t\geqq\alpha|x|$,
(v) $|X_t|>\varepsilon|x|$ *for all* $t\geqq 0$.

There exists a constant $\varepsilon_0>0$ (independent of α) such that if $\varepsilon\leqq\varepsilon_0$, then

$$\lim_{|x|\to\infty}\Pr_x(A_x)=1.$$

Now, proceeding as in Sects. 5, 6 we get the following analogue of Theorems 1–4.

Theorem 5 *Let M be a complete connected Riemannian manifold with bounded geometry such that $\lambda(M)\neq 0$. Then there exists a random variable X_∞ taking values in the space of ends $\Omega(M)$ of the manifold M such that for every $x\in M$,*

$$\lim_{t\to\infty}X_t=X_\infty \qquad \Pr_x\text{-almost surely}$$

in the end topology. The Dirichlet problem for the Laplace-Beltrami operator Δ of the Riemannian metric on M with the boundary values on $\Omega(M)$ admits solution: every continuous function h^* on $\Omega(M)$ can be uniquely extended to a function h which is continuous on $M\cup\Omega(M)$ and harmonic on M.

If in addition M as a metric space is hyperbolic in the sense of Gromov, then the same conclusions hold for the hyperbolic compactification in the place of the end compactification.

Note that the standard example of Gromov hyperbolic Riemannian manifolds is given by simply connected Riemannian manifolds whose negative sectional curvatures are bounded and bounded away from zero. In this case the hyperbolic boundary coincides with the usual visibility boundary and convergence to the boundary was first stated by Prat [Pr], whereas the solvability of the Dirichlet problem at infinity was proved by Anderson [An] and Sullivan [Su]. Our proof seems to be easier even in this case (here the fact that $\lambda(M)<0$ follows from McKean's comparison theorem for the spectrum of the Laplacian – see Chavel [Cha]), as it does not use any local differential geometry and rests only on two global geometric properties: hyperbolicity and Cheeger's inequality (or, equivalently, $\lambda(M)\neq 0$).

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