Geom Dedicata
DOI 10.1007/s10711-006-9104-x

ORIGINAL PAPER

The Poisson boundary of lamplighter random walks on trees

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Received: 1 June 2006/Accepted: 10 October 2006

Abstract Let $\mathbb{T}_q$ be the homogeneous tree with degree $q + 1 \geq 3$ and $\mathcal{F}$ a finitely generated group whose Cayley graph is $\mathbb{T}_q$. The associated lamplighter group is the wreath product $\mathcal{L} \wr \mathcal{F}$, where $\mathcal{L}$ is a finite group. For a large class of random walks on this group, we prove almost sure convergence to a natural geometric boundary. If the probability law governing the random walk has finite first moment, then the probability space formed by this geometric boundary together with the limit distribution of the random walk is proved to be maximal, that is, the Poisson boundary. We also prove that the Dirichlet problem at infinity is solvable for continuous functions on the active part of the boundary, if the lamplighter "operates at bounded range".

Keywords Random walk · Wreath product · Tree · Poisson boundary · Dirichlet problem

Mathematics Subject Classification (2000) 60J50 · 05C05 · 20E08 · 31C20

1 Introduction

Let $\mathbb{T} = \mathbb{T}_q$ be the homogeneous tree with degree $q + 1 \geq 3$. Assume that at each vertex $x \in \mathbb{T}$ there is a lamp which may be switched off or on in r different states of "intensity", encoded by the set $\mathbb{Y} = \{0, \ldots, r - 1\}$, where the state 0 represents "off".

Supported by ESF program RDSES and by Austrian Science Fund (FWF) P15577.

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As an introductory example, consider the following random process: at the beginning, all lamps are switched off. A "lamplighter" person starts a random walk at a vertex of $\mathbb{T}$. With probability θ, she chooses to move, that is, she makes a step to a randomly selected neighbouring vertex (without changing the lamps). With probability $1-\theta$, she chooses to "switch", that is, she randomly modifies the state of the lamp where she stands (and does not move). At each step we observe the actual position in the tree and the configuration of the lamps that are switched on.

A configuration (finite or infinite) is a function $\eta\colon \mathbb{T} \to \mathbb{Y}$. We write $\widehat{\mathcal{C}} = \{\zeta\colon \mathbb{T} \to \mathbb{Y}\} = \mathbb{Y}^{\mathbb{T}}$ for the set of all configurations, and define $\eta \triangle \zeta$ to be the subset of $\mathbb{T}$ where the two configurations η and ζ differ. The configuration $\mathit{0}$ with $\mathit{0}(x) = 0$ for all x corresponds to "all lamps switched off". A configuration η is said to have finite support if $\mathsf{supp}(\eta) = \eta \triangle \mathit{0}$ is a finite set, and we define $\mathcal{C} \subset \widehat{\mathcal{C}}$ as the set of all finitely supported configurations.

Thus, our process evolves on the state space $\mathbb{Y} \wr \mathbb{T} = \mathcal{C} \times \mathbb{T}$, consisting of pairs (η, x), where $x \in \mathbb{T}$ stands for a position of the lamplighter and η for a configuration of lamps.

Let $\mathcal{F}$ be a finitely generated group which acts simply transitively on $\mathbb{T}$ by graph automorphisms. We may view $\mathbb{T}$ as a Cayley graph of $\mathcal{F}$ being a free product of two-element and infinite cyclic groups. Selecting a base point o of the tree, the elements in $x \in \mathcal{F}$ are identified with vertices of $\mathbb{T}$ via the one-to-one correspondence $x \leftrightarrow xo$. The notations x^{-1} and xx' refer to the operations in the group.

Let $\mathcal{L}$ be a finite group acting simply transitively on $\mathbb{Y}$. We identify the underlying set $\mathbb{Y}$ with $\mathcal{L}$ such that 0 corresponds to the group identity in $\mathcal{L}$. Thus, $\mathcal{C}$ becomes a group with point-wise operation $(\eta \cdot \eta')(x) = \eta(x) \cdot \eta'(x)$ taken in $\mathcal{L}$ and unit element $\mathit{0}$.

We form the *wreath product* $\mathcal{G} = \mathcal{L} \wr \mathcal{F}$ in the following way. Every $x \in \mathcal{F}$ acts on $\mathcal{C}$ by the translation T_x defined by $(T_x\eta)(y) = \eta(x^{-1}y)$. The group structure of $\mathcal{G}$ is the one of the semi-direct product of the groups $\mathcal{C}$ and $\mathcal{F}$:

$$\mathcal{L} \wr \mathcal{F} = \mathcal{C} \rtimes \mathcal{F}, \quad \text{with} \quad (\eta, x)(\eta', x') = (\eta \cdot T_x\eta', xx'). \tag{1.1}$$

We may also equip $\mathbb{Y} \wr \mathbb{T}$ with a graph structure, where the neighbourhood relation is given by

$$(\eta, x) \sim (\eta', x') :\iff \begin{cases} x \sim x' \text{ in } \mathbb{T} \text{ and } \eta = \eta', & \text{or} \\ x = x' \text{ in } \mathbb{T} \text{ and } \eta \triangle \eta' = \{x\}. \end{cases} \tag{1.2}$$

Then $\mathbb{Y} \wr \mathbb{T}$ becomes a Cayley graph of $\mathcal{G}$. (It may appear redundant to distinguish between $\mathcal{G}$ and $\mathbb{Y} \wr \mathbb{T}$. However, it is important for our way of thinking to have both a group action and a geometric structure where our process evolves; the basic geometry is provided by the tree.)

Our random process described above is a Markov chain $Z_n = (Y_n, X_n)$ on $\mathbb{Y} \wr \mathbb{T}$, where X_n is the position and Y_n the configuration at time n. Its one-step transition probabilities $p\big((\eta, x), (\eta', x')\big) = \mathsf{Pr}[Z_{n+1} = (\eta', x') | Z_n = (\eta, x)]$ are given by

$$p\big((\eta, x), (\eta', x')\big) = \begin{cases} \theta/(q+1), & \text{if } x \sim x' \text{ and } \eta = \eta', \\ (1-\theta)/(r-1), & \text{if } x = x' \text{ and } \eta \triangle \eta' = \{x\} \\ 0 & \text{in all other cases.} \end{cases} \tag{1.3}$$

Thus, $p\big((\eta, x), (\eta', x')\big) = \mu\big((\eta, x)^{-1}(\eta', x')\big)$, where μ is a probability measure on the group $\mathcal{G} = \mathcal{L} \wr \mathcal{F}$. In particular, we can view Z_n as a random walk on that group. The random walk is transient, that is, with probability 1, it visits every finite subset of $\mathbb{Y} \wr \mathbb{T}$

only finitely many times. Thus, Z_n tends "to infinity", and the purpose of this note is to relate this property in a more detailed way with the underlying structure. Below we shall also consider more general random walks on $\mathcal{G}$.

We shall determine the *Poisson boundary* of a general class of lamplighter random walks on $\mathcal{L} \wr \mathcal{F}$ (resp. $\mathbb{Y} \wr \mathbb{T}$) that includes the basic example (1.3). This boundary can be defined in several equivalent ways, see Kaimanovich and Vershik [19] and Kaimanovich [17]. The Poisson boundary of a random walk on a group is a measure space, determined uniquely up to measure-theoretic isomorphism. One quick definition is to say that it is the space of ergodic components in the trajectory space of the random walk. Another approach is via bounded harmonic functions, see below. Here, we take a more topological viewpoint. We attach to $\mathbb{Y} \wr \mathbb{T}$ a natural boundary Π at infinity, defined purely in geometric terms, such that $(\mathbb{Y} \wr \mathbb{T}) \cup \Pi$ is a metrizable space (not necessarily compact or complete) on which the group $\mathcal{G} = \mathcal{L} \wr \mathcal{F}$ acts by homeomorphisms, and every point in Π is the accumulation point of a sequence in $\mathbb{Y} \wr \mathbb{T}$. We then show that in that topology, (Z_n) converges almost surely to a Π-valued random variable Z_∞. Let ν be the distribution of Z_∞, given that the initial position and configuration of the random walk at time $n = 0$ are o and the zero configuration *0*. The measure ν is often called the *harmonic measure* or *limit distribution*. The pair (Π, ν) provides a model for the behaviour at infinity (in time and space) of the random walk. We give a quite simple proof that this is indeed the Poisson boundary under rather general assumptions. This tells us that up to sets with measure 0, we have found the finest model for distinguishing limit points at infinity of the random walk. The (geometric) tool that we shall use for proving that (Π, ν) is the Poisson boundary is the *strip criterion* of Kaimanovich [17, Sect. 6], [20,Thm. 5.19].

In Sect. 4, we prove that the Dirichlet problem is solvable with respect to this natural geometric boundary. In the spirit of this article, the focus is not on proving the most general result possible.

Let us conclude the introduction with a brief and incomplete overview of recent work on lamplighter random walks and identifications of the Poisson boundary.

The above-mentioned paper of Kaimanovich and Vershik [19] may serve as a major source for the earlier literature, different equivalent definitions of the Poisson boundary, and a wealth of results and methods. There one also finds the first interesting results on the Poisson boundary of lamplighter random walks, namely on $(\mathbb{Z}/2) \wr \mathbb{Z}^d$. The techniques were refined in the subsequent body of work of Kaimanovich, see e.g., [16,17,20] and the references therein. Within the study of random walks on groups, wreath products (lamplighter walks) have been the object of intensive studies in the last decade. Wreath products exhibit interesting types of asymptotic behaviour of n-step return probabilities, see Pittet and Saloff-Coste [25,26], Revelle [28]. The rate of escape has been studied by Lyons, Pemantle and Peres [23], Erschler [10,11], Revelle [27] and, for simple lamplighter walks on trees, by Gilch [14]. For the spectrum of transition operators, see Grigorchuk and Żuk [15], Dicks and Schick [9] and Bartholdi and Woess [1]. The positive harmonic functions for a class of random walks on $(\mathbb{Z}/q) \wr \mathbb{Z}$ have been determined by Woess [30] and Brofferio and Woess [4], who have also determined the full Martin compactification in a "nearest neighbour" case [3]. For the Poisson boundary and bounded harmonic functions for various types of lamplighter random walks, besides [19] and [16], see once more the impressive work of Erschler [12].

Concerning the Dirichlet problem in the discrete setting we refer to Woess [29,Chapter IV] for more information.

2 Convergence to the geometric boundary

The lamplighter graph $\mathbb{Y} \wr \mathbb{T}$ with neighbourhood defined by (1.2) is far from being tree-like (it has one end). It is easy (and well known) to describe the graph metric. A shortest path from (η, x) to (η', x') in the lamplighter graph must be such that the lamplighter person starts at x, walks along the tree and visits every $y \in \eta' \triangle \eta$, where he has to switch the lamp from state $\eta(y)$ to state $\eta'(y)$, and at the end, she must reach x'. Thus, $d\big((\eta, x), (\eta', x')\big) = \ell + |\eta' \triangle \eta|$, where ℓ is the smallest length of a "travelling salesman" tour (walk) from x to x' that visits each element of $\eta' \triangle \eta$. This description of the metric does not require that the base graph is a tree, however note that in a tree, the "travelling salesman" algorithm required for finding such a tour is easy to implement, see e.g., Parry [24] and Ceccherini-Silberstein and Woess [6, Example 2].

Since the group $\mathcal{G} = \mathcal{L} \wr \mathcal{F}$ is non-amenable, the random walks $Z_n = (Y_n, X_n)$ that we consider here are all *transient*, that is, $d(Z_n, Z_0) \to \infty$ almost surely, see [29, Thm. 3.24] for this result going back to Kesten [21,22]. The main question considered here is whether we can describe in a more detailed, geometric way how (Z_n) behaves at infinity.

For this purpose, we first briefly recall the *end compactification* of $\mathbb{T}$, whose graph metric we denote also by $d(\cdot,\cdot)$. A *geodesic path*, resp. *geodesic ray*, resp. *infinite geodesic* in $\mathbb{T}$ is a finite, resp. one-sided infinite, resp. doubly infinite sequence (x_n) of vertices of $\mathbb{T}$ such that $d(x_i, x_j) = |i - j|$ for all i, j. Two rays are *equivalent* if, as sets, their symmetric difference is finite. An *end* of $\mathbb{T}$ is an equivalence class of rays. The space of ends is denoted $\partial\mathbb{T}$, and we write $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. For all $w, z \in \widehat{\mathbb{T}}$ $(w \neq z)$ there is a unique geodesic $\overline{w\,z}$ that connects the two. In particular, if $x \in \mathbb{T}$ and $\mathfrak{u} \in \partial\mathbb{T}$ then $\overline{x\,\mathfrak{u}}$ is the ray that starts at x and represents $\mathfrak{u}$. Furthermore, if $\mathfrak{u}, \mathfrak{v} \in \partial\mathbb{T}$ $(\mathfrak{u} \neq \mathfrak{v})$ then $\overline{\mathfrak{u}\,\mathfrak{v}}$ is the infinite geodesic whose two halves (split at any vertex) are rays that represent $\mathfrak{u}$ and $\mathfrak{v}$, respectively. If $w, z \in \widehat{\mathbb{T}}$, then their *confluent* $c = w \wedge z$ with respect to the root vertex o ($\equiv$ the identity element of $\mathcal{F}$) is defined by $\overline{o\,w} \cap \overline{o\,z} = \overline{o\,c}$. Let $(w|z) = d(o, c)$, which is finite unless $w = z \in \partial\mathbb{T}$. We can define a metric ρ on $\widehat{\mathbb{T}}$ by

$$\rho(w, z) = \begin{cases} q^{-(w|z)}, & \text{if } z \neq w, \\ 0, & \text{if } z = w. \end{cases} \tag{2.1}$$

This makes $\widehat{\mathbb{T}}$ a a compact ultrametric space with $\mathbb{T}$ as a dense, discrete subset. Each isometry $g \in \mathsf{Aut}(\mathbb{T})$ extends to a homeomorphism of $\widehat{\mathbb{T}}$.

For $\mathfrak{u} \in \partial\mathbb{T}$ and $y \in \mathbb{T}$, we define $\mathbb{T}_y(\mathfrak{u})$ as the subtree (branch) of $\mathbb{T}$ which is the component of $\mathfrak{u}$ in $\mathbb{T} \setminus \{y\}$ (that is, every ray that represents $\mathfrak{u}$ has all but finitely many vertices in this component of $\mathbb{T} \setminus \{y\}$).

The natural compactification of $\mathcal{C}$ in the topology of point-wise convergence is the set $\widehat{\mathcal{C}} = \{\zeta : \mathbb{T} \to \mathbb{Y}\}$ of *all*, finitely or infinitely supported, configurations. Since the vertex set of the lamplighter graph is $\mathcal{C} \times \mathbb{T}$, the space $\widehat{\mathbb{Y} \wr \mathbb{T}} = \widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ is a natural compactification, and $\partial(\mathbb{Y} \wr \mathbb{T}) = (\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}) \setminus (\mathcal{C} \times \mathbb{T})$ is a natural "geometric" boundary at infinity of the lamplighter graph. We shall see that this boundary contains many points where the random walk (Z_n) does not accumulate. We define

$$\begin{aligned} \Pi &= \bigcup_{\mathfrak{u} \in \partial\mathbb{T}} \mathcal{C}_{\mathfrak{u}} \times \{\mathfrak{u}\}, \quad \text{where} \\ \mathcal{C}_{\mathfrak{u}} &= \{\zeta \in \widehat{\mathcal{C}} : |\,\mathsf{supp}(\zeta) \setminus \mathbb{T}_y(\mathfrak{u})| < \infty \ \text{ for all } \ y \in \mathbb{T}\}. \end{aligned} \tag{2.2}$$

Thus, a configuration $\zeta \in \widehat{\mathcal{C}}$ is in $\mathcal{C}_{\mathfrak{u}}$ if and only if $\mathsf{supp}(\zeta)$ is finite or else accumulates only at $\mathfrak{u}$.

Since $\mathcal{C}_{\mathfrak{u}}$ is dense in $\widehat{\mathcal{C}}$, the closure of Π is the part $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ of $\partial(\mathbb{Y} \wr \mathbb{T})$. The action of the group $\mathcal{L} \wr \mathcal{F}$ on $\mathbb{Y} \wr \mathbb{T}$ by multiplication from the left extends by homeomorphisms to $\widehat{\mathbb{Y} \wr \mathbb{T}}$ and leaves the Borel subset Π invariant. Indeed, if $g = (\eta, x) \in \mathcal{G} = \mathcal{L} \wr \mathcal{F}$ and $\beta = (\zeta, \mathfrak{u}) \in \widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ then, precisely as in (1.1),

$$g\beta = (\eta, x)(\zeta, \mathfrak{u}) = (\eta \cdot T_x\zeta, x\mathfrak{u})\,. \tag{2.3}$$

(The product of configurations is point-wise in $\mathcal{L}$.) If in addition $\zeta \in \mathcal{C}_{\mathfrak{u}}$, where $\mathfrak{u} \in \partial\mathbb{T}$, then $\eta \cdot T_x\zeta \in \mathcal{C}_{x\mathfrak{u}}$, since multiplying with η modifies $T_x\zeta$ only in finitely many points.

For the basic example (1.3), it is quite clear that Z_n converges almost surely to a random element $Z_\infty = (Y_\infty, X_\infty) \in \Pi$. Indeed, the $\mathbb{T}$-coordinate X_n of Z_n is the random walk on $\mathbb{T}$ with transition probabilities $\widetilde{p}(x, x') = \theta/(q+1)$, if $x' \sim x$, and $\widetilde{p}(x, x) = 1 - \theta$ (and $\widetilde{p}(x, y) = 0$ if $d(y, x) \geq 2$). An elementary argument using only transience yields that X_n converges to a random element $X_\infty \in \partial\mathbb{T}$. Also, only the states of those lamps can be modified which are visited by (X_n), and by transience, each vertex is visited finitely often: after the last visit at a given vertex, the state of lamp sitting there remains unchanged. Therefore Y_n must converge point-wise to a random configuration Y_∞ which can accumulate at no point besides X_∞.

We shall prove an analogous result for a much larger class of random walks on $\mathcal{G} = \mathcal{L} \wr \mathcal{F}$, or equivalently, on $\mathbb{Y} \wr \mathbb{T}$. They are specified by a probability measure μ on the group $\mathcal{G}$, and we suppose that $\mathsf{supp}(\mu)$ generates $\mathcal{G}$. We can model the random walk on the probability space (Ω, Pr), where $\Omega = \mathcal{G}^{\mathbb{N}}$ and $\mathsf{Pr} = \mu^{\mathbb{N}}$, with $\mathbb{N} = \{1, 2, 3, \ldots\}$. The n-th projections $\mathbf{g}_n = (\boldsymbol{\eta}_n, \mathbf{x}_n) : \Omega \to \mathcal{G}$ are a sequence of independent, $\mathcal{G}$-valued random variables with common distribution μ. If $g_0 = (\eta_0, x_0) \in \mathcal{G}$ then the sequence of random variables

$$Z_n = (Y_n, X_n) = g_0\mathbf{g}_1 \cdots \mathbf{g}_n\,,\ n \geq 0$$

is the *right random walk* on $\mathcal{G}$ with law μ and starting point g_0. Its one-step transition probabilities are given by $p(g, g') = \mu(g^{-1}g')$, where $g = (\eta, x)$ and $g' = (\eta', x') \in \mathcal{G}$. Note that

$$X_n = x_0\mathbf{x}_1 \cdots \mathbf{x}_n \quad \text{and} \quad Y_n = \eta_0 \cdot (T_{X_0}\boldsymbol{\eta}_1) \cdots (T_{X_{n-1}}\boldsymbol{\eta}_n). \tag{2.4}$$

In particular, (X_n) is the right random walk on the "base" group $\mathcal{F}$ with starting point x_0 whose law is the projection $\widetilde{\mu}$ of μ ,

$$\widetilde{\mu}(x) = \sum_{\eta \in \mathcal{C}} \mu\big((\eta, x)\big). \tag{2.5}$$

Before stating a result on convergence of (Z_n) to Π, we have to specify additional properties. We say that the lamplighter *operates at bounded range,* if

$$R(\mu) = \max\big\{\min\{d(y, o), d(y, x)\} : \mu\big((\eta, x)\big) > 0\,,\ y \in \mathsf{supp}(\eta)\big\} < \infty\,. \tag{2.6}$$

This means that when the lamplighter person steps from x to x' in $\mathbb{T}$ then while doing so, she can modify the current configuration of lamps only at points which are at bounded distance from x or x'. (Note that here we do not require that μ itself has finite support.)

The law μ of the random walk is said to have *finite first moment,* if

$$\sum_{(\eta,x)\in\mathcal{G}} d\big((0,o),(\eta,x)\big)\,\mu\big((\eta,x)\big) < \infty. \tag{2.7}$$

In this case, it is a well known consequence of Kingman's subadditive ergodic theorem (see e.g., Derriennic [8]) that there are finite constants $\ell(P)$ and $m(P)$ such that

$$\frac{d(Z_n,Z_0)}{n} \to \ell(P) \quad \text{and} \quad \frac{d(X_n,X_0)}{n} \to m(P) \quad \text{almost surely.}$$

It is clear that $\ell(P) \geq m(P)$. Recent [14] and ongoing work by GILCH suggests that $\ell(P) > m(P)$ strictly.

Lemma 2.1 *If P has finite first moment then $m(P) > 0$.*

Proof This follows from the fact that $\mathcal{F}$ is non-amenable, see [29, Thm. 8.14]. Again, for random walks with finite range on a group, this goes back to Kesten [21,22].

Theorem 2.2 *Let $Z_n = (Y_n, X_n)$ be a random walk with law μ on $\mathcal{G} = \mathcal{L} \wr \mathcal{F} \equiv \mathbb{Y} \wr \mathbb{T}$, such that* $\mathsf{supp}(\mu)$ *generates $\mathcal{G}$.*

If the lamplighter operates at bounded range, or if μ has finite first moment, then there is a Π-valued random variable $Z_\infty = (Y_\infty, X_\infty)$ such that $Z_n \to Z_\infty$ in the topology of $\widehat{\mathbb{Y} \wr \mathbb{T}}$, almost surely for every starting point $g_0 = (\eta_0, x_0)$.

Furthermore, the distribution of X_∞ is a continuous measure on $\partial\mathbb{T}$ (it carries no point mass), and consequently the same is true for the distribution of Z_∞ on Π.

Proof We may suppose without loss of generality that $g_0 = id$, where $id = (0, o)$ is the identity of $\mathcal{L} \wr \mathcal{F}$.

The law $\widetilde{\mu}$ of the projected random walk (X_n) on $\mathcal{F}$ is such that its support generates $\mathcal{F}$. Cartwright and Soardi [5] have shown that without any moment assumption, such a random walk on $\mathbb{T} \equiv \mathcal{F}$ converges almost surely to a random end, that is, a $\partial\mathbb{T}$-valued random variable X_∞.

Now suppose first that the lamplighter operates at bounded range. Let (y_n) be an unbounded sequence in $\mathbb{T}$ with $y_n \in \mathsf{supp}(Y_n)$, i.e., y_n is a vertex where the lamp is "on" at time n. Then we see from (2.4) that y_n must be at bounded distance from the initial trajectory $\{X_0, X_1, \ldots, X_n\}$. Therefore, we must have $y_n \to X_\infty$. Indeed, one sees immediately from the definition (2.1) of the metric ρ that the end compactification $\widehat{\mathbb{T}}$ has the following property: if (x_n) and (y_n) are two sequences in $\mathbb{T}$ such that $x_n \to \mathfrak{u} \in \partial\mathbb{T}$ and $\sup_n d(y_n, x_n) < \infty$, then also $y_n \to \mathfrak{u}$.

Next, assume that the random walk has finite first moment. Define the integer-valued random variables $M_n = \max\{d(y,o) : y \in \mathsf{supp}(\boldsymbol{\eta}_n)\}$. They are independent and identically distributed and also have finite first moment. Therefore $M_n/n \to 0$ almost surely. This implies that the following holds with probability 1.

$$\text{If } (y_n) \text{ is a sequence in } \mathbb{T} \text{ such that } y_n \in \mathsf{supp}\big(T_{X_{n-1}}\boldsymbol{\eta}_n\big) \text{ for each } n, \text{ then } d(y_n, X_{n-1})/n \to 0 \quad \text{as} \quad n \to \infty\,. \tag{2.8}$$

Recall that almost surely $X_n \to X_\infty \in \partial\mathbb{T}$ and $d(X_n, o)/n \to m(P)$, with $m(P) > 0$ by Lemma 2.1. Thus, we get for the confluents $c_n = y_n \wedge X_{n-1}$ that $d(c_n, X_{n-1})/n \leq d(y_n, X_{n-1})/n \to 0$, whence $d(c_n, o)/n \to m(P)$. Therefore, for the metric (2.1) of $\widehat{\mathbb{T}}$, we obtain $\rho(y_n, X_{n-1}) \to 0$. Consequently $y_n \to X_\infty$.

Now observe that by (2.4),

$$\mathsf{supp}(Y_n) \subset \bigcup_{k=1}^{n} \mathsf{supp}\big(T_{X_{k-1}}\eta_k\big)\,,$$

which is a union of finite sets. Thus, almost surely by the above, if (y_n) is an unbounded sequence in $\mathbb{T}$ with $y_n \in \mathsf{supp}(Y_n)$, it must converge to X_∞.

Finally, from [5] it is also known (without moment hypothesis) that the distribution of X_∞ is a continuous measure on $\partial\mathbb{T}$. □

3 The Poisson boundary

Under the assumptions of Theorem 2.2, we can define the limit distribution ν of the random walk. This is the probability measure defined for Borel sets $U \subset \Pi$ by

$$\nu(U) = \mathsf{Pr}[Z_\infty \in U \mid Z_0 = id] \tag{3.1}$$

Recall the natural action (2.3) of $\mathcal{G}$ on Π. If $g_0 \in \mathcal{G}$, then

$$\mathsf{Pr}[Z_\infty \in U \mid Z_0 = g] = \nu(g^{-1}U)\,.$$

This implies that ν satisfies the convolution equation $\mu * \nu = \nu$. In particular, the measure space (Π, ν) is a *boundary* of the random walk on $\mathcal{G}$ in the sense of Furstenberg [13].

A general boundary is a probability space (B, λ) such that $\mathcal{G}$ acts on B by measurable bijections and $\mu * \lambda = \lambda$. As is explained in [13], one can then typically (i.e., when B carries a topology and $\mathcal{G}$ acts continuously) construct a topology on $\mathcal{G} \cup \mathsf{B}$ such that (Z_n) converges almost surely to a B-valued random variable whose distribution is λ, given that $Z_0 = id$. This means that (B, λ) is a model for describing in detail how (Z_n) tends to infinity, that is, for distinguishing limit points of (Z_n) as $n \to \infty$. When comparing two boundaries, this has of course to be done modulo sets with measure 0. For example, in the case of the boundary (Π, ν) it may appear more natural to consider ν as a measure on the closure $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$ of Π in $\widehat{\mathbb{Y} \wr \mathbb{T}}$, with the same definition as in (3.1) but Borel sets $U \subset \widehat{\mathcal{C}} \times \partial\mathbb{T}$. Since ν charges only the (dense) subset Π, the measure spaces (Π, ν) and $(\widehat{\mathcal{C}} \times \partial\mathbb{T}, \nu)$ are isomorphic.

In this spirit, our question is now whether the boundary (Π, ν) is *maximal.* This means that for every other boundary (B, λ), up to sets with measure 0, there is a $\mathcal{G}$-equivariant measure-preserving surjection of (Π, ν) onto (B, λ). A more heuristic interpretation of maximality is that (Π, ν) is the finest model for distinguishing limit points at infinity of the random walk. Existence and uniqueness of the maximal boundary is a general fact [19], and it is called the *Poisson boundary*.

Kaimanovich [17, Thm. in Sect. 6.5 on p. 677], [20,Thm. 5.19] has provided a useful geometric criterion for checking maximality. Consider the "reflected" right random walk on $\mathcal{G}$

$$\check{Z}_n = g_0\mathbf{g}_1^{-1}\cdots\mathbf{g}_n^{-1}$$

starting at g_0 (we shall only consider $g_0 = id$). Its law is the probability measure $\check{\mu}$ on $\mathcal{G}$, where $\check{\mu}(g) = \mu(g^{-1})$.

Proposition 3.1 (Kaimanovich) *Let μ be a probability measure on $\mathcal{G}$ with a finite first moment, and let (B,λ) and $(\check{\mathsf{B}},\check{\lambda})$ be a μ- and a $\check{\mu}$-boundary, respectively. Suppose that there is a measurable $\mathcal{G}$-equivariant map S assigning to $(\lambda\times\check{\lambda})$-almost every pair of points $(\beta,\check{\beta})\in\mathsf{B}\times\check{\mathsf{B}}$ a non-empty "strip" $S(\beta,\check{\beta})\subset\mathcal{G}$ such that for the ball $B(id,n)$ of radius n in the metric of $\mathcal{G}$,*

$$\frac{1}{n}\log\big|S(\beta,\check{\beta})\cap B(id,n)\big|\to 0\quad \textit{as}\quad n\to\infty$$

for $(\lambda\times\check{\lambda})$-almost every $(\beta,\check{\beta})\in\Pi\times\check{\Pi}$, then (B,λ) and $(\check{\mathsf{B}},\check{\lambda})$ are the Poisson boundaries of the random walks with law μ and $\check{\mu}$, respectively.

Theorem 3.2 *Let $Z_n=(Y_n,X_n)$ be a random walk with law μ on $\mathcal{G}=\mathcal{L}\wr\mathcal{F}\equiv\mathbb{Y}\wr\mathbb{T}$, such that μ has finite first moment and $\mathsf{supp}(\mu)$ generates $\mathcal{G}$. If Π is defined as in (2.2) and ν is the limit distribution on Π of (Z_n) starting at id, then (Π,ν) is the Poisson boundary of the random walk.*

Proof By Theorem 2.2, each of the random walks (Z_n) and $(\check{Z}_n)$ starting at *id* converges almost surely to a Π-valued random variable. Let ν and $\check{\nu}$ be their respective limit distributions. Then the spaces (Π,ν) and $(\Pi,\check{\nu})$ are boundaries of the respective random walks. If $\beta=(\zeta,\mathfrak{u})\in\Pi$, then let $\mathfrak{u}(\beta)=\mathfrak{u}$. Recall the definition of the component $\mathbb{T}_y(\mathfrak{u})$ of $\mathfrak{u}$ in $\mathbb{T}\setminus\{y\}$, where $y\in\mathbb{T}$ and $\mathfrak{u}\in\partial\mathbb{T}$.

By continuity of ν and $\check{\nu}$ (Theorem 2.2), we have

$$\nu\times\check{\nu}\big(\{(\beta,\check{\beta})\in\Pi\times\Pi\colon \mathfrak{u}(\beta)=\mathfrak{u}(\check{\beta})\}\big)=0.$$

Therefore, we only need to construct the strips $S(\beta,\check{\beta})$ when $\mathfrak{u}(\beta)\neq\mathfrak{u}(\check{\beta})$. Thus, let $\beta=(\zeta,\mathfrak{u}),\check{\beta}=(\check{\zeta},\mathfrak{v})\in\Pi$ and $\mathfrak{u}\neq\mathfrak{v}$. For any vertex y on the (two-sided infinite) geodesic $\overline{\mathfrak{u}\,\mathfrak{v}}$, let $\eta_y(\beta,\check{\beta})$ be the configuration which coincides with ζ on $\mathbb{T}_y(\mathfrak{v})$ and with $\check{\zeta}$ on $\mathbb{T}\setminus\mathbb{T}_y(\mathfrak{v})$. This configuration has finite support, since $\mathsf{supp}(\zeta)$ can only accumulate at $\mathfrak{u}$ and $\mathsf{supp}(\check{\eta})$ can only accumulate at $\mathfrak{v}$, so that $\mathsf{supp}(\zeta)\cap\mathbb{T}_y(\mathfrak{v})$ and $\mathsf{supp}(\zeta')\cap\big(\mathbb{T}\setminus\mathbb{T}_y(\mathfrak{v})\big)$ are finite. Then define

$$S(\beta,\check{\beta})=\big\{\big(\eta_y(\beta,\check{\beta}),y\big)\colon y\in\overline{\mathfrak{u}\,\mathfrak{v}}\big\}.$$

This is a subset of $\mathcal{G}$. We check that the map $(\beta,\check{\beta})\mapsto S(\beta,\check{\beta})$ is $\mathcal{G}$-equivariant: let $g=(\eta,x)\in\mathcal{G}$. We have to show that

$$g\,S(\beta,\check{\beta})=S(g\beta,g\check{\beta}).\tag{3.2}$$

Now, if $y\in\overline{\mathfrak{u}\,\mathfrak{v}}$ then $xy\in\overline{(x\mathfrak{u})\,(x\mathfrak{v})}$. Also, $T_x\eta_y(\beta,\check{\beta})=\eta_{xy}(\beta',\check{\beta}')$, where $\beta'=(T_x\zeta,x\mathfrak{u})$ and $\check{\beta}'=(T_x\check{\zeta},x\mathfrak{v})$. Therefore

$$\eta\cdot\eta_{xy}(\beta',\check{\beta}')=\eta_{xy}(\beta'',\check{\beta}''),\quad\text{where}$$
$$\beta''=(\eta T_x\zeta,x\mathfrak{u})=g\beta\quad\text{and}\quad\check{\beta}''=(\eta T_x\check{\zeta},x\mathfrak{v})=g\check{\beta}.$$

Thus, for $y\in\overline{\mathfrak{u}\,\mathfrak{v}}$,

$$g\big(\eta_y(\beta,\check{\beta}),y\big)=\big(\eta\cdot T_x\eta_y(\beta,\check{\beta}),xy\big)=\big(\eta_{xy}(g\beta,g\check{\beta}),xy\big)$$

This proves (3.2).

Finally, if $(\eta,y)\in S(\beta,\check{\beta})$ and $d\big((0,o),(\eta,y)\big)\le n$ then $d(o,y)\le n$. Since

$$|\{y\in\overline{\mathfrak{u}\,\mathfrak{v}}:d(o,y)\le n\}|\le 2n,$$

we see that all conditions of Proposition 3.1 are satisfied. □

4 The Dirichlet problem at infinity

In this section we shall assume in addition that our random walk on $\mathcal{G}$ is *irreducible* in the sense that its law μ is such that $\mathsf{supp}(\mu)$ generates $\mathcal{G}$ as *semi*group. Equivalently, this means that for every pair of elements $g = (\eta, x)$, $g' = (\eta', x') \in \mathcal{G}$ the probability that the random walk starting at g ever visits g' is > 0.

Also, it will be convenient to consider the limit distribution ν of (3.1) as a Borel measure on the compact set $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$. The irreducibility hypothesis implies that

$$\mathsf{supp}(\nu) = \overline{\Pi}\,, \tag{4.1}$$

that is, the whole of $\overline{\Pi}$ is active (as we shall see below).

With respect to our random walk with transition probabilities $p(g,g') = \mu(g^{-1}g')$ on $\mathcal{G}$, a function $h : \mathcal{G} \to \mathbb{R}$ is called *harmonic,* if it satisfies the weighted mean value property

$$h(g) = \sum_{g'} p(g,g')h(g') \quad \text{for all } g \in \mathcal{G}\,.$$

For $g \in \mathcal{G}$, we define ν_g by $\nu_g(U) = \nu(g^{-1}U)$, where U runs through Borel subsets of $\overline{\Pi}$. It is a basic feature of the Poisson boundary that every bounded harmonic function h on $\mathcal{G}$ has a unique integral representation

$$h(g) = \int_{\widehat{\mathcal{C}} \times \partial\mathbb{T}} f \; d\nu_g\,, \tag{4.2}$$

where $f \in L^\infty(\nu)$, see e.g., [19].

Conversely, once we have convergence of the random walk to the boundary, any function $f \in L^\infty(\nu)$ gives rise to a harmonic function via the integral formula (4.2). In fact, for this we do not need to know that $\overline{\Pi}$ is the Poisson boundary; the only point is that otherwise we will not get *all* bounded harmonic functions via (4.2).

Related to existence of the limit measure on the boundary, there is the question whether the *Dirichlet problem at infinity* is solvable. In our case it reads as follows:

Does every continuous function on $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$ extend continuously to a function on $\mathcal{G} \cup \overline{\Pi}$ which is harmonic on $\mathcal{G}$?

If the answer is positive, then that harmonic function must be given by (4.2), and we would like that whenever f is continuous it should hold that $\lim_{g \to \beta} h(g) = f(\beta)$ for every $\beta \in \overline{\Pi}$. We then say that the *Dirichlet problem at infinity is solvable* for continuous functions on $\overline{\Pi}$.

We note that usually, the Dirichlet problem at infinity refers to the harmonic extension of any continuous function that is given on the *whole* boundary in a (suitable) compactification of the state space, compare with [29, Sect. 20]. In our case, the whole boundary is the set $(\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}) \setminus (\mathcal{C} \times \mathbb{T})$ which contains $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ as a proper, compact subset. However, the complement of $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ is not charged by ν, so that boundary data given on that complement have no effect on the harmonic function h of (4.2), and we cannot expect continuity at those points. Therefore we have to restrict to continuous functions on $\widehat{\mathcal{C}} \times \partial\mathbb{T}$.

The *Green kernel* of the projected random walk (X_n) on $\mathbb{T} \equiv \mathcal{F}$ with law $\widetilde{\mu}$ as in (2.5) is

$$\widetilde{G}(x,y) = \sum_{n=0}^{\infty} \widetilde{p}^{(n)}(x,y) = \sum_{n=0}^{\infty} \widetilde{\mu}^{(n)}(x^{-1}y)\,, \quad x,y \in \mathbb{T}\,,$$

where $\widetilde{p}^{(n)}(\cdot,\cdot)$ denotes n-step transition probabilities and $\widetilde{\mu}^{(n)}$ is the n-th convolution power of $\widetilde{\mu}$. By irreducibility and transience, $0 < \widetilde{G}(x,y) < \infty$. This is the expected number of times that (X_n) visits y, given that $X_0 = 0$.

Lemma 4.1 *If $\widetilde{\mu}$ is irreducible, then the Green kernel vanishes at infinity, that is,*

$$\lim_{d(x,y)\to\infty} \widetilde{G}(x,y) = 0\,.$$

Proof The group $\mathcal{F}$ is non-amenable. Let $S = \mathsf{supp}(\widetilde{\mu})$. Since S generates $\mathcal{F}$ as a semigroup, it is a well-known exercise to show that the subgroup of $\mathcal{F}$ generated by $S \cdot S^{-1}$ is a finite-index normal subgroup, whence also non-amenable. Therefore one can apply Théorème 2 of Derriennic and Guivarc'h [7] and/or Théorème 2 of Berg and Christensen [2] to obtain that the measure $\sum_{n=0}^{\infty} \widetilde{\mu}^{(n)}$ defines a bounded convolution operator on $\ell^2(\mathcal{F})$. It follows that the Green kernel vanishes at infinity. This may also be deduced by applying the main theorem of [22] to $\widetilde{\mu} * \check{\widetilde{\mu}}$. □

Below, we shall need the quantity

$$\widetilde{F}(x,y) = \widetilde{G}(x,y)/\widetilde{G}(y,y) = \mathsf{Pr}[\exists\, n \geq 0 : X_n = y \mid X_0 = 0]\,,$$

which also vanishes at infinity.

Theorem 4.2 *If μ is irreducible and the lamplighter operates at bounded range, then the Dirichlet problem at infinity for continuous functions on $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$ is solvable.*

Proof The typical "probabilistic" method for proving this (see e.g., [29, Sect. 20]) is to show the following: (i) the random walk Z_n converges to the boundary, and (ii) for the corresponding harmonic measure class $\{\nu_g : g = (\eta, x) \in \mathcal{G}\}$, one has

$$\lim_{g\to\beta} \nu_g = \delta_\beta \quad \text{weakly for every } \beta = (\zeta, \mathfrak{u}) \in \widehat{\mathcal{C}} \times \partial\mathbb{T}\,. \tag{4.3}$$

Point (i) is affirmed by Theorem 2.2. For proving (ii), it will be convenient to consider ν_g as a measure on $\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ which charges only the set $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$. Thus, we show that for any neigbourhood U in $\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ of $\beta = (\zeta, \mathfrak{u}) \in \overline{\Pi}$, we have

$$\lim_{g\to\beta} \nu_g(U^c) = 0\,,$$

where $U^c = (\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}) \setminus U$. Here, it is sufficient to take U in a suitable neighbourhood basis of β. Such a basis is obtained as follows: take a vertex $y \in \mathbb{T}$ and let $\widehat{\mathbb{T}}_y(\mathfrak{u})$ be the closure in $\widehat{\mathbb{T}}$ of the subtree $\mathbb{T}_y(\mathfrak{u})$ (the component of $\mathfrak{u}$ in $\mathbb{T} \setminus \{y\}$, see Sect. 2). The family of all $\widehat{T}_y(\mathfrak{u})$, $y \in \mathbb{T}$, is a neighbourhood basis of $\mathfrak{u}$ in $\widehat{\mathbb{T}}$. Also, the family of all sets $\widehat{C}_A(\zeta) = \{\zeta' \in \widehat{\mathcal{C}} : \zeta' = \zeta \text{ on } A\}$, where $A \subset \mathbb{T}$ is finite, is a neighbourhood basis of ζ in $\widehat{\mathcal{C}}$ for the topology of point-wise convergence of configurations. Thus,

$$\{U_{y,A}(\beta) = \widehat{T}_y(\mathfrak{u}) \times \widehat{C}_A(\zeta) : y \in \overline{o\,\mathfrak{u}}\,,\ A \subset \mathbb{T} \text{ finite}\},$$

is a neighbourhood basis of $\beta = (\zeta, \mathfrak{u})$. Consider $U = U_{y,A}(\beta)$. Then

$$U^c \subset V_1 \cup V_2\,, \quad \text{where} \quad V_1 = \left(\widehat{\mathbb{T}} \setminus \widehat{\mathbb{T}}_y(\mathfrak{u})\right) \times \widehat{\mathcal{C}} \quad \text{and} \quad V_2 = \widehat{\mathbb{T}}_y(\mathfrak{u}) \times \left(\widehat{\mathcal{C}} \setminus \widehat{C}_A(\zeta)\right).$$

We now prove that $\nu_g(V_i) \to 0$ for $i = 1, 2$ as $g = (\eta, x) \to \beta$.

Regarding V_1, we first remark that the Dirichlet problem at infinity for continuous functions on $\partial\mathbb{T}$ is solvable for the random walk (X_n) with law $\widetilde{\mu}$ on $\mathcal{F} \equiv \mathbb{T}$, see [5] and [29, Cor. 21.12]. Therefore

$$\lim_{g\to\beta} \nu_g(V_1) = \lim_{x\to\mathfrak{u}} \mathsf{Pr}_x[X_\infty \in \partial\mathbb{T} \setminus \widehat{T}_y(\mathfrak{u})] = 0\,.$$

Regarding V_2, let $R = R(\mu)$ be the bound on the range of (2.6). Set $A^R = \{v \in \mathbb{T} : d(v, A) \le R\}$. Suppose $\eta = \eta(g)$ is sufficiently close to the limit ζ so that $\eta(v) = \zeta(v)$ for every $v \in A$. If the random walk $Z_n = (Y_n, X_n)$ starting at g converges to a limit point in V_2 then (X_n) must visit A^R in order to modify the states of the lamps at the points in A. Therefore

$$\nu_g(V_2) \le \sum_{v\in A^R} \widetilde{F}(x, v)\,,$$

which tends to zero when $x \to \mathfrak{u}$ by Lemma 4.1. This concludes the proof.

5 Final remarks

Theorem 3.2 is yet one more application of the very useful strip criterion of Kaimanovich who, in private communication, has informed us that in the unpublished paper [18] he uses a method in a somewhat similar spirit to the proof of Theorem 3.2 to describe the Poisson boundary for random walks on $\mathcal{L} \wr \mathbb{Z}^d$, where the projection of the random walk onto the integer grid $\mathbb{Z}^d$ has non-zero drift.

Regarding the Dirichlet problem at infinity, we repeat here that this type of question can be asked whenever one has a compactification of the state space in whose topology the random walk converges almost surely to the boundary at infinity. Solvability of the Dirichlet problem for continuous functions on the boundary (or rather its active part, i.e., the support of the limit measure) is by no means the same as having determined the Poisson (or even Martin) boundary, as one finds erroneously stated every now and then in published work. For example, in our case, we know that $(\overline{\Pi}, \nu)$ is the Poisson boundary when μ has finite first moment, while we proved that the corresponding Dirichlet problem is solvable when the lamplighter operates at finite range, but μ need not have finite first moment for that. Thus, we have given positive answers to both questions simultaneously only when μ has finite first moment *and* the lamplighter operates at finite range.

Let us come back to a justification of our (partial and slightly redundant) distinction between the group $\mathcal{L} \wr \mathcal{F}$ and the lamplighter graph $\mathbb{Y} \wr \mathbb{T}$. We can take for $\mathcal{F}$ any group which acts simply transitively on $\mathbb{T}$ and for $\mathcal{L}$ any group that acts simply transitively on $\mathbb{Y}$, but the geometric realization of the Poisson boundary of random walks with finite first moment is always the same: it only depends on $\mathbb{T}$ and the cardinality of $\mathbb{Y}$.

The methods that we have used can be extended to space-homogeneous (in the sense of [20]) lamplighter random walks over hyperbolic graphs, graphs with infinitely many ends, and other classes of transitive base graphs that can be used in the construction of lamplighter graphs according to (1.2). Finally, with additional effort, the result regarding the Dirichlet problem can apparently be extended. The detailed elaboration of these general facts is awaiting future work, while the present note has aimed at giving a short and hopefully readable explanation of the basic aspects.

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