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HARMONIC FUNCTIONS ON INFINITE GRAPHS

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SUNTO. — Su un grafo infinito, localmente finito si considera un « Laplaciano » dato da una matrice stocastica di transizione legata alla struttura del grafo. Si descrive il collegamento fra il cono delle funzioni armoniche positive e lo spazio degli « ends » del grafo. Si presentano i risultati noti per vari classi di grafi, in particolare per grafi di Cayley di gruppi infiniti.

1. - INTRODUCTION.

Let Γ be a graph with countably infinite vertex set S and unoriented edge set E. We assume that Γ is locally finite (each vertex has only finitely many neighbours) and connected (every pair of vertices is connected by some path in Γ). Γ may have loops, but no multiple edges. If x is a vertex and $[x, y] \in E$, then we write $y \sim x$, and the degree of x is $\deg(x) = \#\{y | y \sim x\}$.

We can define the *combinatorial Laplacian* of Γ, compare [Do], [DK] : if f is a real-valued function on S, then

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x)$$

More generally, we shall consider a « Laplacian » of the following type.

$$\Delta f(x) = \sum_{y} p(x, y) f(y) - f(x),$$

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where $\mathcal{P} = (p(x,y))_{x,y\in S}$ is a stochastic transition matrix over S (that is, $p(x,y) \geq 0$ and $\sum_y p(x,y) = 1$ for every x), which is related with the graph structure by the property

(**A**) $p(x,y) > 0$ if and only if $x \sim y$.

(In particular, $p(x,y) > 0$ implies $p(y,x) > 0$).

Definition. - A function h on S is called *harmonic,* if $\Delta h = 0$.

In other words, the value of h at a vertex x must be equal to the mean of the values at its neighbours, where the mean is taken with respect to the weigths $p(x,y)$, $y \sim x$. We are interested in the convex cone $\mathcal{H}^+$ of *positive harmonic functions.* If $e \in S$ is a reference vertex, then $\mathcal{B} = \{h \in \mathcal{H}^+ | h(e) = 1\}$ is a convex base of $\mathcal{H}^+$, compact in the topology of pointwise convergence (by local finiteness). Thus, in view of Choquet's theory of convex cones (see e.g. [Ph]), we want to describe the *extreme elements* of $\mathcal{B}$. The purpose of this survey is to present some results in this direction, with the attempt to reveal the interaction between positive harmonic functions and the geometry of the underlying graph structure.

2. - Random Walk and Martin Boundary.

The transition matrix $\mathcal{P}$ acts as a link between potential theoretic and probabilistic properties: it gives rise to a *random walk* on Γ. This is a Markov chain X_n, $n = 0, 1, 2, \ldots$, of S-valued random variables subject to the transition law

$$Pr\,[X_{n+1} = y \,|\, X_n = x] = p\,(x,y)\,.$$

(See [KSK] for all details concerning Markow chains). In the particular case when $\mathcal{P}$ corresponds to the combinatorial Laplacian ($p(x,y) = 1/\deg(x)$ if $y \sim x$, $p(x,y) = 0$ otherwise), $\{X_n\}$ is called the *simple random walk* on Γ [DS], [GW].

Thus, we start at a vertex of Γ and perform successive random steps along the edges. Now X_n is the random vertex at time n, and the random point where the next step will take us is chosen according to the probability distribution $p(X_n, .)$.

The n-step transition probabilities

$$Pr\,[X_{k+n} = y \,|\, X_k = x] = p^{(n)}\,(x,y)$$

do not depend on k and are just the entries of the matrix power $\mathcal{P}^n$.

The *Green function* is

$$G(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y), \qquad x, y \in S.$$

This is the expected number of visits to vertex y during the lifetime of the random walk starting from x. As Γ is connected, we have the following dichotomy. Either $G(x, y) = \infty$ for all $x, y \in S$, the walk is *recurrent*, or $G(x, y) < \infty$ for all $x, y \in S$, the walk is *transient.*

In the recurrent case, it is rather easy to verify that all positive harmonic functions are *constant* [KSK, Prop. 6-4]. There are several good criteria for recurrence, or transience respectively [Na], [Ly], [DS], [W1], [Do], in particular for the simple random walk. From now on, in addition to (**A**), we make the following assumption.

(**B**) $\mathcal{P}$ gives rise to a transient random walk.

The *Martin kernel* with respect to the reference vertex e (fixed once for all) is

$$K(x, y) = G(x, y)/G(e, y), \qquad x, y \in S$$

Note that $\Delta G(., y) = - \delta^y$, the negative Dirac mass at y, and

$$K(x, y) = \sum_w p(x, w)\, K(w, y) + \delta_y(x)/G(e, y).$$

Thus, for fixed $y \in S$, $K(., y)$ is harmonic in every point except y and has value one at e — it is a good candidate to construct elements in the base $\mathcal{B}$ of $\mathcal{H}^+$. This can be done in the following way (see [KSK, Ch. 10] for the details). One chooses a family $\{q(w) | w \in S\}$ of suitable positive weights and defines a metric on S by

$$\varrho(x, y) = \sum_{w \in S} q(w)\, |K(w, x) - K(w, y)|.$$

The completion $\bar{S}$ of S with respect to this metric is the *Martin compactification* of the random walk, and the set $M = \bar{S} \setminus S$ of new points is the *Martin boundary.* It is characterized by the following three properties.

(I) $\bar{S}$ is compact metrizable and contains S as a dense subset.

(II) The Martin kernel extends to $S \times \bar{S}$ continuously in the second variable (the extension is also denoted by $K(.,.)$).

(III) $K(., z) \equiv K(., z')$ implies $z = z'$ for $z, z' \in S$.

In particular, $\{x_n\}$ is a Cauchy sequence in (S, ϱ) if and only if $\{K(w, x_n)\}$ is Cauchy for every w in S, and the topology of $\bar{S}$ does not depend on the particular choice of the weights $q(w)$. Furthermore, by local finiteness S is discrete in $\bar{S}$ and $K(., z)$ is in $\mathcal{H}^+$ for every z in M. In other words, the points of M can be considered as the « directions of convergence » of the Martin kernel in the second variable.

Now, the *Poisson-Martin representation theorem* says that for every h in $\mathcal{H}^+$ there is a positive Borel measure ν_h on M such that

$$h(x) = \int_M K(x, z)\, \nu_h(dz) \qquad \text{for every } x \text{ in } S.$$

Thus, every extreme point of $\mathcal{B}$ (that is, a point which is not a proper convex combination of two other elements of $\mathcal{B}$, see [KSK, L. 10-30]) is of the type $K(., z)$ with $z \in M$. In general, not every point of M need be extreme in this sense. If M_e is the set of extreme points of M, then the integral representation becomes unique when we consider Borel measures on M_e only. Transience can also be expressed in the following way: for every finite $U \subset S$, with probability one the random walk does not return to U after some finite number of visits. Thus, $\{X_n\}$ tends to « infinity » in Γ almost surely, and the study of $\mathcal{H}^+$ it to some extent linked with the question of how to describe a « good » model of the points attained by $\{X_n\}$ « at infinity ». Indeed, there is an M-valued random variable X_∞ such that

$$\lim_{n \to \infty} X_n = X_\infty \quad \text{almost surely.}$$

The hitting probability on M is defined by

$$\nu(E) = Pr[X_\infty \in E | X_0 = e], \; E \text{ a Borel set in } M.$$

It turns out that ν represents the harmonic function with constant value one, $\nu = \nu_1$. The pair $(\mathrm{Supp}(\nu), \nu)$ is called the *Poisson boundary.* If h is a bounded harmonic function, then there is a function φ in $L^\infty(\nu)$ such that

$$h(x) = \int_M K(x, z)\, \varphi(z)\, \nu(dz) \quad \text{for every } x \text{ in } S.$$

There are several equivalent approaches to Poisson boundary, in particular for random walks on *groups* (Cayley graphs); see [KV] for a good overview and many results. For historical remarks and references concerning Martin boundary, confer the Notes of [KSK].

Still, it is an abstract construction that leads to the Martin boundary. In the next chapter we describe a compactification of the graph which arises naturally from a « geometrical » point of view.

3. - The space of ends of a graph.

A *(simple) path* in Γ is a finite or one-sided infinite sequence of successively contiguous vertices (without repetitions). The *length* of a finite path is its number of edges. The *distance* $d(x, y)$ of two vertices is the length of a shortest path between x and y.

Two infinite simple paths are *equivalent* if, for any finite subset U of S, they can be connected by some finite path lying outside U. An *end* of Γ is an equivalence class of infinite simple paths. The set of all ends is denoted by Ω. If we remove a finite subset U of S, then $\Gamma \setminus U$ splits into finitely many connected components. Let B be the set of vertices of one such component, and adjoin to B every end which can be represented by some infinite simple path lying entirely within B. In this way, we obtain a set $\overline{B} \subset S \cup \Omega$. For $z \in (S \setminus U) \cup \Omega$, we write $C(U, z) = \overline{B}$ if $z \in \overline{B}$: $C(U, z)$ is the component of z after removing U. Varying U (finite) and z, we obtain a basis of a topology, which makes $S \cup \Omega$ a totally disconnected, compact Hausdorff space with Ω as a compact subspace and S as a discrete, open dense subset.

With different notation, this concept was originally introduced by [Fr, 1944], but many graph theorists seem to have been unaware of this earlier work. Later on, the ends where independently introduced by [H1, 1964] and extensively studied, see e.g. [H2]. The topology has also been reintroduced and studied by [Ju], [Po].

At this point, one may naturally ask if the Martin boundary of our random walk on Γ and the space of ends of Γ can be compared. The answer is given by the following theorem, compare [Ta] for the nondiscrete setting.

THEOREM 1 [PW2]. - The identity mapping on S extends to a continuous surjection $\tau : \overline{S} \rightarrow S \cup \Omega$, which maps the Martin boundary $M = \overline{S} - S$ onto Ω.

However, in general τ will not be a homeomorphism, the space of ends usually is to small, see the examples below. Next, we present sufficient conditions under which τ becomes actually a homeomorphism. For a finite subset U of S, its *diameter*, diam(U), is defined as usual with respect to the graph metric d.

Furthermore, we denote

$$\delta(U) = \min \{p(x, y) > 0 \mid x, y \in \tilde{U}, \quad x = y\},$$

where $\tilde{U}$ is the set of all vertices which lie on a shortest path between two elements of U. If $\# U = 1$ then we set $\delta(U) = 1$.

Every end $\omega \in \Omega$ can be described by a sequence $\{U_n\}$ of finite subsets of S with the following properties.

(i) $C(U_{n-1}, \omega) \supseteq U_n \cup C(U_n, \omega)$, and

(ii) $\{C(U_n, \omega)\}$ is a neighbourhood basis at ω.

Such a sequence is called *contracting towards* ω, see Figure 1. It will be called *strictly contracting*, if in addition

(iii) $\tilde{U}_{n-1} \cap U_n = \varnothing$ for every n.

If $\{U_n\}$ is contracting towards ω, then so is every subsequence, and there is a strictly contracting subsequence. If we denote $B_n = \{x \in S \mid d(x, e) \leq n\}$ and $U_n(\omega) = C(B_{n-1}, \omega) \cap B_n$, then $\{U_n(\omega)\}$ is contracting towards ω.

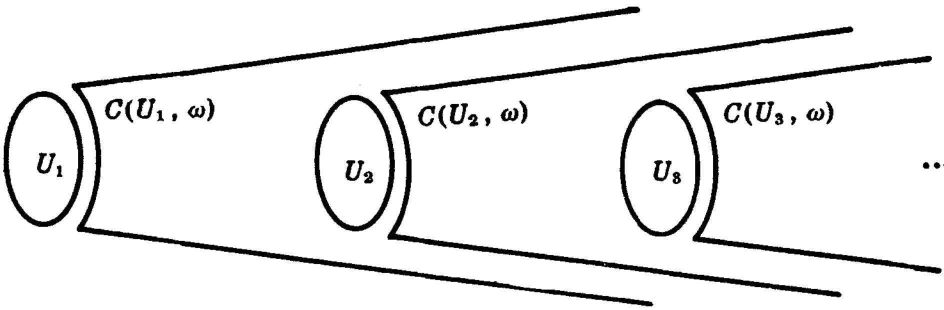

Figure 1

Under assumptions (**A**) and (**B**), the following result holds.

THEOREM 2 [PW2]. - Suppose that for each $\omega \in \Omega$ there is a sequence $\{U_n\}$ of finite sets, strictly contracting towards ω, such that

$$\sum_n \delta(U_n)^{\operatorname{diam}(U_n)} = \infty .$$ Then

(a) the Martin boundary of the random walk coincides with the space of all ends of the graph (τ is a homeomorphism), and

(b) all ends are extreme points of the boundary.

By (*b*), the Poisson-Martin integral representation of positive harmonic functions is unique over Ω. If we define for $\omega \in \Omega$

$$\operatorname{diam}(\omega) = \inf \liminf_{n \to \infty} \operatorname{diam}(U_n), \qquad \delta(\omega) = \inf \liminf_{n \to \infty} \delta(U_n),$$

where the infimum ranges over all sequences $\{U_n\}$ which contract towards ω, then Theorem 2 yields the following.

Corollary. - If $\operatorname{diam}(\omega) < \infty$ and $\delta(\omega) > 0$ for all $\omega \in \Omega$, then (a) and (b) hold.

In particular, if there is an upper bound for the degrees of the vertices of Γ, and if every end has finite diameter, then the Martin boundary of the simple random walk coincides with the space of ends. As a by-product, one obtains that Ω is metrizable in this case. Compare this with [Ju, Thm. 7]!

4. - Further results and examples, random walks on groups.

As a particular case of random walks on locally finite graphs, much attention has been devoted to walks on finitely generated *groups.* If $\mathfrak{G}$ is a group with finite generating set A and identity element e, then its Cayley graph $\Gamma(\mathfrak{G}, A)$ has vertex-set $\mathfrak{G}$, its unoriented edges are $[x, xa] = [xa, x]$, where $x \in \mathfrak{G}$ and $a \in A$. Speaking of a random walk on $\mathfrak{G}$, we always assume that the transition matrix is $\mathfrak{G}$-invariant: $p(x, y) = p(wx, wy)$ for all $w, x, y \in \mathfrak{G}$. The walk is then described by a probability measure μ, $\mu(x) = p(e, x)$, and its convolution powers $\mu^{(n)}$. Considered as a walk on $\Gamma(\mathfrak{G}, A)$, one usually does not assume property $(\mathbf{A})$ of Sect. 1; it suffices that μ has *finite support* and is *irreducible*: $\mathrm{supp}(\mu)$ generates $\mathfrak{G}$ as a semigroup. (For relaxing condition $(\mathbf{A})$ in the general setting, see [PW2]). For the simple random walk on $\Gamma(\mathfrak{G}, A)$, μ is the equidistribution on $A \cup A^{-1}$.

The longstanding problem to classify all *recurrent groups* has been settled recently [Va]: if μ is symmetric ($\mu(x) = \mu(x^{-1})$ for every x) then it gives rise to a recurrent walk if and only if $\mathfrak{G}$ has either $\mathbb{Z}$ or $\mathbb{Z}^2$ as a subgroup of finite index.

For a group, we always use the identity element e as the reference vertex in the Martin kernel. An exponential on $\mathfrak{G}$ is a homomorphism of $\mathfrak{G}$ into the multiplicative group of positive real numbers.

(1) Abelian groups.

For an Abelian group, the extreme points of the basis $\mathcal{B} = \{h \in \mathcal{H}^+ | h(e) = 1\}$ are precisely the exponentials ψ on $\mathfrak{G}$ satisfying $\sum_x \mu(x)\psi(x) = 1$, see [CD]. By [Mr], this carries over to nilpotent groups.

For the *integer lattices*, $\mathfrak{G} = \mathbb{Z}^r$ (Fig. 2a), a geometrical description of the Martin boundary was given in [NS]. Set $\bar{\mu} = \sum_x x\mu(x)$, the mean vector. (Note that we assume $\mathrm{supp}(\mu)$ finite. This can be replaced by a moment condition.)

Case 1. - If $\bar{\mu} = 0$, the identity element, then the walk is recurrent for $r \leq 2$. For $r \geq 3$, it is transient and the Martin boundary consists of a single point [Sp]. Thus for any r, all positive harmonic functions are constant [CD].

Case 2. - If $\bar{\mu} \neq 0$, then the walk is transient for any r. If $r = 1$ then the Martin boundary consists of two points, $M = \{-\infty, +\infty\}$. The Poisson boundary is $\{+\infty\}$ if $\bar{\mu} > 0$ and $\{-\infty\}$ if $\bar{\mu} < 0$.

If $r > 1$ then the Martin boundary is homeomorphic with the unit sphere $\mathfrak{S}_{r-1}$ in r dimensional space. The Martin topology can be obtained by completing $\mathbb{Z}^r$ with respect to the metric $\varrho(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$. The directions of convergence of the Martin kernel can be described in the following way [NS]. For $u \in \mathbb{R}^r$, set

$$\varphi(u) = \sum_x \exp(x \cdot u)\, \mu(x), \quad \operatorname{grad} \varphi(u) = \sum_x x \exp(x \cdot u)\, \mu(x),$$

where $x \cdot u$ is the inner product of u with $x \in \mathbb{Z}^r$.

THEOREM 3 [NS]. - If $z \in \mathfrak{S}_{r-1}$ and $\{y_n\}$ is a sequence in $\mathbb{Z}^r$ such that $|y_n| \to \infty$ and $y_n / |y_n| \to z$, then for any $x \in \mathbb{Z}^r$,

$$\lim K(x, y_n) = \exp(x \cdot u),$$

where u is the unique point in $\mathbb{R}^r$ satisfying

$$\varphi(u) = 1 \text{ and } \frac{\operatorname{grad} \varphi(u)}{|\operatorname{grad} \varphi(u)|} = z.$$

As $\mathbb{Z}^r$ has one end for $r \geq 2$, this theorem yields many simple examples where the Martin boundary is strictly larger than the space of ends. The Poisson boundary, however, has only one point, which corresponds to the direction $z_0 = \bar{\mu}/|\bar{\mu}| \in \mathfrak{S}_{r-1}$ of the mean vector. The corresponding exponential has constant value one, and all bounded harmonic functions are constant.

$\mathbb{Z}^2$

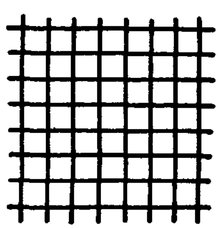

Figure 2a

$\mathbb{F}_2$

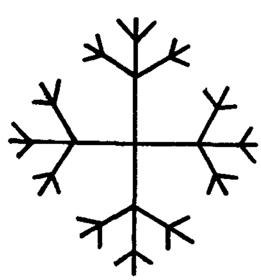

Figure 2b

$\mathbb{Z}_2 * \mathbb{Z}_3$

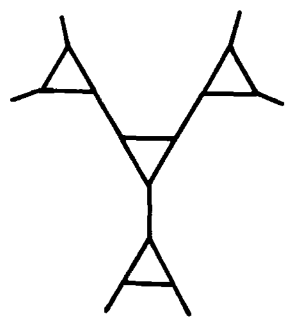

Figure 2c

(2) Free groups and trees.

Trees are typical examples of graphs for which Theorem 2 and the Corollary apply. In particular, if $\mathfrak{F}_r$ is the *free group* with set of free generators $A = \{a_1, ..., a_r\}$ ($r \geq 2$; the case $r = 1$ is covered by (*1*)), then the Cayley graph $\Gamma(\mathfrak{F}_r, A)$ is a homogeneous tree of degree $2r$ (Fig. 2b). Any group-invariant, irreducible random walk on $\mathfrak{F}_r$ is transient [Ke], [DG]. The first result concerning the Martin boundary of $\mathfrak{F}_r$ is due to [DM], who considered invariant walks with property (**A**) on $\Gamma(\mathfrak{F}_r, A)$, in other words, the probability governing the walk is supported by $A \cup A^{-1}$. In [DM], the Martin boundary is characterized as the set of one-sided infinite reduced words over $A \cup A^{-1}$, which is just another description of the space of ends. This result has been extended to all finitely supported invariant walks by [De].

On the other and, [Ca] has considered random walks with properties (**A**) and (**B**) on arbitrary infinite trees, without group-invariance. In this case, the Martin kernel is locally constant on the space of ends. A common generalization of [DM], [De] and [Ca] is given in [PW1], where walks with properties (**A**) and (**B**) are considered on graphs Γ having a *uniformly spanning tree*: this is a tree T with the same vertex set as Γ, such that the metric d_T of T is equivalent with the metric d of Γ ($L^{-1} d_T \leq d \leq L d_T$ for some constant $L > 0$). Again, (**A**) can be relaxed, but in addition, a nonzero lower bound for the nonzero entries of $\mathcal{P}$ is assumed. Under these conditions one also has an analytic version of a « nontangential Fatou theorem », see [PW1].

One may ask which groups allow the application of the Corollary of Theorem 2, i.e., which groups have a Cayley graph whose ends all have finite diameters. These turn out to be precisely the groups having a free subgroup of finite index [W3]. They include, in particular, every free product of finitely many finite groups, for example, the modular group $\mathbb{Z}_2 * \mathbb{Z}_3$ (Fig. 2c). For any invariant, irreducible, finitely supported walk on such a group, Poisson boundary and Martin boundary coincide with the space of ends [De, PW1].

On the other hand, [PW1] exhibit an example of a walk on a tree, invariant under the fixed point-free, but nontransitive action of a finite extension of a free group, where the Poisson boundary is strictly smaller than the Martin boundary.

(3) Fuchsian groups.

A Fuchsian group $\mathfrak{G}$ is a discrete group of conformal automorphisms of the complex unit disc $\mathfrak{D}$. Its limit set Λ is the set of accumulation points of an orbit $\mathfrak{G}\, z_0$, $z_0 \in \mathfrak{D}$. It is independent of z_0 and contained in the unit circle $\mathfrak{S}_1$. In [S2], the Martin boundary is determined for finitely supported, irreducible walks on Fuchsian groups which are non-exceptional, see [S1], [S2] for the definition.

THEOREM 4 [S2]. - If $\mathfrak{G}$ is a non-exceptional Fuchsian group, then there is a continuous surjection $\pi : M \to \Lambda$ which fails to be injective at most at countably many points, where it is two to one. The Poisson boundary coincides with M.

In fact, when $\mathfrak{G}$ is considered as an abstract group, its action on $\mathfrak{D}$ can be always realized in such a way that π is a homeomorphism. By [S2], one could also describe M as the set of « admissible shortest paths to infinity » in a particular Cayley graph of $\mathfrak{G}$. Among the groups covered by Theorem 4, there is for example the group with the presentation

$$\langle a_1, a_2, a_3, a_4 \,|\, a_1 a_2 a_1^{-1} a_2^{-1} a_3 a_4 a_3^{-1} a_4^{-1} = e \rangle \quad \text{(compare [Mg], p. 98).}$$

Its Cayely graph with respect to $A = \{a_1, a_2, a_3, a_4\}$ is made up as the square lattice; instead of four squares, eight octagons meet at each vertex. Note that it has only one end.

(4) Free products.

For certain random walks on a free product $\mathfrak{G} = \mathfrak{G}_1 * \mathfrak{G}_2$ of two (or finitely many) groups, one can describe the Martin boundary in terms of the boundaries of walks on the $\mathfrak{G}_i$ [W2]. In this case, the space Ω of all ends of $\mathfrak{G}$ (the Cayley graph of $\mathfrak{G}$) has an uncountable dense subset Ω_0 of ends with finite diameter. For the mapping of Theorem 1, $\# \tau^{-1}(\{\omega\}) = 1$ for ω in Ω_0, but it may happen that $\# \tau^{-1}(\{\omega\}) = \infty$ for ω in $\Omega \setminus \Omega_0$.

$$\text{For } \mathbb{Z}^2 * \mathbb{Z}^2 = \langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_1^{-1} a_2^{-1} = a_3 a_4 a_3^{-1} a_4^{-1} = e \rangle,$$

the simple random walk on $\Gamma(\mathfrak{G}, A)$, $A = \{a_1, a_2, a_3, a_4\}$, is such an example. Again, the Poisson boundary coincides with the Martin boundary, compare [Ka].

(5) For other approaches and results concerning boundaries in a similar context, see for example [Fu], [AC], [KV], [Le], [Ba]. Of course, the references do not cover all results.

SUMMARY. — On an infinite, locally finite graph, a « Laplacian » is considered which is given by a stochastic transition matrix linked with the graph structure. We describe the interplay between the cone of positive harmonic functions and the space of ends of the graph. We give a survey of known results for several classes of graphs, in particular for Cayley graphs of infinite groups.

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