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A NOTE ON THE NORMS OF TRANSITION OPERATORS ON LAMPLIGHTER GRAPHS AND GROUPS

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Let $L \wr X$ be a lamplighter graph, i.e., the graph-analogue of a wreath product of groups, and let P be the transition operator (matrix) of a random walk on that structure. We explain how methods developed by Saloff-Coste and the author can be applied for determining the ℓ^p-norms and spectral radii of P, if one has an amenable (not necessarily discrete or unimodular) locally compact group of isometries that acts transitively on L. This applies, in particular, to wreath products $K \wr G$ of finitely-generated groups, where K is amenable. As a special case, this comprises a result of Żuk regarding the ℓ^2-spectral radius of symmetric random walks on such groups.

Keywords: Lamplighter graph; wreath product; random walk; transition operator; norm; spectral radius.

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1. Lamplighter Graphs, Groups, and Random Walks

1.1. *Graphs*

In this note, we are thinking of a graph as a finite or countable set, equipped with a symmetric adjancency (or neighbor) relation, denoted by $\sim$. All graphs are locally finite (each point x has a finite number $\deg(x)$ of neighbors) and connected (any pair of points is joined by a path of successive neighbors). The graph distance is denoted $d(\cdot,\cdot)$.

Let X be a (typically infinite) graph, and let (L, o) be a (finite or infinite) graph with *root* o. An L-valued *configuration* on X is a function $\eta : X \to L$ with finite support

$$\operatorname{supp} \eta = \{x \in X \mid \eta(x) \neq o\}.$$

Let $\mathcal{C} = \mathcal{C}(X \to L)$ be the set of all L-valued configurations.

Given $x \in X$, we define an adjacency relation $\sim_x$ on $\mathcal{C}$ by

$$\eta \sim_x \eta' :\Leftrightarrow \eta(x) \sim \eta'(x) \text{ in } L \quad \text{and} \quad \eta(y) = \eta'(y) \text{ for all } y \in X \setminus \{x\}.$$

The *lamplighter graph* $L \wr X$ has vertex set $\mathcal{C} \times X$, and adjacency is given by

$$(\eta, x) \sim (\eta', x') :\Leftrightarrow \begin{cases} x \sim x' & \text{in } X \text{ and } \eta = \eta', \quad \text{or} \\ x = x' & \text{in } X \text{ and } \eta \sim_x \eta'. \end{cases}$$

It is obviously connected, since this is true for X and L. The interpretation is that at each vertex of X, there is a lamp, whose possible states are the vertices of L, including the state "off" that corresponds to the root o of L. Only finitely many lamps may be switched on, and in a single step (that describes adjacency), one may either go to a neighbor of the current vertex in X and leave all lamps unchanged, or one may stay at the actual vertex of X and change the state of the lamp at x to a neighbor state in L.

1.2. *Groups*

An *automorphism* of a graph X is a self-isometry with respect to the graph metric. The group of all automorphisms of X is denoted by $\mathrm{Aut}(X)$. Equipped with the topology of point-wise convergence, it is a totally disconnected, locally compact group, not necessarily discrete, as various examples show (see also below).

Given the lamplighter graph $L \wr X$, let K be a (typically closed) subgroup of $\mathrm{Aut}(L)$, and define

$$\Phi = \Phi(X \to K) = \{\phi : X \to K \mid \mathrm{supp}\, \phi \text{ finite}\},$$

where $\mathrm{supp}\, \phi = \{x \in X : \phi(x) \neq \mathrm{id}_K\}$. Then Φ becomes a group when equipped with the point-wise product in K, i.e.,

$$(\phi_1 \phi_2)(x) = \phi_1(x)\phi_2(x)$$

in the sense of the composition of two automorphisms of L, and

$$\mathrm{id}_\Phi = \iota, \quad \text{where } \iota(x) = \mathrm{id}_K \qquad \forall\, x \in X.$$

The group Φ acts on the set $\mathcal{C}$ of all configurations by

$$\phi\eta(x) = \phi(x)\eta(x)$$

in the sense of the action of an automorphism of L on an element of L. Indeed, $\mathrm{supp}\, \phi\eta \subset \mathrm{supp}\, \phi \cup \mathrm{supp}\, \eta$ is finite. Furthermore, if $\eta \sim_x \eta'$ then $\phi\eta \sim_x \phi\eta'$.

Next, let G be a (typically closed) subgroup of $\mathrm{Aut}(X)$. Every $g \in G$ acts on $\mathcal{C}$ and on Φ. Write T_g for both actions:

$$T_g\eta(x) = \eta(g^{-1}x) \quad \text{and} \quad T_g\phi(x) = \phi(g^{-1}x)$$

for $\eta \in \mathcal{C}$, resp. $\phi \in \Phi$. Observe that $\eta \sim_x \eta' \Rightarrow T_g\eta \sim_{gx} T_g\eta'$.

Finally, we define an action of pairs of elements (ϕ, g) on $L \wr X$, where $g \in G$ and $\phi \in \Phi$:

$$(\phi, g)(\eta, x) := (\phi T_g \eta, gx).$$

This action preserves the adjacency relation of $L \wr X$. Therefore, $(\phi, g) \in \mathrm{Aut}(L \wr X)$. One computes the composition

$$(\phi_1, g_1)(\phi_2, g_2) = (\phi_1 T_{g_1} \phi_2, g_1 g_2).$$

Thus, we obtain the *semi-direct product*

$$\Gamma = \Phi \rtimes G,$$

which is a subgroup of $\mathrm{Aut}(L \wr X)$. The following is straightforward.

Proposition 1.1. (a) *If G acts transitively on X and K acts transitively on L then Γ acts transitively on $L \wr X$.*

(b) *If G and K are closed subgroups of* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(L)$, *respectively, then Γ is closed in* $\mathrm{Aut}(L \wr X)$.

(c) *If both G and K are discrete/finitely generated/compactly generated/amenable/unimodular then so is* Γ.

(d) *If X is a Cayley graph of the group G and L is a Cayley graph of the group K with group identity o, then $L \wr X$ is a Cayley graph of the wreath product* (*lamplighter group*) $\Gamma = K \wr G$.

In the sequel, we shall write ηx for $(\eta, x) \in L \wr X$ and ϕg for $(\phi, g) \in \Gamma = \Phi \rtimes G$.

1.3. *Transition matrices of random walks*

Given any graph X, a random walk (Markov chain) on X is described by a stochastic *transition matrix* $P = \big(p(x,y)\big)_{x,y \in X}$, where $p(x,y)$ is the probability to move in one step from x to y. Then P acts on functions $f : X \to \mathbb{R}$ by

$$Pf(x) = \sum_y p(x,y)\, f(y).$$

When involving a graph structure, we have in mind that P is in some sense adapted to that structure. The basic example is *simple random walk* (SRW), where $p(x,y) = 1/\deg(x)$, if $y \sim x$, and $p(x,y) = 0$, otherwise. In general, P is called *reversible*, or more precisely, m-*reversible*, if m is a strictly positive measure on X such that $\mathsf{m}(x)p(x,y) = \mathsf{m}(y)p(y,x)$ for all $x, y \in X$. SRW is reversible with $\mathsf{m}(x) = \deg(x)$. If G is a group acting on X then P is called G-*invariant*, if $p(gx, gy) = p(x,y)$ for all $x, y \in X$ and $g \in G$. If X is a Cayley graph of G, then G-invariance means that $p(x,y) = \mu(x^{-1}y)$, where μ is a probability measure on G, and we say that P is *induced* by μ.

Given P and a reference measure m on X, we write

$$\sigma_{\mathsf{p}}(P, \mathsf{m}) = \|P\|_{\mathsf{p}\to\mathsf{p}} \quad \text{and} \quad \rho_{\mathsf{p}}(P, \mathsf{m}) = \lim_{n\to\infty} \|P^n\|_{\mathsf{p}\to\mathsf{p}}^{1/n}$$

for the norm and spectral radius of P acting on the weighted space $\ell^{\mathsf{p}}(X, \mathsf{m})$, where $1 < \mathsf{p} < \infty$. Spectral computations for random walks on lamplighter groups and related graphs have attracted some interest recently, see e.g. Grigorchuk and Żuk [4], Dicks and Schick [2], or Bartholdi and Woess [1]. What we are interested in here are the norm and spectral radius of group-invariant transition operators acting on $\ell^{\mathsf{p}}(L \wr X)$, or more generally, on a weighted space $\ell^{\mathsf{p}}(L \wr X, \mathsf{m})$, where m is a suitable measure on the lamplighter graph.

Before this, we explain two ways for constructing transition matrices that are adapted to the structure of $L \wr X$.

We start with transition matrices P_L and P_X on L and X, respectively. We lift P_L to $\bar{P}_L$ on $L \wr X$ by setting

$$\bar{p}_L(\eta\, x, \eta'\, x') = \begin{cases} p_L\big(\eta(x), \eta'(x)\big), & \text{if } x' = x \text{ and } \eta' \equiv \eta \text{ on } X \setminus \{x\}, \\ 0, & \text{otherwise.} \end{cases}$$

We also lift P_X to $\bar{P}_X$ on $L \wr X$ by setting

$$\bar{p}_X(\eta\, x, \eta'\, x') = \begin{cases} p_X(x, x'), & \text{if } \eta' = \eta, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.2. *For the powers of the transition operators $\bar{P}_X$ and $\bar{P}_L$, we have $\overline{P_X^n} = \bar{P}_X^n$ and $\overline{P_L^n} = \bar{P}_L^n$.*

The two basic models on $L \wr X$ are the following.

(1) "Walk or switch". Let $0 < a < 1$. Define P_a on $L \wr X$ by

$$P_a = a\, \bar{P}_X + (1 - a)\, \bar{P}_L.$$

The interpretation of P_a in lamplighter terms is as follows. If the lamplighter stands at x and the actual configuration is η, then he first tosses a coin. If "head" comes up (which happens with probability a) then he makes a random move on X according to the probability distribution $p_X(x, \cdot)$, while leaving all lamps unchanged. If "tail" comes up then he makes no move on X, but modifies the state of the lamp where he stands according to the distribution $p_L\big(\eta(x), \cdot\,\big)$.

(2) "Switch–walk–switch". We define the transition matrix Q on $L \wr X$ by the matrix product

$$Q = \bar{P}_L \bar{P}_X \bar{P}_L.$$

Here, the lamplighter interpretation is as follows. If the lamplighter stands at x and the actual configuration is η, then he first changes the state of the lamp at x according to the probability distribution $p_L\big(\eta(x), \cdot\,\big)$. Subsequently, he makes a step to some point $x' \in X$ according to the probability distribution $p_X(x, \cdot)$, and at last, he changes the state of the lamp at x' according to the probability distribution $p_L\big(\eta(x'), \cdot\,\big)$.

Observation 1.3. (a) If L and X are *regular* and P_L and P_X are the transition matrices of the respective SRWs, then the SRW on $L \wr X$ is P_a with

$$a = \frac{\deg_X}{\deg_X + \deg_L},$$

where $\deg_L$ and $\deg_X$ are the constant vertex degrees of L and X, respectively.

(b) Suppose that $K \le \mathrm{Aut}(L)$ and $G \le \mathrm{Aut}(X)$. If P_L is K-invariant and P_X is G-invariant then P_a and Q are Γ-invariant, where Γ is defined as above.

(c) Suppose that L is a Cayley graph of K, that X is a Cayley graph of G, and that $p_L(k, l) = \nu(k^{-1}l)$ and $p_X(x, y) = \mu(x^{-1}y)$, where ν and μ are probability measures on K and G, respectively. Then P_a is induced by the convex combination $a \cdot \mu + (1 - a) \cdot \nu$, and Q is induced by the convolution $\nu * \mu * \nu$.

Here, μ and ν are seen as probability measures on Γ, which is legitimate, since both G and K embed naturally into Γ.

1.4. *Reference measures*

If m_X and m_L are measures on X and L, respectively, and

$$\mathsf{m}_L(o) = 1, \tag{1.1}$$

then we construct a measure $\mathsf{m} = \mathsf{m}_L \wr \mathsf{m}_X$ on $L \wr X$ by setting

$$\mathsf{m}(\eta\, x) = \mathsf{m}_X(x)\mathsf{m}_{\mathcal{C}}(\eta), \quad \text{where } \mathsf{m}_{\mathcal{C}}(\eta) = \prod_{y \in X} \mathsf{m}_L(\eta(y)). \tag{1.2}$$

This choice is natural because of the following.

Lemma 1.4. *If P_X is m_X-reversible and P_L is m_L-reversible, then both P_a and Q are m-reversible on $L \wr X$.*

2. Norms and Spectral Radii

Let X be a general graph, G a closed subgroup of $\mathrm{Aut}(X)$, and P a G-invariant transition matrix on X. Furthermore, we say that a reference measure m on X is *G-compatible,* if it satisfies

$$\mathsf{m}(gx)/\mathsf{m}(x) = \mathsf{m}(gy)/\mathsf{m}(y) \quad \text{for all } x, y \in X,\ g \in G. \tag{2.1}$$

We remark that this property is satisfied in the typical case when P is G-invariant, m-reversible and irreducible (as a positive matrix, i.e., for every $x, y \in X$ there is n such that $p^{(n)}(x, y)$, the (x, y)-entry of P^n, is > 0).

We also remark that every irreducible, m-reversible P acts as a self-adjoint operator on the weighted Hilbert space $\ell^2(X, \mathsf{m})$, so that the spectral radius $\rho_2(P, \mathsf{m})$ coincides with the norm $\|P\|_{2\to 2}$ on that space. Furthermore, in this case, for all $x, y \in X$,

$$\rho_2(P, \mathsf{m}) = \limsup_{n\to\infty} p^{(n)}(x, y)^{1/n}$$

describes the exponential decay of the n-step transition probabilities. Most of these simple facts can be found in [12].

Given G as above, let G_x denote the stabilizer of $x \in X$. We introduce the following function $t = t_G$ on $X \times X$:

$$t(x, y) = |G_x y|/|G_y x|.$$

Here, $G_x y = \{gy : g \in G_x\}$, and $|\cdot|$ denotes cardinality. The function t is G-invariant. It is linked to the *modular function* of the group G, that is, the function that describes the change from left to right Haar measure. In particular, when G is unimodular then we even have $t(gx, hy) = t(x, y)$ for all $x, y \in X$ and $g, h \in G$. See Schlichting [9], Trofimov [11] and Saloff-Coste and Woess [7, Lemma 1].

We can consider the family of orbits X_i, $i \in I$ (a suitable index set) of the G-action on X, and the factor space $I = G\backslash X$.

Given $1 < \mathsf{p} < \infty$, following Saloff-Coste and Woess [8], we now define a non-negative, but not necessarily stochastic matrix $A_{\mathsf{p}}(P, \mathsf{m}) = \big(a_{\mathsf{p}}(i,j)\big)_{i,j\in I}$ by

$$a_{\mathsf{p}}(i,j) = \sum_{y\in X_j} (t(y,x)\mathsf{m}(x)/\mathsf{m}(y))^{1/\mathsf{p}} p(x,y), \quad x \in X_i. \tag{2.2}$$

This is independent of the specific choice of $x \in X_i$ by (2.1) and G-invariance of t.

Theorem 2.1 (Saloff-Coste and Woess [8]). *Suppose that P is G-invariant and that the reference measure m satisfies* (2.1). *If the group G is amenable then the norm $\sigma_{\mathsf{p}}(P, \mathsf{m}) = \|P\|_{\mathsf{p}\to\mathsf{p}}$ and the spectral radius $\rho_{\mathsf{p}}(P, \mathsf{m})$ of P acting on $\ell^{\mathsf{p}}(X, \mathsf{m})$ coincide with the norm and spectral radius of $A_{\mathsf{p}}(P, \mathsf{m})$ acting on the space $\ell^{\mathsf{p}}(I)$,*

$$\sigma_{\mathsf{p}}(P, \mathsf{m}) = \|A_{\mathsf{p}}(P, \mathsf{m})\|_{\mathsf{p}\to\mathsf{p}\mathsf{p}} \quad \textit{and} \quad \rho_{\mathsf{p}}(P, \mathsf{m}) = \rho_{\mathsf{p}}\big(A_{\mathsf{p}}(P, \mathsf{m})\big).$$

In particular, when G also acts transitively, then norm and spectral radius coincide.

This result has its roots in Soardi and Woess [10], who considered SRW in the transitive case, with $\mathsf{p} = 2$.

The purpose of this note is to outline how Theorem 2.1 applies to lamplighter random walks on $L \wr X$ in the place of X, with suitable subgroups of the group $\Gamma = \Phi(X \to K) \rtimes G$ in the place of G.

Without any assumption of group-invariance, the following holds.

Lemma 2.2. *Let P_X, m_X on X and P_L, m_L on L be given. With m constructed as in* (1.2), *one has*

$$\sigma_{\mathsf{p}}(\bar{P}_X, \mathsf{m}) = \sigma_{\mathsf{p}}(P_X, \mathsf{m}_X) \quad \textit{and} \quad \sigma_{\mathsf{p}}(\bar{P}_L, \mathsf{m}) = \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L).$$

The same holds for spectral radii in the place of norms.

Proof. The following notation will be useful: if $\eta \in \mathcal{C}$ then define

$$\eta^{l_1,\ldots,l_n}_{x_1,\ldots,x_n}(y) = \begin{cases} l_i, & \text{if } y = x_i \ \ (i = 1, \ldots, n), \\ \eta(y), & \text{otherwise,} \end{cases}$$

where $x_1, \ldots, x_n \in X$ are *distinct* and $l_1, \ldots, l_n \in L$.

Furthermore, $\delta \in \mathcal{C}$ denotes the configuration with $\delta(x) = o$ for all $x \in X$.

We start by showing that $\sigma_{\mathsf{p}}(\bar{P}_L, \mathsf{m}) \le \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)$. Let $f \in \ell^{\mathsf{p}}(L \wr X, \mathsf{m})$. Note that every configuration can be written as η^l_x, where $l \in L$ and η is such that

$\eta(x) = o$, with arbitrary $x \in X$. We compute

$$\begin{aligned}
\|\bar{P}_L f\|_{\mathsf{p}}^{\mathsf{p}} &= \sum_{x\in X} \sum_{\eta:\eta(x)=o} \sum_{l\in L} \big|\bar{P}_L f(\eta_x^l\, x)\big|^{\mathsf{p}}\, \mathsf{m}_X(x)\, \mathsf{m}_{\mathcal{C}}(\eta_x^l) \\
&= \sum_{x\in X} \sum_{\eta:\eta(x)=o} \left(\sum_{l\in L} \left| \sum_{l'\in L} p_L(l,l') f(\eta_x^{l'}\, x) \right|^{\mathsf{p}} \mathsf{m}_L(l) \right) \mathsf{m}_X(x)\, \mathsf{m}_{\mathcal{C}}(\eta) \\
&\le \sum_{x\in X} \sum_{\eta:\eta(x)=o} \left(\sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)^{\mathsf{p}} \sum_{l\in L} |f(\eta_x^l\, x)|^{\mathsf{p}}\, \mathsf{m}_L(l) \right) \mathsf{m}_X(x)\, \mathsf{m}_{\mathcal{C}}(\eta) \\
&= \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)^{\mathsf{p}}\, \|f\|_{\mathsf{p}}^{\mathsf{p}}.
\end{aligned}$$

To show equality, we choose a reference point $x_0 \in X$ and embed $\ell^{\mathsf{p}}(L, \mathsf{m}_L)$ isometrically into $\ell^{\mathsf{p}}(L \wr X, \mathsf{m})$ via $f \mapsto \bar{f}$, where

$$\bar{f}(\eta\, x) = \begin{cases} f(l)\, \mathsf{m}_X(x_0)^{-1/\mathsf{p}}, & \text{if } x = x_0 \text{ and } \eta = \delta_{x_0}^l \ (l \in L), \\ 0, & \text{otherwise.} \end{cases}$$

One verifies easily that $\bar{P}_L \bar{f} = \overline{P_L f}$ for all $f \in \ell^{\mathsf{p}}(L, \mathsf{m}_L)$. Therefore $\sigma_{\mathsf{p}}(\bar{P}_L, \mathsf{m}) = \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)$.

For $\bar{P}_X$, the proof is analogous. In this case, one embeds $\ell^{\mathsf{p}}(X, \mathsf{m}_X)$ isometrically into $\ell^{\mathsf{p}}(L \wr X, \mathsf{m})$ via $f \mapsto \bar{f}$, where $\bar{f}(\eta\, x) = f(x)$ if $\eta = \delta$, and $\bar{f}(\eta\, x) = 0$ otherwise. □

We deduce that the inequalities

$$\sigma_{\mathsf{p}}(Q, \mathsf{m}) \le \sigma_{\mathsf{p}}(P_X, \mathsf{m}_X)\, \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)^2 \quad \text{and}$$
$$\sigma_{\mathsf{p}}(P_a, \mathsf{m}) \le a\, \sigma_{\mathsf{p}}(P_X, \mathsf{m}_X) + (1-a)\, \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)$$

hold always. An analogous inequality for spectral radii is not obvious.

Theorem 2.3. *Let P_X and m_X be arbitrary. Suppose that K is a locally compact group that acts transitively on L, that P_L is K-invariant, and that the reference measure m_L on L is normalized by* (1.1) *and K-compatible.*

If K is amenable *then by Theorem* 2.1,

$$\sigma_{\mathsf{p}}(P_L, \mathsf{m}_L) = \rho_{\mathsf{p}}(P_L, \mathsf{m}_L) = \sum_{l\in L} \big(t_K(l,o)\mathsf{m}_L(l)\big)^{1/\mathsf{p}} p_L(o,l),$$

and furthermore, if m is given by (1.2), *then the* "switch–walk–switch" *operator Q on $L \wr X$ satisfies*

$$\sigma_{\mathsf{p}}(Q, \mathsf{m}) = \sigma_{\mathsf{p}}(P_X, \mathsf{m}_X)\, \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)^2 \quad \textit{and} \quad \rho_{\mathsf{p}}(Q, \mathsf{m}) = \rho_{\mathsf{p}}(P_X, \mathsf{m}_X)\, \rho_{\mathsf{p}}(P_L, \mathsf{m}_L)^2,$$

while the "walk or switch" *operator P_a satisfies*

$$\sigma_{\mathsf{p}}(P_a, \mathsf{m}) = \sigma_{\mathsf{p}}(a\, P_X + c\, I_X, \mathsf{m}_X) \quad \textit{and} \quad \rho_{\mathsf{p}}(P_a, \mathsf{m}) = \rho_{\mathsf{p}}(a\, P_X + c\, I_X, \mathsf{m}_X),$$

where $c = (1-a)\sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)$ and I_X is the identity operator over X.

Proof. If K is amenable, then so is the group $\Phi = \Phi(X \to K)$, since it is a direct sum of countably many copies of K. It embeds into $\mathrm{Aut}(L \wr X)$ by $\phi \mapsto (\phi, \mathrm{id}_X)$. The factor space is $\Phi\backslash(L \wr X) \cong X$, and the different orbits are the sets $\mathcal{O}_x = \{\eta\, x : \eta \in \mathcal{C}\}$. The stabilizer of $\eta\, x \in L \wr X$ under the action of Φ is independent of x; it is the direct sum of the stabilizers of all $\eta(y)$ $(y \in X)$ under the action of K, i.e.,

$$\Phi_{\eta\, x} = \{\phi \in \Phi : \phi(y) \in K_{\eta(y)}\ \forall\, y \in X\}.$$

Therefore, one computes for $\eta \in \mathcal{C}$ and $x, x' \in X$

$$t_\Phi(\eta\, x', \delta\, x) = \prod_{y \in \mathrm{supp}\eta} t_K\bigl(\eta(y), o\bigr).$$

When applying (2.2) to Q and P_a, we have to replace I with X and X_i with $\mathcal{O}_x$.

(1) "Switch–walk–switch". When $x' \neq x$, we have $q(\delta\, x, \eta\, x') \neq 0$ only when $\eta = \delta^{l,l'}_{x,x'}$ for some $l, l' \in L$. In this case,

$$q(\delta\, x, \delta^{l,l'}_{x,x'}\, x') = p_L(o,l)\, p_X(x,x')\, p_L(o,l').$$

Therefore

$$\begin{aligned} a_{\mathsf{p}}(x,x') &= \sum_{l,l' \in L} \left(t_\Phi(\delta^{l,l'}_{x,x'}\, x', \delta\, x)\, \frac{\mathsf{m}(\delta\, x)}{\mathsf{m}(\delta^{l,l'}_{x,x'}\, x')} \right)^{1/\mathsf{p}} p_L(o,l)\, p_X(x,x')\, p_L(o,l') \\ &= p_X(x,x') \left(\frac{\mathsf{m}_X(x)}{\mathsf{m}_X(x')} \right)^{1/\mathsf{p}} \sum_{l,l' \in L} \left(t_K(l,o) \frac{\mathsf{m}_L(o)}{\mathsf{m}_L(l)} \cdot t_K(l',o) \frac{\mathsf{m}_L(o)}{\mathsf{m}_L(l')} \right)^{1/\mathsf{p}} \\ &\quad \times p_L(o,l)\, p_L(o,l') \\ &= p_X(x,x') \left(\frac{\mathsf{m}_X(x)}{\mathsf{m}_X(x')} \right)^{1/\mathsf{p}} \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)^2. \end{aligned}$$

When $x' = x$, we have $q(\delta\, x, \eta\, x) \neq 0$ only when $\eta = \delta^l_x$ for some $l \in L$. In this case,

$$q(\delta\, x, \delta^l_x x) = \sum_{l' \in L} p_L(o,l') p_X(x,x')\, p_L(l',l).$$

Note that $t_K(l,o) = t_K(l,l') t_K(l',o)$ by [7, Lemma 1]. By transitivity of the action of K on L, Theorem 2.1 yields that

$$\sigma_{\mathsf{p}}(P_L, \mathsf{m}_L) = \sum_{l \in L} \bigl(t_K(l,l') \mathsf{m}_L(l') / \mathsf{m}_L(l) \bigr)^{1/\mathsf{p}} p_L(l',l) \quad \text{for every} \quad l' \in L.$$

Therefore

$$\begin{aligned} a_{\mathsf{p}}(x,x) &= \sum_{l,l' \in L} \left(t_\Phi(\delta^l_x\, x, \delta\, x)\, \frac{\mathsf{m}(\delta\, x)}{\mathsf{m}(\delta^l_x\, x)} \right)^{1/\mathsf{p}} p_L(o,l')\, p_X(x,x)\, p_L(l',l) \\ &= p_X(x,x) \sum_{l,l' \in L} \left(t_K(l,l') \frac{\mathsf{m}_L(l')}{\mathsf{m}_L(l)} \cdot t_K(l',o) \frac{\mathsf{m}_L(o)}{\mathsf{m}_L(l')} \right)^{1/\mathsf{p}} p_L(o,l')\, p_L(l',l) \\ &= p_X(x,x) \sigma_{\mathsf{p}}(P_L, \mathsf{m}_L)^2. \end{aligned}$$

Thus, $a_{\mathsf{p}}(x,x') = \sigma_{\mathsf{p}}(P_L,\mathsf{m}_L)^2\, p_X(x,x')\big(\mathsf{m}_X(x)/\mathsf{m}_X(x')\big)^{1/\mathsf{p}}$ for all $x,x' \in X$. Since the mapping $f \mapsto f\,\mathsf{m}_X^{-1/\mathsf{p}}$ is a Banach space isomorphism from $\ell^{\mathsf{p}}(X)$ (with the counting measure) to $\ell^{\mathsf{p}}(X,\mathsf{m}_X)$, Theorem 2.1 yields that

$$\begin{aligned}\|A_{\mathsf{p}\to\mathsf{p}}(Q,\mathsf{m})\|_{\mathsf{p}} &= \sigma_{\mathsf{p}}(P_L,\mathsf{m}_L)^2\sigma_{\mathsf{p}}(P_X,\mathsf{m}) \quad \text{and}\\ \rho_{\mathsf{p}}\big(A_{\mathsf{p}}(Q,\mathsf{m})\big) &= \sigma_{\mathsf{p}}(P_L,\mathsf{m}_L)^2\rho_{\mathsf{p}}(P_X,\mathsf{m}_X).\end{aligned}$$

Since $\sigma_{\mathsf{p}}(P_L,\mathsf{m}_L) = \rho_{\mathsf{p}}(P_L,\mathsf{m}_L)$, this concludes the proof for Q.

(2) "Walk or switch". When $x' \neq x$, we have $p_a(\delta\,x,\eta\,x') \neq 0$ only when $\eta = \delta$, and in this case, $p_a(\delta\,x,\delta\,x') = a\,p_X(x,x')$. Therefore

$$a_{\mathsf{p}}(x,x') = a\left(t_\Phi(\delta\,x',\delta\,x)\,\frac{\mathsf{m}(\delta\,x)}{\mathsf{m}(\delta\,x')}\right)^{1/\mathsf{p}} p_X(x,x') = a\,p_X(x,x')\left(\frac{\mathsf{m}_X(x)}{\mathsf{m}_X(x')}\right)^{1/\mathsf{p}}.$$

When $x' = x$, we have

$$\begin{aligned}p_a(\delta\,x,\delta\,x) &= a\,p_X(x,x) + (1-a)\,p_L(o,o) \quad \text{and}\\ p_a(\delta\,x,\delta_x^l\,x) &= (1-a)\,p_L(o,l)\,, \quad l \neq o,\end{aligned}$$

while $p_a(\delta\,x,\eta\,x) = 0$ in all other cases. Therefore,

$$\begin{aligned}a_{\mathsf{p}}(x,x) &= a\,p_X(x,x) + (1-a)\sum_{l\in L}\left(t_\Phi(\delta_x^l\,x,\delta\,x)\,\frac{\mathsf{m}(\delta\,x)}{\mathsf{m}(\delta_x^l\,x)}\right)^{1/\mathsf{p}} p_L(o,l)\\ &= a\,p_X(x,x) + (1-a)\sum_{l\in L}\left(t_K(l,o)\,\frac{\mathsf{m}_L(o)}{\mathsf{m}_L(l)}\right)^{1/\mathsf{p}} p_L(o,l)\\ &= a\,p_X(x,x) + (1-a)\,\sigma_{\mathsf{p}}(P_L,\mathsf{m}_L).\end{aligned}$$

Using once more the isomorphism $f \mapsto f\,\mathsf{m}_X^{-1/\mathsf{p}}$ from $\ell^{\mathsf{p}}(X)$ to $\ell^{\mathsf{p}}(X,\mathsf{m}_X)$, we obtain the proposed results for P_a. □

Remark 2.4. Regarding the norms of Q and P_a, it appears reasonable to suspect that the conclusions of Theorem 2.3 might hold under more general assumptions, e.g., without assuming transitivity or amenability of the group K that acts on L. However, at the moment, the author does not see a proof without specific assumptions of this type.

We now list a few special cases that may be of interest.

Corollary 2.5. *Assume all hypotheses of Theorem* 2.3.

(a) *If in addition*, m_L *is the counting measure, and* K (*besides being amenable*) *is also* unimodular $(t \equiv 1)$ *then*

$$\begin{aligned}\sigma_{\mathsf{p}}(Q,\mathsf{m}) &= \sigma_{\mathsf{p}}(P_X,\mathsf{m}_X), & & \sigma_{\mathsf{p}}(P_a,\mathsf{m}) = \sigma_{\mathsf{p}}\big(a\,P_X + (1-a)\,I_X,\mathsf{m}_X\big),\\ \rho_{\mathsf{p}}(Q,\mathsf{m}) &= \rho_{\mathsf{p}}(P_X,\mathsf{m}_X), & \textit{and} \quad & \rho_{\mathsf{p}}(P_a,\mathsf{m}) = \rho_{\mathsf{p}}\big(a\,P_X + (1-a)\,I_X,\mathsf{m}_X\big).\end{aligned}$$

(b) *In particular, suppose that X and L are* Cayley graphs *of the finitely-generated groups G and K, and that μ and ν are probability measures on G and K, respectively. If K is amenable, then the transition ($\equiv$ convolution) operators on $\Gamma = K \wr G$ induced by $\nu * \mu * \nu$ and $a \cdot \mu + (1-a) \cdot \nu$ satisfy*

$$\|\nu * \mu * \nu\|_{\mathsf{p}\to\mathsf{p}} = \|\mu\|_{\mathsf{p}\to\mathsf{p}}, \qquad \|a\cdot\mu + (1-a)\cdot\nu\|_{\mathsf{p}\to\mathsf{p}} = \|a\cdot\mu + (1-a)\cdot\delta_{\mathrm{id}_G}\|_{\mathsf{p}\to\mathsf{p}},$$
$$\rho_{\mathsf{p}}(\nu * \mu * \nu) = \rho_{\mathsf{p}}(\mu), \quad \text{and} \quad \rho_{\mathsf{p}}\big(a\cdot\mu + (1-a)\cdot\nu\big) = \rho_{\mathsf{p}}\big(a\cdot\mu + (1-a)\cdot\delta_{\mathrm{id}_G}\big).$$

(*The norms and spectral radii on the left-hand sides are taken over Γ, while on the right-hand sides they are over G. The reference measures are the counting measures.*)

(c) *In the general statement of Theorem* 2.3, *if K is amenable and P_X is m_X-reversible, then for $\mathsf{p} = 2$ one gets*

$$\sigma_2(P_a, \mathsf{m}) = \rho_2(P_a, \mathsf{m}) = a\,\rho_2(P_X, \mathsf{m}_X) + (1-a)\,\rho_2(P_L, \mathsf{m}_L).$$

(*Recall that in this case $\sigma_2(P_X, \mathsf{m}_X) = \rho_2(P_X, \mathsf{m}_X)$ by reversibility, and $\sigma_2(P_L, \mathsf{m}_L) = \rho_2(P_L, \mathsf{m}_L)$ by Theorem* 2.1.)

Żuk [13] has used a completely different method to prove statement (b) in the special case when $\mathsf{p} = 2$ and μ and ν are finitely supported, symmetric probability measures on the respective groups.

In Theorem 2.3, we have assumed that the group K is amenable and acts transitively on L, without any specific assumptions on X and P_X. Dually, one might try to use transitivity and amenability of G acting on X, without specific assumptions on L and P_L. However, this turns out to be much more complicated, since the factor space of the action of G on $L \wr X$ is isomorphic with $G_{x_0}\backslash\mathcal{C}$, where $x_0 \in X$ is arbitrary and G_{x_0} is the stabilizer of x_0 in G. (Recall that G acts on $\mathcal{C}$.) Even when X is a Cayley graph of G, in which case the factor space is $\cong \mathcal{C}$ and $t_G \equiv 1$, the "reduced" operators $A_{\mathsf{p}}(Q, \mathsf{m})$ and $A_{\mathsf{p}}(P_a, \mathsf{m})$ on $\ell^2(\mathcal{C})$ according to (2.2) appear to be too complicated for the purpose of simplifying the computation of norms and spectral radii.

However, if we are in the situation of Theorem 2.3, and *in addition* we have an amenable group G acting (not necessarily transitively) on X, such that P_X is G-invariant and m_X is G-compatible, then Theorem 2.1 can be used for explicit computations. In particular, there are cases when a Cayley graph of a non-amenable group admits a transitive action of an amenable (non-unimodular) group of automorphisms (isometries), or such an action with only finitely many orbits. In conclusion, we shall now exhibit, resp. recall, a simple example of this type.

Example 2.6. Let $X = \mathbb{T}_r$, the homogeneous tree with degree $r+1$, and $L = \mathbb{T}_s$, and consider the corresponding lamplighter graph $\mathbb{T}_s \wr \mathbb{T}_r$.

Since $\mathbb{T}_r$ is the Cayley graph of the group $\Gamma_r = \langle a_1, \dots, a_{r+1} \mid a_i^2 = \mathrm{id}\rangle$ with respect to the generating set $\{a_1, \dots, a_{r+1}\}$, the graph $\mathbb{T}_s \wr \mathbb{T}_r$ is a Cayley graph of $\Gamma_s \wr \Gamma_r$. (If r, resp. s, is odd, we can also take the free group.) Let P_X and P_L be the corresponding simple random walks, and let m_X, m_L and m be the respective counting measures.

There are different ways for computing the ℓ^{p}-spectral radii. For $\mathsf{p} = 2$, this goes back to Kesten [5]. For arbitrary p, see e.g. Pytlik [6] or Figà-Talamanca and Steger [3]. The simplest computation uses Theorem 2.1: we choose an end of $\mathbb{T}_r$ and consider the group G of all automorphisms (isometries) of the tree which fix that end. This group is amenable (and nonunimodular) and acts transitively on $\mathbb{T}_r$. Thus, the single-orbit-case of Theorem 2.1 applies, and one obtains

$$\rho_{\mathsf{p}}\big(\mathrm{SRW}(\mathbb{T}_r)\big) = \frac{r^{1/\mathsf{p}} + r^{1/\mathsf{q}}}{r+1}, \quad \text{where } \frac{1}{\mathsf{p}} + \frac{1}{\mathsf{q}} = 1.$$

For details of the computation and a figure, see [8].

The hypotheses of Theorem 2.3 are satisfied. For the switch–walk–switch model on $\mathbb{T}_s \wr \mathbb{T}_r$, we get

$$\rho_{\mathsf{p}}(Q) = \sigma_{\mathsf{p}}(Q) = \frac{r^{1/\mathsf{p}} + r^{1/\mathsf{q}}}{r+1}\left(\frac{s^{1/\mathsf{p}} + s^{1/\mathsf{q}}}{s+1}\right)^2.$$

As mentioned before, SRW on $\mathbb{T}_s \wr \mathbb{T}_r$ is P_a with $a = \frac{r+1}{r+s+2}$. Since G is amenable and acts transitively, we get from Theorem 2.1 that $\rho_{\mathsf{p}}(a\,P_X + c\,I_X) = a\rho_{\mathsf{p}}(P_X) + c$. Therefore

$$\rho_{\mathsf{p}}\big(\mathrm{SRW}(\mathbb{T}_s \wr \mathbb{T}_r)\big) = \sigma_{\mathsf{p}}\big(\mathrm{SRW}(\mathbb{T}_s \wr \mathbb{T}_r)\big) = \frac{r^{1/\mathsf{p}} + r^{1/\mathsf{q}} + s^{1/\mathsf{p}} + s^{1/\mathsf{q}}}{r+s+2}.$$

In the same way, one can compute the norms and spectral radii of random walks on $\mathbb{T}_{r_1} \wr \cdots \wr \mathbb{T}_{r_k}$ for arbitrary $k \geq 2$. □

Further examples of the same type may be constructed by using those given in [7] and [8] (such as, e.g., the buildings associated with the matrix group $\mathrm{PGL}(n)$ over a local field).

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