ON THE RIESZ DECOMPOSITION FOR MARKOV CHAINS

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Abstract. This is an addendum to Chapter 6 of the author's book [4]. The Riesz decomposition theorem for arbitrary superharmonic functions is elaborated; in the available Markov chain literature it is usually given only for positive functions.

0. Introduction

In the literature on the discrete potential theory associated with Markov chains – see e.g. Kemeny, Snell and Knapp [2], Revuz [3], Woess [4] – the Riesz decomposition theorem for superharmonic functions is usually only stated and proved for non-negative functions. Here, we explain the necessary small adaptations for dealing with functions that may be unbounded below as well as above. No originality is claimed; the purpose of the notes is to fill a small gap in the literature in a way that does not require a large amount of general or abstract potential theory.

1. Superharmonic functions

Throughout, we use the notation of [4]. We consider a Markov chain (X, P), where X is countably infinite and the transition matrix $P = \big(p(x,y)\big)_{x,y \in X}$ is substochastic.

For $f : X \to \mathbb{R}$, we let as ususal

$$Pf(x) = \sum_y p(x,y) f(y)\,, \quad x \in X\,. \tag{1.1}$$

if that sum is defined in the sense of Lebesgue integration. That is, for each x, at least one of $Pf_+(x)$ and $Pf_-(x)$ has to be finite, and then $Pf(x) = Pf_+(x) - Pf_-(x)$. Here, $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$.

For $A \subset X$ and $x \in A$, we define recursively

$$N_1(x,A) = \{y \in A : p(x,y) > 0\} \quad \text{and} \quad N_k(x,A) = \bigcup_{y \in N_1(x,A)} N_{k-1}(y,A)\,.$$

We think of $N_1(x,A)$ as the set of *forward neighbours* of x in A. When $A = X$, we just write $N_k(x) = N_k(x,X)$. For any $A \subset X$, we let

$$A^o = \{x \in A : N_1(x) \subset A\} \quad \text{and} \quad \vartheta A = A \setminus A^o\,.$$

(1.2) Remark. In classical potential theory, superharmonic functions are usually assumed to take values in $(-\infty\,, \infty]$. Typically, the value $+\infty$ occurs only only on sets of measure 0 (e.g., the pole of a Green function). In the discrete setting, this does not make sense: if u is superharmonic and $u(x) = \infty$ for some x, then irreducibility implies that $u \equiv \infty$ everywhere. Even without the assumption of irreducibility, the set $\{u = \infty\}$

Date: May 25, 2011.

2000 *Mathematics Subject Classification.* 60J45; 31C20 .

Key words and phrases. Markov chain, superharmonic functions, Riesz decomposition.

has to be a union of irreducible classes; namely, if $u(y) = \infty$ and $p^{(k)}(x,y) > 0$ then also $u(x) = \infty$.

Therefore, here we restrict attention to functions with values in $\mathbb{R}$ only.

Also, in classical potential theory, superharmonic functions are assumed to be lower semicontinuous; see [1, p. 58]. We shall need a substitute for that condition. This is part (a) of the following definition.

(1.3) Definition. Let $A \subset X$ and $u : A \to \mathbb{R}$.

(a) The function u is called *locally bounded below on A,* if for every $x \in A^o$ and $k \in \mathbb{N}$,

$$\inf\{u(y) : y \in N_k(x,A)\} > -\infty\,.$$

(b) The function is called *superharmonic on A,* if it is locally bounded below on A and

$$Pu(x) \le u(x) \quad \text{for all} \;\; x \in A^o.$$

(c) It is called *harmonic on A,* if both u and $-u$ are superharmonic on A.

Note that $Pu(x)$ is well defined for any function u that is locally bounded from below, because $Pu_- < \infty$. Also note that when the Markov chan has *finite range,* that is, $N_1(x)$ is finite for every x, then any real function u on X, resp. $A \subset X$, is locally bounded below.

Recall from [4, (2.15) & (2.16)] the restriction P_{A^o} of P to the set A^o and the associated Green kernel

$$G_{A^o}(x,y) = \sum_{n=0}^{\infty} p_{A^o}^{(n)}(x,y)\,.$$

We consider both as being defined over $A \times A$, or equivalently, over $X \times X$ with value 0 outside of $A \times A$. By a slight abuse of terminology, we say that the set A is *transient,* if $G_{A^o}(x,y) < \infty$ for all $x, y \in A$. In stochastic terms, this means that the Markov chain governed by P, when started in some point of A^o, will almost surely visist any finite subset of A^o only finitely often. In the potential theoretic sense, this means that the set A, or rather A^o, possesses a Green function. Because of discreteness of X, we will have to include both A and A^o in our considerations, instead of speaking about open sets as in classical potential theory.

In the notation of [4, Chapter 6], a function u is harmonic, resp. superharmonic on A if and only if it has this property for the modified Markov chain on A, where the outgoing transition probabilities remain the same at every point of A^o, while all points of ϑA are made absorbing:

$$\widetilde{p}_A(x,y) = \begin{cases} p(x,y)\,, & \text{if} \;\; x \in A^o\,,\; y \in A\,,\\ 1\,, & \text{if} \;\; x \in \vartheta A\,,\; y = x\,,\\ 0\,, & \text{otherwise.} \end{cases}$$

Then a function u is superharmonic on A if and only if $\widetilde{P}_A u \le u$ in every point of A. Note that

$$G_{A^o}(x,y) = \widetilde{G}_A(x,y) \quad \text{for all } x, y \in A^o\,, \tag{1.4}$$

where $\widetilde{G}_A$ denotes the ordinary Green kernel of the Markov chain $(A, \widetilde{P}_A)$.

The following is analogous to the Riesz decomposition of positive superharmonic functions, see [4, Chapter 6], although here we do not assume irreducibility nor stochasticity of P.

(1.5) Proposition. *Suppose that $u : A \to \mathbb{R}$ is superharmonic and globally bounded below on A. Then one has the unique decomposition*

$$u(x) = G_{A^o} f(x) + h(x) \quad \textit{for all } \; x \in A\,,$$

where

$$f(x) = \begin{cases} u(x) - Pu(x)\,, & \textit{if } x \in A^o\,,\\ 0\,, & \textit{otherwise,} \end{cases}$$

and h is harmonic on A.

Proof. Compare with [4, Thm. 6.43]. We work with $\widetilde{P}_A$ and $\widetilde{G}_A$. Note that $\widetilde{P}_A u(x) = u(x)$ for every $x \in \vartheta A$.

By assumption, there is $M \le 0$ (finite) such that $u \ge M$ on A. Since P is substochastic, the sequence $\bigl(\widetilde{P}_A^n u(x)\bigr)_{n \ge 0}$ is bounded below by M for every $x \in A$. Since $\widetilde{P}_A u \le u$ on A, the sequence is monotonically decreasing. Let

$$h(x) = \lim_{n\to\infty} \widetilde{P}_A^n u(x)\,,$$

so that $h \le u$ on A and $h = u$ on ϑA. By dominated convergence, $\widetilde{P}_A h = h$ on A, so that h is harmonic on A in the sense of Definition 1.3.

With $f = u - \widetilde{P}_A u \ge 0$, we get by "telescoping"

$$u - \widetilde{P}_A^{n+1} u = \sum_{k=0}^{n} P_A^k f\,.$$

Letting $n \to \infty$, we get $u - h = \widetilde{G}_A f$.

Now note that $f \equiv 0$ on ϑA, and $\widetilde{G}_A f = G_{A^o} f$ on A by (1.4). □

When u is not globally bounded below on A, the above proof fails because the decreasing sequence is not necessarily bounded below. Thus, as in the classical case, the formulation of the general Riesz decomposition a bit is more involved, but it is an immediate consequence of Proposition 1.5.

(1.6) Theorem. *Suppose that $u : X \to \mathbb{R}$ is superharmonic. Then for every $A \subset X$ on which u is golbally bounded below, one has the unique decomposition*

$$u(x) = G_{A^o} f(x) + h_A(x) \quad \textit{for all } \; x \in A\,,$$

where

$$f(x) = \begin{cases} u(x) - Pu(x)\,, & \textit{if } x \in A^o\,,\\ 0\,, & \textit{otherwise,} \end{cases}$$

and h_A is harmonic on A.

(1.7) Remarks. (a) In the classical situation, the analogous sets $A \subset \mathbb{R}^d$ are usually taken to be open with compact closure. Since superharmonic functions are assumed to be lower semicontinuous in that case, they are (globally) bounded below on such sets.

(b) The typical analogue to the classical situation would be to take A finite here. Then any superharmonic function will of course be bounded below on A. When P has finite range, this makes sense.

But otherwise, when P does not have finite range, it may well be that $A^o = \emptyset$ for any finite $A \subset X$, so that working with finite sets leads to no useful information. Instead, we can work with sets A of the form $N_k(x)$, or finite unions of such sets. They may be infinite, but when u is superharmonic on X then it is globally bounded below on such A.

Let us now consider the case when P is irreducible, has finite range, and is also stochastic. Then it is reasonable in the Riesz decomposition theorem to take finite sets A which are "connected" in the sense that the restricted transition matrix P_{A^o} is irreducible on A^o. When we now consider the Markov chain $(A, \widetilde{P}_A)$, then its essential classes are the one-point sets $\{x\}$, where $x \in \vartheta A$. Thus, case **(III)** of [4, pp 171–172] applies and yields that the harmonic function appearing in the decomposition of a superharmonic function u on A is given by

$$h_A(x) = \sum_{y \in \vartheta A} u(y)\, \nu_x(y)\,,\ x \in A^o\,,$$

where ν_x is the hitting distribution of on ϑA of the Markov chain starting in x. This is the probability measure on ϑA for which $\nu_x(y)$ is the probability that y is the first point in ϑA visited by the chain starting in x. In other words, h_A is the solution of the Dirichlet problem on A with preassigned boundary data $u(y)$, $y \in \vartheta A$. Compare this also with [4, Thm. 6.7].

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