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Topological Groups and Recurrence of Quasi Transitive Graphs

Estratto dal
«Rendiconti del Seminario Matematico e Fisico di Milano»
Vol. LXIV (1994)

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Conferenza tenuta il 28 novembre 1994

ABSTRACT. We present all steps which are necessary in order to classify all locally finite, infinite graphs which carry a quasi transitive random walk that is recurrent. Some new and/or simpler proofs are given. Most of them rely on the fact that autmomorphism groups of locally finite graphs are locally compact with respect to the topology of pointwise convergence - this allows the use of integration on these groups.

1. Introduction and main theorem

Let X be a locally finite, infinite connected graph. A *random walk* on X is an X-valued Markov chain $(Z_n)_{n\geq 0}$ induced by a stochastic transition matrix $P = (p(x,y))_{x,y\in X}$ which is in some way (to specify more precisely) adapted to the graph structure.

The most typical example is the *simple random walk* (SRW) on X, whose one-step transition probabilities are given by

$$p(x,y) = \begin{cases} \frac{1}{\deg(x)}\,, & \text{if } y \sim x\,; \\ 0\,, & \text{otherwise.} \end{cases}$$

Here, $\sim$ indicates neighbourhood, and $\deg(x)$ denotes the number of neighbours of vertex $x \in X$. (Our way of thinking of a graph is that of a set of vertices X together with a symmetric *neighbourhood* relation $\sim$, also considered as the *edge set* $E = E(X) \subset X \times X$.)

In general, we always require *irreducibility:* for every $x, y \in X$, there is $n \geq 0$ such that $p^{(n)}(x,y)$ (the probability to be in y after n steps when starting from x) is > 0. For the SRW, irreducibility follows from connectedness. The random walk, or the pair (X, P), is called *recurrent* if

$$G(x, y|1) = \infty,$$

where

$$G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)\, z^n \quad (z \in \mathbb{C}) \text{ [the \textit{Green function}]},$$

for some ($\Longleftrightarrow$ every) $x, y \in X$. If (X, P) is not recurrent, then it is called *transient.*

The graph X carries a natural discrete metric: $d(x,y) = d_X(x,y)$ is the minimal length (number of edges) of some path in X connecting x and y. The *automorphism group* $\mathrm{AUT}(X)$ is the group of all self-isometries of X with respect to this metric. We say that X is *quasi transitive,* if $\mathrm{AUT}(X)$ acts with finitely many orbits, and *transitive,* if there is just one orbit. Also, we denote by

$$\mathrm{AUT}(X, P) = \{\gamma \in \mathrm{AUT}(X) : p(\gamma x, \gamma y) = p(x, y) \text{ for all } x, y\}$$

the automorphism group of (X, P), and we say that the random walk (or (X, P)) is (quasi) transitive if this group acts (quasi) transitively. This is a strong adaptendness property.

These groups are locally compact with respect to the topology of pointwise convergence (i.e., $\gamma_n \to \gamma \iff$ for every x, $\gamma_n x = \gamma x$ for all but finitely many n). For any closed subgroup Γ of $\mathrm{AUT}(X)$, a subbase of the neighbourhood filter at the identity ι is given by the stabilizers Γ_x of the vertices $x \in X$ in Γ. In particular, a subset of Γ is compact if and only if it has finite orbits. (See e.g. Woess [W2] for more details.)

If X_i, $i \in I$, are the orbits of Γ, then we can define the factor graph $\Gamma \backslash X$. Its vertex set is I, and $i \sim j$ in $\Gamma \backslash X$ if some ($\Longleftrightarrow$ every) vertex in X_i has some neighbour in X_j.

By $\mathbb{Z}^d$ we denote the (additive) group of integer elements in $\mathbb{R}^d$, that is, the free abelian group on d generators.

MAIN THEOREM – *Let (X, P) be irreducible, quasi transitive and recurrent. Then each of the following holds.*

(I) *the* SRW *on X is recurrent, and*

(II) *there is a compact normal subgroup $\mathfrak{K}$ of* $\mathrm{AUT}(X,P)$ *such that the factor group $\tilde{\Gamma} = \mathrm{AUT}(X,P)/\mathfrak{K}$ is discrete, finitely generated, and contains a subgroup isomorphic with $\mathbb{Z}$ or $\mathbb{Z}^2$ which has finite index in $\tilde{\Gamma}$ and hence acts quasi transitively on the factor graph $\mathfrak{K}\backslash X$.*

This generalizes Varopoulos' [V1], [V2], [V-S-C] solution of a famous problem of Kesten [Ke], namely, the classification of all finitely generated, infinite groups which carry a recurrent random walk (they have a finite index subgroup isomorphic with $\mathbb{Z}$ or $\mathbb{Z}^2$; Gromov's [Gr] classification of all groups with polynomial growth is crucial). The extension given by above theorem was indicated in the survey by Woess [W3] and proved (for strongly reversible random walks) by Saloff-Coste [SC].

Here, we present all necessary steps leading to the theorem and give some new proofs, involving integration on $\mathrm{AUT}(X)$, which simplify and/or extend previous results (in particular of [SC]). This is a considerably enlarged version of the author's "inaugural lecture" as a professor of Probability at the University of Milano. The paper was written during a visit at the University of Salzburg in May 1995. The author acknowledges the hospitality of the Institute of Mathematics and of Mrs. D. Eaton, Salzburg.

2. Some recurrence criteria

Let P be the stochastic transition matrix of an irreducible Markov chain on X. We let P act on functions $f : X \to \mathbb{R}$ by $Pf(x) = \sum_y p(x,y)f(y)$, whenever this sum is absolutely convergent. We denote by $\mathbb{P}_x$ the probability measure governing the Markov chain with $Z_0 = x$, and by $\mathbb{E}_x$ the associated expectation. We have $G(x,y|1) = \mathbb{E}_x(|\{n : Z_n = y\}|)$. For $y \in X$, consider the stopping time $\mathrm{s}^y = \inf\{n \geq 0 : Z_n = y\}$ and define

$$f^{(n)}(x,y) = \mathbb{P}_x[Z_n = y]\,, \quad F(x,y|z) = \sum_{n=0}^{\infty} f^{(n)}(x,y)z^n \ (z \in \mathbb{C})\,.$$

Thus, $F(x,y) = F(x,y|1)$ is the probability of ever reaching y when starting in x. It is well known and easy to prove that

$$G(x,y|z) = F(x,y|z)G(y,y|z) \,. \tag{1}$$

A measure (thought of as a row vector) $\nu : X \to (0\,,\,\infty)$ is called *excessive* (or P-excessive) if $\nu P \le \nu$, where $\nu P(y) = \sum_x \nu(x)p(x,y)$. If $\nu P = \nu$, then it is called *invariant.* The following well known recurrence criteria can be found in every book on Markov chains, for example Kemeny, Snell and Knapp [K-S-K].

THEOREM 2.1 – *The following are equivalent.*

(a) *(X,P) is recurrent.*

(b) *$F(x,y) = 1$ for all $x, y \in X$.*

(c) *P has an invariant measure ν such that every P-excessive measure is a constant multiple of ν.*

(X,P) is called *reversible,* if there is a measure m on x such that $m(x) > 0$ and $a(x,y) = a(y,x)$ for all x,y, where $a(x,y) = m(x)p(x,y)$. In particular, m is P-invariant. The associated *Dirichlet space* $\mathcal{D}(X,P)$ consists of all functions $f : X \to \mathbb{R}$ with finite *Dirichlet norm*

$$D_P(f) = \frac{1}{2} \sum_{x,y \in X} (f(x) - f(y))^2 a(x,y) \,.$$

(If P is the SRW on the graph X, then we write $D(f)$ or $D_X(f)$.) This is a quasi-norm, its kernel consisting of the constants. If we choose a reference point (root) $o \in X$, then $\|f\|_o^2 = D_P(f) + f(o)^2$ defines a norm. Different roots give rise to equivalent norms, and $\mathcal{D}(X,P)$ is a Banach (indeed Hilbert) space. By $\ell_0(X)$ we denote the space of finitely supported functions $X \to \mathbb{R}$, and by $\mathcal{D}_0(X,P)$ its closure in the Dirichlet space.

The *capacity* of $x \in X$ is

$$\operatorname{cap}(x) = \inf\{D(f) : f \in \ell_0(X)\,,\ f(x) \equiv 1\} \,.$$

The following transience criteria are due to Yamasaki [Y1], [Y2] and Varopoulos [V1] (in analogy with the continuous setting of Brownian motion), see also Soardi and Yamasaki [S-Y].

THEOREM 2.2 – *For a reversible Markov chain, the following statements are equivalent.*

(a) *(X, P) is transient.*

(b) *For some ($\Longleftrightarrow$ every) $x \in X$, one has* $\operatorname{cap}(x) > 0$.

(c) *The constant function* $\mathbf{1}$ *is not in* $\mathcal{D}_0(X, P)$.

For further equivalent statements, see the survey [W3]. However, note that in [W3], there is a mistake in the definition of extremal length (the weights of the edges are missing), and that the corresponding criterion is known only in the case when P has finite range.

Criteria (b) and (c) show that transience/recurrence relies only on properties of the Dirichlet norm. Thus, one can derive a variety of comparison results for recurrence of different reversible Markov chains by comparing the Dirichlet norms of finitely supported functions.

One can also compare non reversible and reversible Markov chains. Let P be the transition matrix of an irreducible Markov chain (not necessarily reversible) over X with excessive measure ν. On the real Hilbert space $\ell^2(X, \nu)$ with inner product

$$(f, g)_\nu = \sum_{x \in X} f(x)g(x)\nu(x),$$

P acts as a contraction, as one can deduce from the inequality

$$(Pf, f)_\nu \le \frac{1}{2}\sum_x\sum_y \nu(x)p(x,y)f(x)^2 + \frac{1}{2}\sum_x\sum_y \nu(x)p(x,y)f(y)^2 \le (f, f)_\nu\,. \tag{2}$$

The following theorem has a row of forerunners; the version proved here is an extension of Theorem 1 of Chen [Ch].

THEOREM 2.3 – *Let P and Q be irreducible transition matrices over X, such that P has excessive measure ν and Q is reversible with associated invariant measure m. Suppose that*

(1) $\sup_X m(x)/\nu(x) < \infty$, *and*

(2) *there is $\varepsilon_0 > 0$ such that $P \ge \varepsilon_0 Q$ elementwise.*

Then recurrence of P implies recurrence of Q.

The extension is that [Ch] also requires that $\inf_X m(x)/\nu(x) > 0$. Indeed, I have severe doubts about the very last inequality in the proof of [Ch, Thm. 1].

Proof – One needs the following popular lemma of Baldi [B-L-P] (originally used in the classification of connected Lie groups carrying a recurrent random walk).

LEMMA – *Let $\mathcal{H}$ be a real Hilbert space, and let T_1, T_2 be two invertible linear operators on $\mathcal{H}$ such that T_1 is self-adjoint and $(T_2 f, f) \geq (T_1 f, f) \geq 0$ for all $f \in \mathcal{H}$. Then*

$$(T_1^{-1} f, f) \geq (T_2^{-1} f, f) \quad \textit{for all } f \in \mathcal{H}.$$

Coming back to the proof, we have $u(x) = m(x)/\nu(x) \leq C$ for some constant $C > 0$. Define $\bar{P} = \frac{1}{2}(I + P)$ and $\bar{Q} = (1 - \frac{1}{2C}u)I + \frac{1}{2C}u\,Q$, that is,

$$\bar{q}(x,y) = \Big(1 - \frac{1}{2C}u(x)\Big)\delta_x(y) + \frac{1}{2C}u(x)q(x,y)\,.$$

Then $\bar{P}$ and $\bar{Q}$ are stochastic, irreducible contractions of the real Hilbert space $\ell^2(X,\nu)$. Furthermore, $\bar{Q}$ is reversible with associated invariant measure ν, and ν is $\bar{P}$-excessive. Also, $\bar{P} \geq \varepsilon_1 \bar{Q}$ with $\varepsilon_1 > 0$. By (2),

$$\tfrac{1}{1-\varepsilon_1}\,((\bar{P} - \varepsilon_1\bar{Q})f, f)_\nu \leq (f,f)_\nu \quad \text{for all } f \in \ell^2(X,\nu)\,.$$

Consequently, if $0 < z \leq 1$, then

$$((I - z\bar{P})f, f)_\nu \geq \varepsilon_1\,((I - z\bar{Q})f, f)_\nu \geq 0$$

on $\ell^2(X,\nu)$. Thus, for $0 < z < 1$, the operators $T_1 = \varepsilon_1(I - z\bar{Q})$ and $T_2 = I - z\bar{P}$ satisfy the assumptions of Baldi's Lemma. We have $(I - z\bar{P})^{-1}f(x) = \sum_y G_{\bar{P}}(x,y|z)f(y)$ (analogously for $\bar{Q}$), and setting $f = \delta_x$, we get

$$(1 - \tfrac{z}{2})G_P(x,x|\tfrac{z}{2-z}) = G_{\bar{P}}(x,x|z) \leq \tfrac{1}{\varepsilon_1}G_{\bar{Q}}(x,x|z) \text{ for every } z \in (0\,,\,1)\,.$$

Letting $z \to 1$ from below, we obtain $G_P(x,x) \leq G_{\bar{Q}}(x,x)/\varepsilon_1$ for every X. Thus, recurrence of (X,P) implies recurrence of $(X,\bar{Q})$.

Now let $f \in \ell_0(X)$. Then

$$\begin{aligned} D_{\bar{Q}}(f) &= \frac{1}{2}\sum_{x,y}(f(x) - f(y))^2\nu(x)\bar{q}(x,y) \\ &\geq \frac{1}{2}\sum_{x,y}(f(x) - f(y))^2\nu(x)\frac{u(x)}{2C}q(x,y) = \frac{1}{2C}D_Q(f). \end{aligned}$$

From Theorem 2.2 we now deduce that recurrence of $(X,\bar{Q})$ implies recurrence of (X,Q). □

3. Recurrence and unimodularity

Let Γ be a closed subgroup of $\mathrm{AUT}(X)$. (In the sequel, all subgroups in consideration are tacitly assumed to be closed.) Being locally compact, Γ carries a *left Haar measure* $d\gamma$. For a Borel subset B of Γ, we write $|B|$ for its measure. We shall also need the *modular function* Δ of Γ. Recall that this is a homomorphism from Γ to the multiplicative group of positive reals satisfying $|B\gamma| = \Delta(\gamma)|B|$ for every Borel set in Γ and $\int_\Gamma g(\gamma^{-1})\Delta(\gamma^{-1})\,d\gamma = \int_\Gamma g(\gamma)\,d\gamma$ for integrable functions g on Γ. The group Γ is called *unimodular* when $\Delta \equiv 1$.

If $x \in X$, then the stabilizer Γ_x is open and compact, so that $0 < |\Gamma_x| < \infty$. For $\gamma \in \Gamma$, we have $\Gamma_{\gamma x} = \gamma\Gamma_x\gamma^{-1}$, whence

$$|\Gamma_{\gamma x}| = \Delta(\gamma)|\Gamma_x| \tag{3}$$

Note that there are various examples of graphs with non unimodular automorphism group, see e.g. Soardi and Woess [S-W].

We shall use the following formula:

$$\int_\Gamma f(\gamma x)\,d\gamma = \sum_y \int_{\{\gamma:\gamma x = y\}} f(y)\,d\gamma = |\Gamma_x| \sum_{y\in\Gamma x} f(y)\,, \tag{4}$$

whenever $f : X \to \mathbb{R}$ is a function for which the sum (integral) converges absolutely. (Note the difference between the stabilizer $\Gamma_x \subset \Gamma$ and the orbit $\Gamma x \subset X$.) Analogously, if $e = [x,y] \in E(X)$, then Γ_e is the set of all $\gamma \in \Gamma$ satisfying $\{\gamma x, \gamma y\} = \{x, y\}$. Again, Γ_e is open and compact ($\Gamma_x \cap \Gamma_y$ is a subgroup with index 1 or 2), and the analogues of formulae (3) and (4) remain valid for functions defined on $E(X)$.

For a subgroup Γ of $\mathrm{AUT}(X, P)$, let X_i, $i \in I$, be the orbits of Γ on X. We define the transition matrix $\tilde{P}$ of the *factor chain* on I by

$$\tilde{p}(i, j) = \sum_{w \in X_j} p(x, w),$$

where $x \in X_i$ is arbitrary. It inherits irreducibility from P.

LEMMA 3.1 – *Let* Γ *be a subgroup of* $\mathrm{AUT}(X, P)$. *Suppose that the factor chain* $(I, \tilde{P})$ *has an excessive measure* $\tilde{\nu}$. *Then*

$$\nu(x) = \tilde{\nu}(i)\,|\Gamma_x|, \quad x \in X_i,$$

defines an excessive measure for P *on* X. *If* $\tilde{\nu}$ *is invariant then so is* ν.

Proof – Choose a reference point o_i in each orbit X_i. Let $y \in X_j$. Using (3) and (4), we compute

$$\begin{aligned}
\sum_{x \in X} \nu(x) p(x, y) &= \sum_{i \in I} \tilde{\nu}(i) \sum_{x \in X_i} |\Gamma_x| p(x, y) = \sum_{i \in I} \tilde{\nu}(i) \int_\Gamma \frac{|\Gamma_{\gamma o_i}|}{|\Gamma_{o_i}|} p(\gamma o_i, y)\, d\gamma \\
&= \sum_{i \in I} \tilde{\nu}(i) \int_\Gamma \Delta(\gamma^{-1}) p(o_i, \gamma^{-1} y)\, d\gamma = \sum_{i \in I} \tilde{\nu}(i) \int_\Gamma p(o_i, \gamma y)\, d\gamma \\
&= \sum_{i \in I} \tilde{\nu}(i) \sum_{w \in X_j} |\Gamma_y| p(o_i, w) = |\Gamma_y| \sum_{i \in I} \tilde{\nu}(i) \tilde{p}(i, j) \\
&\le |\Gamma_y|\, \tilde{\nu}(j) = \nu(y).
\end{aligned}$$

If $\tilde{\nu}$ is invariant, then the inequality is an equality. □

THEOREM 3.2 – *If* (X, P) *is recurrent then every subgroup* Γ *of* $\mathrm{AUT}(X, P)$ *is unimodular.*

Proof – *First step.* We suppose that Γ acts transitively.

First observe that for every positive $z < 1$ and every $x \in X$, the measure $\nu(y) = G(x,y|z)/G(x,x|z)$ satisfies $\nu P \leq \frac{1}{z}\nu$. Now, by transitivity, $G(x,x|z) = G(y,y|z)$ for all x,y. Using (1) we get, in the recurrent as well as in the transient case,

$$F(x,y) = \lim_{z \to 1-} \frac{G(x,y|z)}{G(y,y|z)} = \lim_{z \to 1-} \frac{G(x,y|z)}{G(x,x|z)}.$$

From the above and Fatou's lemma we infer that for every x, the measure $\nu(y) = F(x,y)$ is excessive.

On the other hand, by Lemma 3.1, $y \mapsto |\Gamma_y|$ defines an invariant measure.

Now suppose that (X,P) is recurrent. Then, by Theorem 2.1, $F(x,y) = 1$ for all x,y. Also, by Theorem 2.1, up to multiplication with a constant there is a unique excessive measure, which has to be invariant. Consequently $|\Gamma_y| = C\,F(x,y) = C > 0$ for all y. This yields $\Delta(\gamma) = |\Gamma_y|/|\Gamma_{\gamma y}| = 1$ for all $\gamma \in \Gamma$.

Note that we did not really need the graph structure, but only that Γ is a group of permutations of X with compact point stabilizers.

Second step. Coming to the general case, let $Y \subset X$ be one of the orbits of Γ on X. Define the stopping time

$$\mathbf{t}^Y = \min\{n \geq 1 : Z_n \in Y\}.$$

By recurrence, it is $\mathbb{P}_x$-almost surely finite for every x. The *induced random walk* (Y, P^Y) has transition probabilities

$$p^Y(x,y) = \mathbb{P}_x[Z_{\mathbf{t}^Y} = y], \quad x,y \in Y.$$

Recurrence of (X,P) implies recurrence of the induced walk: if $x,y \in Y$, then $F(x,y)$ is the same for the original and for the induced random walk. Now, Γ acts transitively on Y. Let $\mathfrak{K} = \bigcap_Y \Gamma_y$. This is a compact normal subgroup, and $\Gamma/\mathfrak{K}$ is a transitive group of permutations of Y with compact point stabilizers. It leaves P^Y invariant, so that we are in the situation of the first step. Therefore $\Gamma/\mathfrak{K}$, and also Γ, must be unimodular. □

Note that in the last theorem quasi transitivity was not required. We remark that Theorem 3.2 can also be deduced from Thm. 51 of

Guivarc'h, Keane and Roynette [G-K-R] concerning random walks on locally compact groups. However, the proof in [G-K-R] is too long; a shorter proof can be given by "extrapolating" the above first step, see §6.

We now apply Theorem 2.3 and Theorem 3.2 to quasi transitive graphs.

THEOREM 3.3 – *If (X, P) is quasi transitive and recurrent, then the* SRW *on* X *is recurrent.*

Proof – Let $\Gamma = \mathrm{AUT}(X, P)$. Recurrence implies that Γ is unimodular. Quasi transitivity and local finiteness imply that Γ acts with finitely many orbits on the set of ordered pairs (x, y) such that $x \sim y$. For each pair, let $n_{x,y}$ be minimal among all integers n with $p^{(n)}(x, y) > 0$, and let $\varepsilon_{x,y} = p^{(n_{x,y})}(x, y)$. Then $K = \max\{n_{x,y} : x \sim y\} < \infty$ and $\varepsilon_0 = \min\{\varepsilon_{x,y} : x \sim y\} > 0$. Define $\hat{P} = (\frac{1}{2}(I + P))^K$. Then $\hat{P}$ is recurrent (a very easy exercise using the Green function), irreducible, and if $x \sim y$, then

$$\hat{p}(x, y) \geq \frac{\varepsilon_0}{2^K} q(x, y),$$

where Q stands for the SRW on X. In order to verify the other hypothesis of Theorem 2.3, first recall that the SRW is reversible with $m(x) = \deg(x)$, which is bounded above by quasi-transitivity.

Next, let $(I, \tilde{P})$ be the factor chain of (X, P) modulo Γ. As I is finite, $\tilde{P}$ is recurrent and has an invariant (probability) measure $\tilde{\nu}$. Lemma 3.1 yields existence of a P-invariant measure ν on X. As Γ is unimodular, this ν is constant on each orbit by (3). Therefore $\inf_X \nu(x) > 0$, and we obtain $\sup_X m(x)/\nu(x) < \infty$. Theorem 2.3 now tells us that also Q must be recurrent. □

Thus, we have proved statement (I) of the Main Theorem.

Next, we shall look for a class of random walks which have to be recurrent when the SRW is recurrent. A reversible Markov chain with associated invariant measure m is called *strongly reversible,* if $\inf_X m(x) > 0$ and $\sup_X m(x) < \infty$.

LEMMA 3.4 – *If (X,P) is quasi transitive, recurrent and reversible, then it must be strongly reversible, and m is constant on each orbit of* $\mathrm{AUT}(X,P)$.

Proof – Let $\Gamma = \mathrm{AUT}(X,P)$. From Lemma 3.1 and Theorem 3.2 we know that P has an invariant measure ν which is constant on each orbit of Γ.

Now suppose that P is reversible with respect to m. Let $x, y \in X$, and choose n such that $p^{(n)}(x,y) > 0$. If $\gamma \in \Gamma$ then

$$m(x)p^{(n)}(x,y) = m(y)p^{(n)}(y,x)$$

and

$$m(\gamma x)p^{(n)}(x,y) = m(\gamma y)p^{(n)}(y,x)\,.$$

Dividing, we get $m(\gamma x)/m(x) = m(\gamma y)/m(y)$, that is, the function $g(\gamma) = m(\gamma x)/m(x)$ does not depend on $x \in X$. This means that g is an exponential on Γ: it satisfies $g(\beta\gamma) = g(\beta)g(\gamma)$. We have two possibilities: either $g \equiv 1$, in which case m is constant on each orbit of Γ, or g and m are unbounded.

By recurrence, it must be $m = C\nu$ for some $C > 0$, and the first case must hold. □

We say that (X,P) has *finite second moment,* if

$$M_2(P) = \sup_{x \in X} \mathbb{E}_x(d(Z_1,x)^2) < \infty\,.$$

THEOREM 3.5 – *Let X be a quasi transitive graph. Then the* SRW *on X is recurrent if and only if some ($\Longleftrightarrow$ every) strongly reversible, quasi transitive random walk on X with finite second moment is recurrent.*

Proof – The "if" part follows from Theorem 3.2.

Conversely, let (X,P) be strongly reversible with associated invariant measure m. If $f \in \ell_0(X)$ and $e = [x,y] \in E = E(X)$, then we write $|\nabla f(e)| = |f(x) - f(y)|$.

For arbitrary $x, y \in X$, write $\Pi(x,y)$ for the (finite) set of geodesics (paths of length $d(x,y)$) from x to y, and $\Pi_e(x,y)$ for all paths in

$\Pi(x,y)$ containing edge e. Using the Cauchy-Schwarz inequality, we have

$$(f(x)-f(y))^2 \le \frac{1}{|\Pi(x,y)|}\sum_{\pi\in\Pi(x,y)}\sum_{e\in\pi}(\nabla f(e))^2 d(x,y).$$

Hence

$$\begin{aligned} D_P(f) &\le \frac{1}{2}\sum_{x,y\in X} m(x)p(x,y)\frac{1}{|\Pi(x,y)|}\sum_{\pi\in\Pi(x,y)}\sum_{e\in\pi}(\nabla f(e))^2 d(x,y) \\ &= \sum_{e\in E}(\nabla f(e))^2\phi(e), \end{aligned}$$

with

$$\phi(e) = \frac{1}{2}\sum_{x,y\in X} m(x)p(x,y)d(x,y)\frac{|\Pi_e(x,y)|}{|\Pi(x,y)|}.$$

We obtain $D_P(f) \le \sup_E \phi(e)\, D(f)$, where $D(\cdot)$ is the Dirichet norm associated with the SRW.

If (X,P) is recurrent, then $\Gamma = \mathrm{AUT}(X,P)$ is unimodular (Theorem 3.2). We use quasi transitivity to show that ϕ is bounded above: then recurrence of the SRW implies recurrence of (X,P) by virtue of Theorem 2.2. Let X_i, $i\in I$, be the finitely many orbits of Γ in X. For every i, choose $o_i \in X_i$. Let e be an edge of X. Then, applying formula (4) and Lemma 3.4,

$$\begin{aligned} \phi(e) &= \sum_i \frac{1}{2|\Gamma_{o_i}|}\sum_y \int_\Gamma m(\gamma o_i)p(\gamma o_i,y)d(\gamma o_i,y)\frac{|\Pi_e(\gamma o_i,y)|}{|\Pi(\gamma o_i,y)|}\,d\gamma \\ &= \sum_i \frac{m(o_i)}{2|\Gamma_{o_i}|}\sum_y \int_\Gamma p(o_i,\gamma^{-1}y)d(o_i,\gamma^{-1}y)\frac{|\Pi_{\gamma^{-1}e}(o_i,\gamma^{-1}y)|}{|\Pi(o_i,\gamma^{-1}y)|}\,d\gamma \\ &= \sum_i \frac{m(o_i)}{2|\Gamma_{o_i}|}\sum_y\sum_w \int_{\{\gamma:\gamma w=y\}} p(o_i,w)d(o_i,w)\frac{|\Pi_{\gamma^{-1}e}(o_i,w)|}{|\Pi(o_i,w)|}\,d\gamma \\ &= \sum_i \frac{m(o_i)}{2|\Gamma_{o_i}|}\sum_w p(o_i,w)d(o_i,w)\int_\Gamma \frac{|\Pi_{\gamma e}(o_i,w)|}{|\Pi(o_i,w)|}\,d\gamma. \end{aligned}$$

(We have used unimodularity in the last identity.) Using (4) once more,

$$\int_\Gamma \frac{|\Pi_{\gamma e}(o_i,w)|}{|\Pi(o_i,w)|}\,d\gamma = |\Gamma_e| \sum_{e'\in\Gamma e} \frac{|\Pi_{e'}(o_i,w)|}{|\Pi(o_i,w)|} = |\Gamma_e| \sum_{e'\in\Gamma e} \sum_{\substack{\pi\in\Pi(o_i,w)\\ \pi\ni e'}} \frac{1}{|\Pi(o_i,w)|}$$

$$= |\Gamma_e| \frac{1}{|\Pi(o_i,w)|} \sum_{\pi\in\Pi(o_i,w)} |\{e'\in\Gamma e : e'\in\pi\}| \le |\Gamma_e|\, d(o_i,w)\,.$$

Therefore,

$$\phi(e) \le |\Gamma_e| \sum_i \frac{m(o_i)}{2|\Gamma_{o_i}|} \mathbb{E}_{o_i}(d(Z_1,o_i)^2)\,.$$

As X is locally finite, Γ also has only finitely many orbits on $E(X)$, and (by unimodularity, see (3)) $|\Gamma_e|$ is constant on each orbit. We get that $\phi(e)$ is bounded above by the finite number

$$M_2(P)\Big(\max_{e\in E}|\Gamma_e|\Big) \sum_i \frac{m(o_i)}{2|\Gamma_{o_i}|}\,. \qquad \square$$

The last result extends an inequality for Dirichlet norms proved for random walks on discrete groups by Varopoulos [V2]. It also extends the one indicated in [SC, §3, Remark 2]: there, Saloff-Coste assumes amenability (non-validity of a strong isoperimetric inequality). But amenability of a quasi transitive graph implies unimodularity of every quasi transitive group of automorphisms, see Soardi and Woess [S-W] and Salvatori [Sl].

4. Growth and isoperimetric inequalities

Let X be a locally finite, connected graph. Recall that the invariant measure associated with the SRW is given by

$$m(A) = \sum_{x\in A} \deg(x)\,.$$

For $x \in X$ and $n \ge 0$, the *n-ball* centered at x is $B(x,n) = \{y \in X : d(y,x) \le n\}$. The *growth function of X at x* is $V(x,n) = m(B(x,n))$.

We set

$$V(n) = V_X(n) = \inf_X V(x,n) \quad \text{and} \quad \bar{V}(n) = \bar{V}_X(n) = \sup_X V(x,n) .$$

We say that the graph X has *polynomial growth,* if $\bar{V}(n) \le C\, n^d$ for some $C, d > 0$ and all $n \ge 1$. If X is vertex transitive, then $V(x,n) = V(n) = \bar{V}(n)$ is independent of x, and if X is quasi transitive, then $\bar{V}(n)/V(n)$ is bounded.

In general, two postive functions $V_0(\cdot), V_1(\cdot)$ on $\mathbb{N}$ are said to be *equivalent,* if

$$V_i(n) \le c_1\, V_{1-i}(c_2 n) \qquad (i = 0, 1)$$

for all n, where $c_1, c_2 > 0$.

The main result linking growth with the algebraic structure of quasi transitive, infinite graphs is the following.

THEOREM 4.1 – *Suppose that* Γ *is a closed subgroup of* $\mathrm{AUT}(X)$ *which acts quasi transitively. If there are constants* $C, d > 0$ *such that* $\bar{V}(n) \le C\, n^d$ *for infinitely many* n*, then the following holds.*

(a) *There is a compact normal subgroup* $\mathfrak{K}$ *of* Γ *such that* $\Gamma/\mathfrak{K}$ *is a finitely generated, discrete group which has a nilpotent subgroup with finite index.*

(b) $\Gamma/\mathfrak{K}$ *acts with finite vertex stabilizers on the factor graph* $\mathfrak{K}\backslash X$.

(c) *There is an integer* $d(X) \le d$ *such that*

$$C_1\, n^{d(X)} \le V(n) \le \bar{V}(n) \le C_2\, n^{d(X)}$$

for all $n \ge 1$*, where* $0 < C_1, C_2 < \infty$.

(d) *If* $d < 3$ *then* $d(X) \in \{1, 2\}$ *and* $\Gamma/\mathfrak{K}$ *has a finite index subgroup isomorphic with* $\mathbb{Z}$ *or* $\mathbb{Z}^2$.

Outline of proof – (Compare with [W2]). This uses (1) the formula of Bass [Ba] for the growth degree of a finitely generated nilpotent group, (2) Gromov's [Gr] famous classification of the finitely generated groups with polynomial growth, and (3) the extension to compactly generated topological groups due to Losert [Lo]. (Previous to [Lo], there was the paper by Trofimov [Tr] on transitive graphs with polynomial growth.)

If Γ is any locally compact group, generated by a compact, symmetric neighbourhood U of the identity, then the associated growth function is $V_\Gamma(n) = V_{\Gamma,U}(n) = |U^n|$, where $|\cdot|$ is left Haar measure. Change to another generating neighbourhood gives rise to an equivalent growth function.

Now, if $\Gamma \leq \mathrm{AUT}(X)$ acts quasi transitively on the graph X, then one can find a neighbourhood U of the identity such that the corresponding growth function $V_\Gamma(n)$ is equivalent with $V_X(n)$. By [Lo, Thm. 2], Γ has a compact normal subgroup $\mathfrak{K}$ such that $\tilde{\Gamma} = \Gamma/\mathfrak{K}$ is a (possibly 0-dimensional) Lie group. As it is totally disconnected, it must be discrete and finitely generated, and it must have finite vertex stabilizers on $\tilde{X} = \mathfrak{K}\backslash X$. This proves (b).

We get that $\tilde{\Gamma}$, $\tilde{X}$ and X have equivalent growth functions (compare with Theorem 5.3 below). Now Gromov's theorem implies that $\tilde{\Gamma}$ has a nilpotent subgroup $\mathfrak{N}$ with finite index, and the growth function of $\mathfrak{N}$ is equivalent with that of $\tilde{\Gamma}$. By [Ba], the exponent of polynomial growth of $\mathfrak{N}$ is integer: this is $d(X) = d(\mathfrak{N})$, and we get (a) and (c). (d) also follows from [Ba]: if $d(\mathfrak{N}) \in \{1,2\}$, then $\mathfrak{N}$ has a finite index subgroup ismorphic with $\mathbb{Z}^{d(\mathfrak{N})}$.

Note that both [Gr] and [Lo] formulate their result under the assumption $V_\Gamma(n) \leq C\,n^d$ for *all* n. However, checking the proofs, one sees that one only needs this for infintely many n. □

For $A \subset X$, we define ∂A as the set of all edges in $E(X)$ having one endpoint in A and the other in $X \setminus A$. Then $m(A)$ and $|\partial A|$ are discrete analogues for volume and surface area. Given a function $\mathfrak{F} : \mathbb{N} \to \mathbb{R}^+$, we say that X satisfies a *$\mathfrak{F}$-isoperimetric inequality* $IS_{\mathfrak{F}}$, if there is a constant $\kappa > 0$ such that

$$\mathfrak{F}(m(A)) \leq \kappa\,|\partial A|$$

for every finite $A \subset X$.

If, in particular, $\mathfrak{F}(t) = t^{1-1/d}$ (with $d \geq 0$) then we speak of a *d-dimensional isoperimetric inequality,* in short IS_d. Note that if X has bounded vertex degrees, then IS_1 is equivalent with X being infinite.

IS_d is equivalent with a *d-dimensional Sobolev inequality.* For a function $f : X \to \mathbb{R}$, its *Sobolev norm* (with respect to the SRW) and its norm in $\ell^p(X,m)$ are

$$S_X(f) = S(f) = \frac{1}{2} \sum_{x,y \in X} |f(x) - f(y)| \quad \text{and} \quad \|f\|_p = \left(\sum_{x \in X} |f(x)|^p m(x) \right)^{1/p},$$

respectively, whenever these sums converge. (If $p = \infty$ then we mean the sup-norm.) The following is well known, see e.g. [SC].

PROPOSITION 4.2 – *X satisfies IS_d if and only if*

$$\|f\|_{\frac{d}{d-1}} \le \kappa \, S_P(f) \quad \textit{for every } f \in \ell_0(X)\,.$$

(*The constant κ is the same as in IS_d.*)

Next, we link growth with isoperimetric inequalities. The following is a simple exercise.

LEMMA 4.3 – *If X satisfies IS_d then $V(n) \ge C\,n^d$ for some $C > 0$.*

Under an additionl condition, there is a converse to the last lemma. For $f : X \to \mathbb{R}$, define

$$P_n f(x) = \frac{1}{V(x,n)} \sum_{y \in V(x,n)} f(y) m(y)\,.$$

We say that X is *smooth* if there is a constant $\eta < \infty$ such that

$$\|f - P_n f\|_1 \le \eta \, n \, S(f) \quad \text{for all } f \in \ell_0(X) \text{ and all } n\,.$$

With the growth function, we associate the right semicontinuous functions

$$\mathfrak{f}(t) = \min\{n : V(n) > t\} \quad \text{and} \quad \mathfrak{F}(t) = t/\mathfrak{f}(2t)\,, \quad t \ge \tfrac{1}{2} \inf_X m(x)\,.$$

We remark that [SC] defines $\mathfrak{f}$ with a "$\ge$" but then uses right semicontinuity. For this reason, we (re-)prove the following; see also [C-S2].

PROPOSITION 4.4 – *If X is smooth (with constant η) then it satisfies $IS_{\mathfrak{F}}$ with constant $\kappa = 2\eta$.*

Proof – Let $A \subset X$ be finite. Then, for arbitrary n,

$$m(A) = m[\mathbf{1}_A \geq 1] \leq m[|\mathbf{1}_A - P_n\mathbf{1}_A| \geq 1/2] + m[P_n\mathbf{1}_A \geq 1/2].$$

(Here, we denote $[f \geq c] = \{x \in X : f(x) \geq c\}$.) By Markov's inequality, and using smoothness,

$$m[|\mathbf{1}_A - P_n\mathbf{1}_A| \geq 1/2] \leq 2\|\mathbf{1}_A - P_n\mathbf{1}_A\|_1 \leq 2n\,\eta\,|\partial A|.$$

Now choose $n = \mathfrak{f}(2m(A))$. Then $V(x,n) > 2m(A)$ and consequently $P_n\mathbf{1}_A(x) \leq m(A)/V(x,n) < 1/2$ for every $x \in X$. We get

$$m(A) \leq 2\eta\,\mathfrak{f}(2m(A))\,|\partial A|. \qquad \square$$

COROLLARY 4.5 – *If X is smooth then it satisfies IS_d $(d \geq 1)$ if and only if $V(n) \geq C\,n^d$ for all n $(C > 0)$.*

Next, we show that this applies to quasi transitive graphs. In analogy with Theorem 3.5 above, we simplify (and extend) the proof of [SC, Thm. 2.1].

THEOREM 4.6 – *If $\mathrm{AUT}(X)$ has a unimodular subgroup Γ which acts quasi transitively, then X is smooth.*

Proof – Recall the notation of the proof of Theorem 3.5. Let $f \in \ell_0(X)$. For $x, y \in X$ and $\pi \in \Pi(x,y)$ we have $|f(y) - f(x)| \leq \sum_{e\in\pi} |\nabla f(e)|$.

$$\|f - P_nf\|_1 = \sum_{x\in X} \left| \frac{1}{V(x,n)} \sum_{y\in B(x,n)} (f(x) - f(y))m(y) \right| m(x)$$

$$\leq \sum_{x\in X} \frac{m(x)}{V(x,n)} \sum_{y\in B(x,n)} \frac{m(y)}{|\Pi(x,y)|} \sum_{\pi\in\Pi(x,y)} \sum_{e\in\pi} |\nabla f(e)|$$

$$= \sum_{e\in E} |\nabla f(e)| \left(\sum_{x\in X} \frac{m(x)}{V(x,n)} \sum_{y\in B(x,n)} \frac{m(y)|\Pi_e(x,y)|}{|\Pi(x,y)|} \right).$$

Hence $\|f - P_nf\|_1 \leq (\sup_{E(X)} \eta(e,n))\,S(f)$, where

$$\eta(e,n) = \sum_{x\in X} \frac{m(x)}{V(x,n)} \sum_{y\in B(x,n)} \frac{m(y)|\Pi_e(x,y)|}{|\Pi(x,y)|}.$$

We show that $\eta(e,n) \le n\,\eta$ for some finite η.

Choose $o_i \in X_i$, where X_i, $i \in I$, are the finitely many orbits of Γ in X. Let $e \in E(X)$. Then, applying (4) and unimodularity,

$$\eta(e,n) = \sum_i \frac{1}{|\Gamma_{o_i}|} \int_\Gamma \frac{m(\gamma o_i)}{V(\gamma o_i, n)} \sum_{y \in B(\gamma o_i, n)} \frac{m(y)|\Pi_e(\gamma o_i, y)|}{|\Pi(\gamma o_i, y)|}\, d\gamma\,.$$

$$= \sum_i \frac{1}{|\Gamma_{o_i}|} \int_\Gamma \frac{m(o_i)}{V(o_i, n)} \sum_{w \in B(o_i, n)} \frac{m(w)|\Pi_{\gamma^{-1}e}(o_i, w)|}{|\Pi(o_i, w)|}\, d\gamma\,.$$

$$= \sum_i \frac{1}{|\Gamma_{o_i}|} \frac{m(o_i)}{V(o_i, n)} \sum_{w \in B(o_i, n)} m(w) \int_\Gamma \frac{|\Pi_{\gamma e}(o_i, w)|}{|\Pi(o_i, w)|}\, d\gamma\,.$$

As in the proof of Theorem 3.5,

$$\int_\Gamma \frac{|\Pi_{\gamma e}(o_i, w)|}{|\Pi(o_i, w)|}\, d\gamma \le |\Gamma_e|\, d(o_i, w)\,,$$

which is bounded by $n\,|\Gamma_e|$. We obtain $\eta(e,n) \le n\,\eta(e)$, where

$$\eta(e) = |\Gamma_e| \sum_i \frac{m(o_i)}{|\Gamma_{o_i}|}$$

As Γ acts on $E(X)$ with finitely many orbits, and once more using unimodularity in the edge-version of (3), we see that $\eta = \sup_E \eta(e)$ is finite.

□

We remark that the constant η obtained here can be "translated" into the one given in [SC].

5. Rough isometries

Let (X,d) and (X',d') be two metric spaces. A *rough isometry* is a mapping $\varphi : X \to X'$ such that

$$A^{-1}d(x,y) - A^{-1}B \le d'(\varphi x, \varphi y) \le A\,d(x,y) + B$$

for all $x, y \in X$, and

$$d'(x', \varphi X) \le B$$

for every $x' \in X'$, where $A \geq 1$ and $B \geq 0$.

In this case we say that the two spaces are *roughly isometric.*

We can construct a *rough inverse* $\bar{\varphi}$ of φ: for every $x' \in X'$, choose $x \in X$ such that $d'(x', \varphi x) \leq B$, and set $\bar{\varphi} x' = x$. Then one easily works out that $\bar{\varphi}$ is a rough isometry with constants $A' = A$ and $B' = \max\{3B, (2A+1)B\}$. Furthermore, $d(\bar{\varphi}\varphi x, x) \leq (A+1)B$ for every $x \in X$, and $d'(\varphi\bar{\varphi}x', x') \leq B$ for every $x' \in X'$. It is obvious that the composition of two rough isometries is again a rough isometry. Hence, to be roughly isometric is an equivalence relation between metric spaces.

In the sequel, we shall only consider graphs, and rough isometries will always refer to their discrete metrics induced by neighbourhood. Finite connected graphs are roughly isometric with a singleton. Here, we always consider infinite graphs.

The following is easy to prove; for (b), see e.g. [Sl].

PROPOSITION 5.1 – (a) *Let* $\Gamma \leq \mathrm{AUT}(X)$ *act quasi transitively, and let* $\mathfrak{K}$ *be a compact normal subgroup of* Γ. *Then the factor graph* $\mathfrak{K}\backslash X$ *is roughly isometric with* X.

(b) *Every quasi transitive graph is roughly isometric with a vertex transitive graph.*

(The vertex transitive graph in (b) is constructed over one of the orbits of the autmorphism group on the graph.) If Γ is a finitely generated discrete group and S is a finite, symmetric set of generators then the *Cayley graph* $X(\Gamma, S)$ of Γ with respect to S has vertex set $X \equiv \Gamma$, and $x \sim y \iff x^{-1}y \in S$. Then $\Gamma \leq \mathrm{AUT}(X)$, acting by left multiplication. Considering the growth of Γ is the same as considering the growth of $X(\Gamma, S)$; different generating sets give rise to equivalent growth functions. The typical Cayley graph of $\mathbb{Z}^d$ is the d-dimensional grid.

The following is easily deduced from Thm. 4 of Sabidussi [Sb].

PROPOSITION 5.2 – *If* $\Gamma \leq \mathrm{AUT}(X)$ *acts on* X *quasi transitively and with finite vertex stabilizers, then* Γ *is fintely generated and* X *is roughly isometric with any Cayley graph of* Γ.

Rough isometries preserve growth, isoperimetric inequalities and recurrence; see e.g. Kanai [K1], [K2].

THEOREM 5.3 – *Let* X *and* X' *be roughly isometric graphs with bounded vertex degrees. Then we have the following.*

(a) *The growth functions of* X *and* X' *are equivalent.*

(b) X *satisfies* IS_d *if and only if* X' *satisfies* IS_d.

(c) *The* SRW *on* X *is recurrent if and only if the* SRW *on* X' *is recurrent.*

Typically, one can prove (c) by comparing the relative Dirichlet norms and applying Theorem 2.2.

With these results, one can easily prove the following converse of the Main Theorem.

COROLLARY 5.4 – *Suppose that* X *is quasi transitive. If there is a compact normal subgroup* $\mathfrak{K}$ *of* $\mathrm{AUT}(X)$ *such that the factor group* $\mathrm{AUT}(X)/\mathfrak{K}$ *is discrete, finitely generated, and contains a finite index subgroup isomorphic with* $\mathbb{Z}$ *or* $\mathbb{Z}^2$, *then the* SRW *on* X *is recurrent.*

Proof – By the above results, X is roughly isometric with one of the grids $\mathbb{Z}^d$, where $d \in \{1,2\}$. The simple random walks on these grids are very well known to be recurrent, see the classical paper by Pólya [Po]. □

6. Conclusion

For an almost transitive graph X, we now have the following from Theorem 4.1 and §5: either X is roughly isometric with $\mathbb{Z}$ or $\mathbb{Z}^2$, or $V_X(n) \geq Cn^3$ for all n. To conclude the proof of the Main Theorem, we must show that in the second case, every almost transitive random walk on X has to be transient. By Theorem 3.3, it is enough to show this for the SRW. Since Varopoulos' breakthrough of 1986 [V2], several different methods have been found. Let me start with my favourite.

Variant 1. Transient subtrees. The following extends the isoperimetric inequalities of §4. Let X be a locally finite graph. Given $\mathfrak{F} : \mathbb{N} \to \mathbb{R}^+$ and a "root" $o \in X$, we say that X satisfies $IS_{\mathfrak{F},o}$ if there is $\kappa > 0$ such that

$$\mathfrak{F}(m(A)) \leq \kappa \, |\partial A|$$

for every finite, connected subgraph A of X containing o. Thomassen [T1], [T2] has proved the following remarkable result by purely graph theoretical methods.

THEOREM 6.1 – *Let X be a graph with bounded vertex degrees satisfying $IS_{\mathfrak{F},o}$, where $\mathfrak{F}$ is non decreasing and*

$$\sum_{n=0}^{\infty} \mathfrak{F}(n)^{-2} < \infty \, .$$

Then X has a subtree T such that the SRW *on T is transient.*

Recall that a tree is a connected graph without (non trivial) cycles. Thomassen uses $IS_{\mathfrak{F},o}$ with the cardinality of A in the place of $m(A) = \sum_A \deg(x)$, and with the set of vertices in A having a neghbour in $X \setminus A$ in the place of ∂A. For his proof, vertex degrees do not have to be bounded (local finiteness is enough). We remark that already the construction of a transient subtree of the grid $\mathbb{Z}^3$ is a non-trivial task, see Doyle and Snell [D-S] and Gerl [Ge].

Also note that for any connected subgraph Y of X, one has $D_Y(f) \leq D_X(f)$ for all $f \in \ell_0(Y)$. Hence, by Theorem 2.2, transience of the SRW on Y implies transience of the SRW on X.

Proof 1 of the Main Theorem – If X is quasi transitive and $V(n) \geq Cn^3$, then X satisfies IS_3 by Theorem 4.6 and Corollary 4.5. Therefore $IS_{\mathfrak{F},o}$ holds with $\mathfrak{F}(t) = t^{2/3}$ and arbitrary o. Now X has a transient subtree by Theorem 6.1, and the SRW on X must be transient. Consequently every other almost transitive random walk on X must be transient as well (Theorem 3.3). □

Variant 2. Isoperimetric inequalities and decay of transition probabilities. In one of the most significant steps on his way towards the classification of the recurrent groups, Varopoulos [V1] has proved the following (in a more general form than stated here).

THEOREM 6.2 – *If X is a locally finite graph satisfying IS_d $(d \geq 1)$ then the transition probabilities of the* SRW *satisfy*

$$\sup_{x,y} p^{(n)}(x,y)/\deg(y) = O(n^{-d/2}).$$

Proof 2 of the Main Theorem – If X is quasi transitive and $V(n) \geq C\,n^3$, then X satisfies IS_3 by Theorem 4.6 and Corollary 4.5. By Theorem 6.2, $p^{(n)}(o,o) \leq C\,n^{-3/2}$, and $G(o,o|1) < \infty$. □

Variant 3. Growth and decay of transition probabilities. The following property is a variant of smoothness.

$$(*) \quad \begin{array}{l} \text{There is } \eta > 0 \text{ such that} \\ \|f - P_n f\|_2^2 \leq n^2 \eta\, D(f) \quad \text{for all } f \in \ell_0(X),\ n \in \mathbb{N}. \end{array}$$

For the following theorem, see [SC] and the preceding versions by Carlen, Kusuoka and Stroock [C-K-S], Coulhon and Saloff-Coste [C-S1] and Varopoulos, Saloff-Coste and Coulhon [V-S-C].

THEOREM 6.3 – *If X is a locally finite graph satisfying $(*)$ and $V(n) \geq C\,n^d$ $(d \geq 1)$, then*

$$\sup_{x} p^{(2n)}(x,x) = O(n^{-d/2}).$$

Now the following is proved almost exactly like Theorem 4.6.

THEOREM 6.4 – *If* AUT(X) *has a unimodular subgroup Γ which acts quasi transitively, then X satisfies $(*)$.*

Proof 3 of the Main Theorem – Let X be quasi transitive and $V(n) \geq C\,n^3$. If AUT(X) is non unimodular, then the SRW is transient by Theorem 3.2. Otherwise, X satisfies $(*)$ by Theorem 6.4, and we may use Theorem 6.3 to conclude as in Proof 2. □

Variant 4. Lifting the random walk to the automorphism group. This is the method which has been (vaguely) indicated in [W3].

By Lemma 5.1 and Theorem 5.3, it is sufficient to prove the Main Theorem in the case when X is vertex-transitive. More generally, let (X,P) be irreducible and transitive, and let $\Gamma = \mathrm{AUT}(X,P)$. Choose $o \in X$ and define

$$\Phi(\gamma) = \frac{1}{|\Gamma_o|} p(o,\gamma o) \quad \text{and} \quad \mu(d\gamma) = \Phi(\gamma)\, d\gamma\,, \quad \gamma \in \Gamma\,. \tag{5}$$

Then one gets the following.

PROPOSITION 6.5 – *μ is a probability measure on Γ, and its n-th convolution powers satisfy*

$$\mu^{(n)}(\Gamma_o) = p^{(n)}(o,o)\,.$$

Furthermore, P is symmetric ($p(x,y) = p(y,x)$ for all x,y) if and only if Φ is symmetric ($\Phi(\gamma^{-1}) = \Phi(\gamma)$ for all γ).

This simple, but useful fact was proved by Woess [W1]; the construction goes back to Soardi and Woess [S-W]. Varopoulos [V3] has extended his results on the decay of transition probabilities to locally compact groups.

THEOREM 6.6 – *Let Γ be a locally compact, unimodular group, and let U be a compact, symmetric neighbourhood of the identity which generates Γ. Suppose that $V_\Gamma(n) \geq C\,n^d$ for all n. If Φ is a symmetric probability density in $L^2(\Gamma)$ with compact support satisfying* $\inf_U \Phi(\gamma) > 0$ *then*

$$\|\Phi^{(n)}\|_\infty = O(n^{-d'/2}) \quad \textit{for every } d' < d\,.$$

(Here, $\Phi^{(n)}$ is the n-th convolution power, and the norms refer to Haar measure on Γ.)

Although this is not completely within the main theme of this paper, I next give an outline of a simpler proof than in [G-K-R] that a recurrent locally compact group must be unimodular.

THEOREM 6.7 – *Let Γ be a locally compact group with countable base of the topology and μ a Borel probability measure on Γ such that*

$\langle \operatorname{supp} \mu \rangle^{-} = \Gamma$. *If* μ *is recurrent, i.e.,*

$$\sum_{n=1}^{\infty} \mu^{(n)}(U) = \infty \tag{6}$$

for some nonempty, open $U \subset \Gamma$ *with compact closure, then* Γ *is unimodular.*

Outline of proof – By a theorem of Loynes (see [G-K-R, Thm. 22]), either $\sum_{n\geq 1} \mu^{(n)}$ is a Radon measure or else (6) holds for every U. In particular, recurrence implies that μ is irreducible in the sense that supp μ generates Γ as a closed semigroup.

For $0 < z < 1$, we can define a Borel measure on Γ by $\nu_z = \sum_{n\geq 1} z^n \mu^{(n)}$. It satisfies $\nu_z * \mu = \mu * \nu_z \leq z\,\nu_z$. Choose a nonzero continuous function $f_0 : \Gamma \to \mathbb{R}^+$ with compact support. By irreducibility (recurrent or not),

$$C_z = \{\nu : \text{a Radon measure on } \Gamma \text{ such that } \nu * \mu \leq z\nu\,,\ \mu * \nu \leq z\nu$$
$$\text{and} \quad \int_\Gamma f_0\, d\nu = 1\}$$

is compact in the vague topology and non empty by the above. Hence also C_1 is nonempty. Let $\nu_0 \in C_1$.

In the recurrent case, like in the discrete setting (Theorem 2.1), there is a Radon measure ν_l such that $\mu * \nu_l = \mu$, and every Radon measure with $\mu * \nu \leq \nu$ is a multiple of ν_l. Now the equation is satisfied by left Haar measure, and consequently ν_0 is left Haar measure (normed by $\int f_0\, d\nu_0 = 1$).

Analogously, there is a measure ν_r such that $\nu_r * \mu = \nu_r$ such that every measure with $\nu * \mu \leq \nu$ is a multiple of ν_r. Thus, $\nu_0 = C\,\nu_r$ must be right Haar measure.

We get that left and right Haar measure coincide. □

Proof 4 of the Main Theorem – Let X be transitive and $V(n) \geq C\,n^3$. Let Q be the SRW and $P = \frac{1}{2}(I+Q)$. Then the SRW is transient if and only if P is transient. Let $\Gamma = \mathrm{AUT}(X)$, and define Φ and μ as in (5). Then

$$U = \mathrm{supp}\ \Phi = \{\gamma \in \Gamma : d(\gamma o, o) \le 1\}$$

is a compact, open, symmetric neighbourhood of the identity, and (see [W2])

$$U^n = \{\gamma \in \Gamma : d(\gamma o, o) \le n\}\,,$$

so that U generates Γ, and

$$V_\Gamma(n)/|\Gamma_o| = V_X(n)/\deg\,,$$

where deg is the (constant) vertex degree in X. Furthermore, Φ is continuous and constant on U.

If Γ is non unimodular, then we use Proposition 6.5 and Theorem 6.7 (or Theorem 3.2) to obtain

$$G_P(o,o|1) = 1 + \sum_{n \ge 1} \mu^{(n)}(\Gamma_o) < \infty\,.$$

Otherwise, all conditions of Theorem 6.6 are satisfied (with $d = 3$), and $\|\Phi^{(n)}\|_\infty \le \bar{C}\, n^{-5/2}$. By Proposition 6.5,

$$p^{(n)}(o,o) = \int_{\Gamma_o} \Phi^{(n)}(\gamma)\, d\gamma \le \bar{C}\, |\Gamma_o|\, n^{-5/2}\,,$$

so that P is transient. □

Variant 5. A transient random walk with finite second moment. In the preceding variants, we have not used the full strength of Theorem 3.5. This will be done here. Recall Lemma 5.1 and Theorem 5.3, and let X be a vertex transitive graph with $V(n) \ge C\, n^3$. Following [V2] and Ancona [An], we now construct a transitive random walk (X, P) with finite second moment. Let $(\lambda_k)_{k \ge 1}$ be a sequence of positive numbers such that $\sum_k \lambda_k = 1$ and $\sum_k k^2 \lambda_k < \infty$. We define a transition matrix P by

$$p(x,y) = \sum_{k=1}^{\infty} \lambda_k \frac{1}{V(k)} \mathbf{1}_{B(x,k)}(y)\,.$$

It is easy to see that $M_2(P) \le \sum_k k^2 \lambda_k < \infty$.

PROPOSITION 6.8 – *If X is vertex transitive, $V(n) \geq C\,n^3$ and $\mathrm{AUT}(X)$ is unimodular, then the λ_k can be chosen such that*

$$\sup_{x,y} p^{(n)}(x,y) = O(n^{-1-\varepsilon}) \quad \textit{for every}\ \varepsilon < 1/2\,.$$

Outline of proof – The proof follows [An, Lemme 4.3 & Cor. 4.4]. First of all, note that $\mathrm{AUT}(X,P) = \mathrm{AUT}(X)$. By Lemma 3.1, $\nu(x) = 1$ defines a P-excessive measure, that is, P is doubly stochastic. Let E be the matrix over X with all entries equal to one. Then $E\,P = P\,E = E$.

Decompose $P = Q+R$, and let $\|Q\|_1 = \sup_x \sum_y q(x,y)$ and $\|R\|_\infty = \sup_{x,y} r(x,y)$. We claim that

$$P^n \leq (\|Q\|_1^n + n\|R\|_\infty)E \quad \text{elementwise for all } n \in \mathbb{N}\,.$$

This is true for $n = 1$, and if it holds for $n-1$, then

$$\begin{aligned} P^n = (Q+R)P^{n-1} &\leq Q\,P^{n-1} + \|R\|_\infty\,E\,P^{n-1} \\ &\leq (\|Q\|_1^{n-1} + (n-1)\|R\|_\infty)Q\,E + \|R\|_\infty\,E\,. \end{aligned}$$

Noting that $Q\,E \leq \|Q\|_1 E$, we see that the statement also holds for n.

Now we define Q and $R = P - Q$ by

$$q(x,y) = \sum_{k=1}^{m} \lambda_k \frac{\deg}{V(k)} \mathbf{1}_{B(x,k)}(y)\,.$$

(Again, deg is the constant vertex degree.) Then we get from the above that

$$\sup_{x,y} p^{(n)}(x,y) \leq \left(\sum_{k=1}^{m-1} \lambda_k\right)^n + n \sum_{k=m}^{\infty} \lambda_k \frac{\deg}{V(k)}\,.$$

Setting $\lambda_k = (k\sqrt{\log k})^{-3}$ for $k \geq 2$ and $\lambda_1 = 1 - \sum_{k\geq 2} \lambda_k$, the proof now proceeds as in [An]. □

Proof 5 of the Main Theorem – Suppose without loss of generality (Theorem 3.2) that $\mathrm{AUT}(X)$ is unimodular. We may apply Proposition 6.8 to find a transient random walk which is transitive and has finite second moment. By Theorem 3.5, also the SRW is transient. □

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Pervenuta in Redazione il 5 giugno 1995