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# A SHORT COMPUTATION OF THE NORMS OF FREE CONVOLUTION OPERATORS

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ABSTRACT. Akemann and Ostrand in 1976 gave a formula for the norms of free convolution operators on the $L^2$-space of a discrete group. Using random walk techniques and generating functions, a short and elementary computation of this formula is given.

**1. Introduction and statement of results.** Let $\mathbf{G}$ be a discrete group with unit element $e$ and let $\alpha$ be a complex-valued function on $\mathbf{G}$ supported by a finite set $X$ which has the *Leinert property* (see [**1**] for the definition, resp. Lemma 1 below). The main result of the—by now classical—article [**1**] by Akemann and Ostrand is the following:

THEOREM 1.

$$\|\alpha\| = \min\left\{2t + \sum_{x\in X}\left(\sqrt{t^2 + |\alpha(x)|^2} - t\right)\middle| t \geqslant 0\right\},$$

*where* $\|\alpha\|$ *denotes the norm of* $\alpha$ *as a* (*left*) *convolution operator on* $L^2(\mathbf{G})$ (*called "free" operator in* [**1**]).

The aim of this paper is to give a short and elementary proof of this theorem. From the prerequisites of [**1**] only the following result is used without proof.

LEMMA 1. *$X \subseteq \mathbf{G}$ has the Leinert property iff $X = y \cdot (Y \cup \{e\})$, where $Y$ is a free set in $\mathbf{G}$ and $y \in X$.*

Indeed, $X$ has the Leinert property if and only if there are no nontrivial relations among the elements of $y^{-1}X$ for any fixed $y \in X$ or, equivalently, if and only if $X$ is a left translate of a subset $Y \cup \{e\}$ of $\mathbf{G}$, where $Y$ has no nontrivial relations.

Let $\check{\alpha}(x) = \alpha(x^{-1})$ for $x \in \mathbf{G}$. $\check{\alpha}$ gives the adjoint convolution operator. We use the formula $\|\alpha\| = \|\mu\|^{1/2}$, where $\mu = \check{\alpha} * \alpha$. Let $\mu^{(n)}$ denote the $n$th convolution power of $\mu$, $\mu^{(0)} = \delta_e$.

LEMMA 2. $\|\mu\| = \lim_{n\to\infty} \mu^{(n)}(e)^{1/n}$.

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Thus $\|\alpha\|$ is the inverse of the radius of convergence $r$ of the Taylor series

$$G(z) = \sum_{n=0}^{\infty} \mu^{(n)}(e) z^{2n} \qquad (z \in \mathbf{C}). \tag{1}$$

We shall prove the following result:

THEOREM 2. *$G(z) = P(zG(z))$, where*

$$P(t) = 1 + \frac{1}{2} \sum_{x \in X} \left( \sqrt{1 + 4|\alpha(x)|^2 t^2} - 1 \right) \qquad (t \in \mathbf{C}).$$

As $G(z)$ has positive real coefficients, $r$ is its smallest positive singularity. The same technique as in **[3]** yields

COROLLARY 1. $r^{-1} = \inf\{P(t)/t \,|\, t > 0\}$.

If $X$ has more than two elements (the interesting case), then the infimum is a minimum; otherwise, it is attained as $t \to \infty$. The result of Corollary 1 can be easily transformed into the formula of Theorem 1.

**2. Proofs.**

PROOF OF LEMMA 2. Let $|\alpha|(x) = |\alpha(x)|$ and $|\mu| = |\check{\alpha}| * |\alpha|$. By Lemma 1, $\mu^{(n)}(e)$ is the sum over all products

$$\check{\alpha}\left(x_{i_1}^{-1} y^{-1}\right) \alpha(y x_{i_2}) \cdots \check{\alpha}\left(x_{i_{2n-1}}^{-1} y^{-1}\right) \alpha(y x_{i_{2n}}), \tag{2}$$

where $x_{i_1}, \ldots, x_{i_{2n}} \in Y \cup \{e\}$ and $x_{i_1}^{-1} x_{i_2} \cdots x_{i_{2n-1}}^{-1} x_{i_{2n}} = e$. As $Y$ is a free set, there must be a bijection between the $x_{i_j}$ with even index $j$ and those with odd index. Therefore the product in (2) is equal to the corresponding product in the sum that gives $|\mu|^{(n)}(e)$. This yields

$$\mu^{(n)}(e) = |\mu|^{(n)}(e). \tag{3}$$

This gives

$$|\mu|^{(n)}(e)^{1/n} = \mu^{(n)}(e)^{1/n} = \left\langle \mu^{(n)} * \delta_e, \delta_e \right\rangle^{1/n}$$
$$\leqslant \left\| \mu^{(n)} \right\|^{1/n} = \|\mu\| \leqslant \| |\mu| \|.$$

On the other hand, as $X$ is finite, **[2, 4]** yield

$$\| |\mu| \| = \lim_{n \to \infty} |\mu|^{(n)}(e)^{1/n},$$

and, by another application of (3), the lemma is proved. □

Thus we have also proved Theorem III G of **[1]**. If $\beta = c\alpha$, $c > 0$ and $\nu = \check{\beta} * \beta$ then it is clear that $\nu^{(n)} = c^{2n} \mu^{(n)}$. This, Lemma 1, (3) and the finiteness of $X$ justify the following assumptions for the proof of Theorem 2 (without loss of generality).

*Assumptions.* (i) $\mathbf{G} = \mathbf{F}_s$ is the free group on $Y = \{x_1, \ldots, x_s\}$, and the support of $\alpha$ is $X = \{e\} \cup Y$.

(ii) $\alpha$ is a probability distribution, i.e., $\alpha(e) = \alpha_0$, $\alpha(x_i) = \alpha_i$, $i = 1, \ldots, s$, where $\alpha_i > 0$ (real) and $\sum_{i=0}^{s} \alpha_i = 1$.

Now consider a sequence of $\mathbf{G}$-valued random variables $X_n$, $n = 0, 1, 2, \ldots$, constituting a Markov process with transition probabilities

$$\Pr[X_{n+1} = y \mid X_n = x] = \begin{cases} \check{\alpha}(x^{-1}y) & \text{if } n \text{ is even,} \\ \alpha(x^{-1}y) & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

This gives an "alternating random walk" on the homogeneous tree of degree $2s$ that represents $\mathbf{F}_s$. We have $\mu^{(n)}(e) = \Pr[X_{2n} = e \mid X_0 = e]$. Let $p^{(2n-1)} = \Pr[X_{2n} = e \mid X_1 = e]$ and $H(z) = \sum_{n=1}^{\infty} p^{(2n-1)} z^{2n-1}$. Besides $\mu^{(n)}(e)$ and $p^{(2n-1)}$ we need the following "taboo probabilities" and their generating functions for $i = 1, \ldots, s$:

$$f_i^{(n)} = \Pr\left[X_n = e;\, X_m \neq e \text{ for } m = 1, \ldots, n-1;\, X_1 = x_i^{-1} \middle| X_0 = e\right], \qquad f_i^{(0)} = 0,$$

$$a_i^{(2n)} = \Pr\left[X_{2n+1} = e;\, X_m \neq x_i \text{ for } m = 2, \ldots, 2n \mid X_1 = e\right], \qquad a_i^{(0)} = 1,$$

$$b_i^{(2n-1)} = \Pr\left[X_{2n-1} = e;\, X_m \neq x_i \text{ for } m = 1, \ldots, 2n-2 \mid X_0 = e\right],$$

$$F_i(z) = \sum_{n=0}^{\infty} f_i^{(2n)} z^{2n}, \quad A_i(z) = \sum_{n=0}^{\infty} a_i^{(2n)} z^{2n}, \quad B_i(z) = \sum_{n=1}^{\infty} b_i^{(2n-1)} z^{2n-1}.$$

Furthermore, let $f^{(n)} = \sum_{i=1}^{s} f_i^{(n)}$ and $F(z) = \sum_{i=1}^{s} F_i(z)$. Note that $f_i^{(2n-1)} = 0$ for all $n \geqslant 1$, as the $(2n-1)$st step cannot lead from $x_i^{-1}$ to $e$ with positive probability.

LEMMA 3. (a) $G(z) = 1 + F(z)G(z) + \alpha_0 z H(z)$,
(b) $H(z) = F(z)H(z) + \alpha_0 z G(z)$,
(c) $A_i(z) = 1 + (F(z) - F_i(z))A_i(z) + \alpha_0 z B_i(z)$,
(d) $B_i(z) = F(z)B_i(z) + \alpha_0 z A_i(z)$,
(e) $F_i(z) = \alpha_i^2 z^2 A_i(z)$, $i = 1, \ldots, s$.

PROOF. This is obvious from the following relations

$$\mu^{(n)}(e) = \sum_{k=0}^{n} f^{(2k)} \mu^{(n-k)}(e) + \alpha_0 p^{(2n-1)} \quad \text{for } n \geqslant 1, \quad \mu^{(0)}(e) = 1. \tag{5}$$

Here and in (6), (7) and (8) the summation on the right is taken over all possible instants of first return to $e$, which are $2k$ ($k = 0, \ldots, n$, resp. $n-1$) and 1. In (6) and (7) we use the fact that the symmetry between $\alpha$ and $\check{\alpha}$ gives us

$$f_i^{(2k)} = \Pr\left[X_{2k+1} = e;\, X_m \neq e \text{ for } m = 2, \ldots, 2k;\, X_2 = x_i \mid X_1 = e\right].$$

$$p^{(2n-1)} = \sum_{k=0}^{n-1} f^{(2k)} p^{(2n-2k-1)} + \alpha_0 \mu^{(n-1)}(e), \tag{6}$$

$$a_i^{(2n)} = \sum_{k=0}^{n} \left(f^{(2k)} - f_i^{(2k)}\right) a_i^{(2n-2k)} + \alpha_0 b_i^{(2n-1)} \quad \text{for } n \geqslant 1,\ a_i^{(0)} = 1. \tag{7}$$

$$b_i^{(2n-1)} = \sum_{k=0}^{n-1} f^{(2k)} b_i^{(2n-2k-1)} + \alpha_0 a_i^{(2n-2)} \quad \text{for } n \geqslant 1. \tag{8}$$

$$f_i^{(2n)} = \alpha_i^2 a_i^{(2n-2)} \quad \text{for } n \geqslant 1,\ f_i^{(0)} = 0. \tag{9}$$

In (9) we have used the fact that the transitions from $x$ to $y$ and from $e$ to $x^{-1}y$ have the same probability. □

PROOF OF THEOREM 2. Let $F_0(z) = \alpha_0^2 z^2/(1 - F(z))$. From (a) and (b) we obtain

$$G(z) = 1/\big(1 - F(z) - F_0(z)\big). \tag{10}$$

Replacing $1 - F(z)$ by $\alpha_0^2 z^2/F_0(z)$, (10) gives a quadratic equation for $F_0(z)$ which has the solution

$$F_0(z) = \left(\sqrt{1 + 4\alpha_0^2 z^2 G(z)^2} - 1\right)/2G(z). \tag{11}$$

Among the two solutions this is the proper one because $G(0) = 1$ and $F_0(0) = 0$. (c) and (d) yield $A_i(z) = 1/(F_i(z) + 1/G(z))$, and by (e),

$$F_i(z) = \alpha_i^2 z^2/(F_i(z) + 1/G(z)), \tag{12}$$

which gives, like (11),

$$F_i(z) = \left(\sqrt{1 + 4\alpha_i^2 z^2 G(z)^2} - 1\right)/2G(z) \quad \text{for } i = 1, \ldots, s. \tag{13}$$

If we write (10) in the form $G(z) = 1/(1 - \sum_{i=0}^{s} F_i(z))$ and use (11) and (13), we then obtain the proposed equation for $G(z)$. □

PROOF OF COROLLARY 1. This is proved exactly like Proposition 3 in [**3**]. For the sake of completeness, the lines of the proof are indicated.

By Theorem 2, $w = G(z)$ solves $\mathscr{F}(z,w) = 0$, where

$$\mathscr{F}(z,w) = P(zw) - w, \tag{14}$$

and this function is analytic for positive $z, w$. The radius of convergence $r$ is the smallest positive singularity of $G(z)$.

For positive real $t$, the convex curve $y = P(t)$ approaches the asymptote $y = (\sum_{x \in X}|\alpha(x)|)t - (s-1)/2$. For $0 < z < r$, $G(z)$ is positive real and can be illustrated as the ordinate of the point of intersection of the line $y = (1/z)t$ with $y = P(t)$ in the real $(t, y)$-plane. If there are two points of intersection, by continuity, the proper one is the one closer to the origin.

If $s = 0$ or $s = 1$, then for each positive $z < (\sum_{x \in X}|\alpha(x)|)^{-1}$ we find exactly one solution, and the angle of intersection is nonzero. By the implicit function theorem, applied to (14), the solution $G(z)$ is analytic at $z$. For larger positive real $z$, there is no real solution at all, hence

$$r^{-1} = \sum_{x \in X} |\alpha(x)| = \lim_{t \to \infty} (P(t)/t) = \inf\{P(t)/t \mid t > 0\}, \qquad s = 0, 1. \tag{15}$$

If $s \geq 2$, then the asymptote passes below the origin, and there is a unique positive solution $\theta$ of the equation $tP'(t) = P(t)$. As above, for positive $z < \theta/P(\theta)$ we can find an analytic solution $G(z)$ which is positive real, whereas for larger $z$ there is no such solution. Thus

$$r^{-1} = P(\theta)/\theta = \min\{P(t)/t \mid t > 0\}, \qquad s \geq 2. \quad \square \tag{16}$$

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