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Functional iterations and periodic oscillations for simple random walk on the Sierpiński graph

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Abstract

We use methods from asymptotic combinatorics and functional iterations to give a rigorous proof of the fluctuating behaviour of the n-step transition probabilities for the simple random walk on the Sierpiński graph. © 1997 Elsevier Science B.V.

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1. Introduction

The Sierpiński gasket is a well known planar fractal which has been studied from different points of view since 1915, when Sierpiński introduced it as an example of a curve all of whose points are ramification points. Since the 1980s a notion of Brownian motion on this fractal (and later other "nested fractals", cf. Lindstrøm, 1990) has been developed. For an excellent introduction to this subject we refer to Barlow and Perkins (1988). The diffusion on fractals is defined as the weak limit of properly chosen rescalings of the simple random walk $(X_n)_{n\geqslant 0}$ on the "Sierpiński graph".

Besides the study of Brownian motion on this fractal, a theory of calculus has been developed for functions on the Sierpiński gasket. The Laplacian as the infinitesimal generator of the diffusion on nested fractals (cf. Lindstrøm, 1990) has been studied extensively. An equivalent approach to study this operator is to consider the limit of transition operators on graphs approximating the fractal (cf. Kigami, 1989). We refer to Barlow and Perkins (1988), Kigami and Lapidus (1993) and Lapidus (1994) for a detailed discussion of the distribution of eigenvalues of the Laplacian for a class of compact self-similar fractals (including the gasket).

This note is devoted to a rigorous study of the asymptotic behaviour of the n-step transition probabilities on the Sierpiński graph $\mathscr{G}$. A detailed construction of this graph is given in Barlow and Perkins (1988), so that we just refer to it as in Fig. 1.

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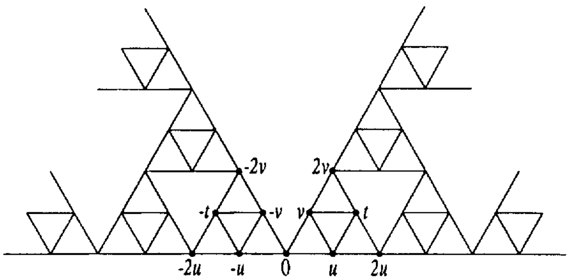

Fig. 1. The Sierpiński graph $\mathscr{G}$ around the origin.

The one-step transition probabilities are $p(x,y)=\frac{1}{4}$ whenever x and y are neighbours (and zero, otherwise). The n-step transition probabilities

$$p^{(n)}(x,y)=\mathbb{P}[X_n=y\,|\,X_0=x] \tag{1.1}$$

have been studied by various methods and with varying degrees of rigor. Rammal (1984) and Rammal and Toulouse (1983) use a physical argument to derive a functional equation for the Green function. In Friedberg and Martin (1986), a heuristic (using implicitly a Tauberian argument, which cannot be used here, as we shall see below) is presented in order to explain that the transition probabilities behave like $n^{-\log 3/\log 5}$ when $n\to\infty$.

Recently, this has been (rigorously) confirmed by Jones (1996). More precisely, inspired by the methods of Barlow and Perkins (1988), Jones (1996) gives upper and lower bounds for the transition densities of continuous-time random walk on the graph which are uniform in x and y; the same methods apply for discrete time. Both in Barlow and Perkins (1988) and Jones (1996) the constants involved in the upper and lower bounds are different.

Our result shows why these constants had to be different. We shall prove that

$$p^{(n)}(x,y)=n^{-\log 3/\log 5}F(\log n/\log 5)(1+\mathrm{o}(1)), \tag{1.2}$$

as $n\to\infty$, where F is a continuous, non-constant periodic function of period 1. We use the method of "singularity analysis" due to Flajolet and Odlyzko (1990). This method will also give an expression for the Fourier-coefficients of the function F, which in principle makes it possible to compute them numerically. This periodicity phenomenon seems to reflect the self similarity of the fractal. We remark here that a similar phenomenon occurs in the computation of the Hausdorff measure of certain geometrically defined subsets of the (finite) Sierpiński gasket in Grabner (1993) and Grabner and Tichy (1997). However, to our knowledge this is the first (non-trivial) example of a random walk on a locally finite graph where oscillations of this type have been detected, compare with Section 6 in Woess (1994). On the other hand, see Cartwright (1988) for a non locally finite example related with the Cantor distribution. Similar periodicity phenomena occur repeatedly in the study of diffusion on fractals: the main term in the asymptotics of the counting function of eigenvalues of the Laplacian contains a periodic factor (cf. Barlow and Kigami, 1997; Kigami and Lapidus, 1993; Lapidus, 1994) and branching processes used to model hitting times for the Brownian motion exhibit (tiny) periodic fluctuations (cf. Barlow and Perkins, 1988; Biggins and Bingham, 1991).

2. A functional equation for the Green function

The *Green function* of the random walk is the probability generating function (PGF) of the n-step transition probabilities from x to y in $\mathscr{G}$:

$$G(x,y|z)=\sum_{n=0}^{\infty} p^{(n)}(x,y)z^n, \quad z\in\mathbb{C}.$$

We first use elementary path-arguments to derive the same functional equation as in Rammal (1984). Viewing $\mathscr{G}$ as in Fig. 1 as a graph in the plane, $2\mathscr{G}$ denotes the graph obtained by multiplying each point of $\mathscr{G}$ by two. It is isomorphic with $\mathscr{G}$. Its vertices are also vertices of $\mathscr{G}$, and in $\mathscr{G}$, the subset $2\mathscr{G}$ is recurrent (visited by the random walk infinitely often with probability one).

A *path* in $\mathscr{G}$ is simply a sequence $\omega=[x_0,x_1,\ldots,x_n]$ such that $x_{i+1}\sim x_i$ for all i ($\sim$ denotes neighbourhood). The length of ω is $|\omega|=n$. This includes paths of length zero. We write $\Omega(x,y)$ for the set of all paths from x to y, and $\Lambda(x,y)$ for those paths from x to y which meet y only at the end. For a path ω, we define its *weight* $W(\omega|z)=z^{|\omega|}$, where $z\in\mathbb{C}$, and for a set Ω of paths, we set $W(\Omega|z)=\sum_{\omega\in\Omega}W(\omega|z)$. Then we have

$$G(x,y|z)=W(\Omega(x,y)|z/4).$$

Now let $x,y\in\mathscr{G}$ and consider a path $\omega=[2x=x_0,x_1,\ldots,x_n=2y]$ in $\mathscr{G}$. Define $\tau_j=\tau_j(\omega)$ by

$$\tau_0=0 \text{ and for } j\geqslant 1 \quad \tau_j=\min\{i>\tau_{j-1}: x_i\in 2\mathscr{G}, x_i\neq x_{\tau_{j-1}}\}, \tag{2.1}$$

$0\leqslant j\leqslant k$, where $k=k(\omega)$ is the maximal index for which the last set is nonempty; $\tau_k\leqslant n$ and $x_{\tau_k}=2y$. The *shadow* of ω in $2\mathscr{G}$ is $[x_{\tau_0},\ldots,x_{\tau_k}]$, which in $\mathscr{G}$ corresponds to

$$\sigma(\omega)=[\tfrac{1}{2}x_{\tau_0},\ldots\tfrac{1}{2}x_{\tau_k}]\in\Omega(x,y).$$

For vertices $x,y\in\mathscr{G}$, $x\sim y$, we now define

$$\Omega_0(2x)=\sigma^{-1}[x]=\{\omega\in\Omega(2x,2x):k(\omega)=0\}$$

and

$$\Lambda_1(2x,2y)=\sigma^{-1}[2x,2y]\cap\Lambda(2x,2y)=\{\omega\in\Lambda(2x,2y):k(\omega)=1\}.$$

Lemma 1. (a) $W(\Omega_0(2x)|z/4)=1+f(z)$, *where* $f(z)=2z^2/(8-2z-3z^2)$.
(b) $W(\Lambda_1(2x,2y)\,|\,z/4)=\frac{1}{4}\phi(z)$, *where* $\phi(z)=z^2/(4-3z)$.

Proof. (a) $\Omega_0(2x)$ consists of all paths starting and ending at $2x$ which do not leave the subgraph $\mathscr{G}_0=\mathscr{G}_0(2x)$ of $\mathscr{G}$ shown in Fig. 2. For different x, these graphs are

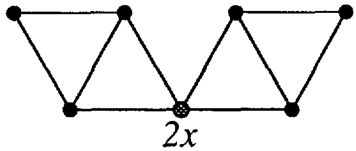

Fig. 2. The graph $\mathscr{G}_0$.

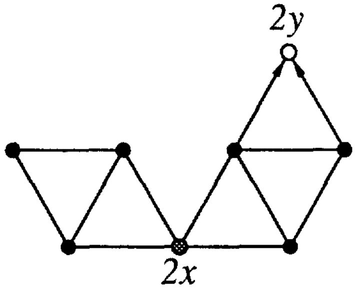

Fig. 3. The graph $\mathscr{G}_1$.

isomorphic. If A_0 is the adjacency matrix of $\mathscr{G}_0$, then the element at position $(2x,2x)$ of A_0^n is the number of paths in $\mathscr{G}_0$ starting and ending at $2x$. Thus, $W(\Omega_0(2x)|\frac{z}{4})$ is the $(2x,2x)$-element of $(I-(z/4)A_0)^{-1}$, where I denotes the identity matrix. We omit the elementary computation, which can be simplified by using the symmetries of $\mathscr{G}_0$.

(b) Analogously, $\Lambda_1(2x,2y)$ consists of all paths which start at $2x$, meet $2y$ only at the end and do not leave the graph $\mathscr{G}_1=\mathscr{G}_1(2x,2y)$ shown in Fig. 3. Once more the graphs are isomorphic for different pairs of neighbours.

Write A_1 for the adjacency matrix of $\mathscr{G}_1$, and $G_1(u,v|z)$ for the element at position (u,v) of $(I-(z/4)A_1)^{-1}$. Then, using the fact that every path from $2x$ to $2y$ decomposes into a path in $\Lambda(2x,2y)$ and a path in $\Omega(2y,2y)$,

$$W(\Lambda_1(2x,2y)|z/4)=G_1(2x,2y|z)/G_1(2y,2y|z),$$

which is equal to $\phi(z)$ by routine calculations. [As a matter of fact, we used *Maple* for this task.] □

Let $x,w,y\in\mathscr{G}$. If $\omega_1\in\Omega(x,w)$ and $\omega_2\in\Omega(w,y)$ then we can join the two and obtain a path $\omega=\omega_1\circ\omega_2\in\Omega(x,y)$. If $A\subset\Omega(x,w)$ and $B\subset\Omega(w,y)$ then we write $A\circ B=\{\omega_1\circ\omega_2:\omega_1\in A,\omega_2\in B\}$. We get $W(A\circ B|z)=W(A|z)\,W(B|z)$.

Now take $\omega=[x=x_0,x_1,\dots,x_n=y]\in\Omega(x,y)$. Then we can decompose

$$\sigma^{-1}(\omega)=\Lambda_1(2x,2x_1)\circ\Lambda_1(2x_1,2x_2)\circ\cdots\circ\Lambda_1(2x_{n-1},2y)\circ\Omega_0(2y).$$

Therefore, using Lemma 1,

$$W(\sigma^{-1}(\omega)|z/4)=\left(\frac{\phi(z)}{4}\right)^n(1+f(z)). \tag{2.1}$$

On the other hand, $\sigma^{-1}(\Omega(x,y))=\Omega(2x,2y)$. We obtain

$$W(\Omega(2x,2y)|z/4)=\sum_{\omega\in\Omega(x,y)} W(\sigma^{-1}(\omega)|z/4)=(1+f(z))W\left(\Omega(x,y)\;\middle|\;\frac{1}{4}\phi(z)\right).$$

We have proved the following.

Proposition 1. $G(2x,2y|z)=(1+f(z))G(x,y|\phi(z))$.

Remark. We have chosen to deduce Proposition 1 via the combinatorics of paths in order to underline its elementary nature in a way which can be easily understood not only by probabilists, but also by physicists, combinatorialists, and others. (2.1) is the "substitution construction" for generating functions, see Goulden and Jackson (1983). The arguments can of course be shortened slightly by using stopping times, compare with Barlow and Perkins (1988) and Jones (1996). If we write $(X_n)_{n\geqslant 0}$ for the random walk (sequence of $\mathscr{G}$-valued random values), then (2.1) corresponds to considering the stopping times

$$t_0=0,\quad t_j=\min\{n>t_{j-1}:X_n\in 2\mathscr{G},X_n\neq X_{t_{j-1}}\},$$

where it is supposed that $X_0\in 2\mathscr{G}$. Lemma 1(b) says that the increments t_j-t_{j-1} are i.i.d. with PGF

$$\mathbb{E}(z^{t_1})=\phi(z)$$

(which is of course known). Furthermore, writing

$$s_j=\max\{n<t_j:X_n=X_{t_{j-1}}\},\quad j\geqslant 1,$$

Lemma 1(a) implies that the differences s_j-t_{j-1} are also i.i.d. with PGF

$$\mathbb{E}(z^{s_1})=1+f(z).$$

3. Functional iterations and singularity analysis

In order to prove (1.2) for the return probabilities to the origin, we now work with the functional equation of Proposition 1 for $x=y=0$. It is similar to the equation studied in Odlyzko (1982), the only difference being that Odlyzko's equation is additive, whereas ours is multiplicative. Nevertheless, the same ideas apply for the solution. We first derive a formal power series expression for $G(z)=G(0,0|z)$ by iterating the functional equation of Proposition 1, which becomes

$$G(z)=(1+f(z))\,G(\phi(z)). \tag{3.1}$$

Notice that the Taylor expansion of $\phi(z)$ starts with z^2, and that $G(0)=1$. Thus we have

$$G(z)=\lim_{N\to\infty}\prod_{n=0}^{N-1}(1+f(\phi^{(n)}(z)))G(\phi^{(N)}(z))=\prod_{n=0}^{\infty}(1+f(\phi^{(n)}(z))), \tag{3.2}$$

where $\phi^{(n)}(z)$ is the nth iterate of $\phi(z)$, i.e.

$$\phi^{(0)}(z) = z, \qquad \phi^{(n+1)}(z) = \phi^{(n)}(\phi(z)).$$

In order to prove that this product converges in "almost all" points of the complex plane, we have to investigate the behaviour of the iterates $\phi^{(n)}(z)$. This will be done by the following lemma.

Lemma 2. *The sequence $\phi^{(n)}(z)$ tends to 0 for all z in the complex plane except for a Cantor subset of $(-\infty, -4] \cup [1, \infty)$. Convergence is quadratic for $|z| < 1$. Furthermore, 1 is a repelling fixed point of ϕ.*

Proof. We first notice that for $\psi(z) = 1/z$ we have

$$\tilde{\phi}(z) := \psi \circ \phi \circ \psi(z) = 4z^2 - 3z.$$

Thus it suffices to prove that for all $z \in \mathbb{C} \setminus [-\frac{1}{4}, 1]$ the iterates of $\tilde{\phi}$ diverge to ∞ and to study the points in the interval $[-\frac{1}{4}, 1]$.

It is immediate that

$$|4z^2 - 3z| > |z| \quad \text{for } |z| > 1 \quad \text{and} \quad |4z^2 - 3z - 1| > |z - 1| \quad \text{for } |z - 1| > \tfrac{3}{2},$$

which implies divergence to ∞ for the iterates on the union of the two regions $\{z : |z| > 1\}$ and $\{z : |z - 1| > \frac{3}{2}\}$. For the remaining region $\{z : |z - 1| \leqslant \frac{3}{2}, |z| \leqslant 1\}$ we note that $\Im \tilde{\phi}(x + iy) = (2x - 3)y$ which implies $|\Im \tilde{\phi}(z)| > |\Im z|$ for $\Im z \neq 0$ and therefore divergence outside the interval $[-\frac{1}{2}, 1]$. It is easy real analysis to see that the iterates of $\tilde{\phi}$ diverge on $[-\frac{1}{2}, -\frac{1}{4})$. Theorem 9.8.1 in Beardon (1991) yields the statement on the nature of the Julia set of ϕ. The quadratic convergence comes from the fact that $\phi(z)$ starts with z^2. We have $|\phi(z)| < |z|^2$ for $|z| < 1$. Finally, $\phi(1) = 1$ and $\phi'(1) = 5$. □

It is now an immediate consequence of Lemma 2 that the product expansion (3.2) converges uniformly on all compact subsets of the Fatou set of ϕ.

The next objective in order to prove (1.2) is the analysis of the singularity $z = 1$ of (3.2). We summarize the result in the following lemma.

Proposition 2. *The function $G(z)$ has the local singular expansion*

$$G(z) = (1 - z)^{\eta - 1} K\left(\frac{\log(1 - z)}{\log 5}\right)(1 + \mathcal{O}_{\alpha,\varepsilon}(|z - 1|^{1-\varepsilon}))$$

for $|\arg(1 - z)| \leqslant \alpha < \pi$ and for all $\varepsilon > 0$, where $\eta = \log 3/\log 5$ and $K(s)$ is a periodic function of period 1, which is holomorphic in the strip $|\Im s| < \pi/\log 5$.

Proof. We imitate the construction used in Odlyzko (1982) to study the nature of the singularity. We substitute

$$G(z) = \tilde{G}(z)(1 - z)^{\eta - 1} \tag{3.3}$$

into (3.1). This yields the functional equation

$$\tilde{G}(z)=(1+f(z))\left(\frac{1-z}{1-\phi(z)}\right)^{-\eta+1}\tilde{G}(\phi(z)) \tag{3.4}$$

and by the same arguments as above the overconvergent product expansion

$$\tilde{G}(z)=\prod_{k=0}^{\infty}(1+f(\phi^{(k)}(z)))\left(\frac{1-\phi^{(k)}(z)}{1-\phi^{(k+1)}(z)}\right)^{-\eta+1}.$$

We notice that $\phi(z)$ has a local inverse around $z=1$ ($\phi'(1)=5\neq 0$) which we denote by $\phi^{(-1)}(z)$. This function has an attracting fixed point at $z=1$. We introduce the functions

$$G^*(z)=\prod_{k=-\infty}^{-1}(1+f(\phi^{(k)}(z)))\left(\frac{1-\phi^{(k)}(z)}{1-\phi^{(k+1)}(z)}\right)^{-\eta+1},$$

$$H(z)=\tilde{G}(z)G^*(z). \tag{3.5}$$

G^* is holomorphic in some neighbourhood of $z=1$ and therefore $H(z)$ is holomorphic in $\{z\in\mathbb{C}\backslash[1,\infty):|z-1|<c\}$.

By the general theory of functional iterations (cf. Beardon, 1991), there exists a function $\xi(z)$ holomorphic in a neighbourhood of $z=1$ such that

$$\xi\circ\phi^{(-1)}\circ\xi^{(-1)}(z)=1+\tfrac{1}{5}(z-1)\quad\text{and}\quad\xi(1)=\xi'(1)=1.$$

We use this function to define $K(z)=H(\xi^{(-1)}(z))$, which is a holomorphic function in $\{z\in\mathbb{C}\backslash[1,\infty):|z-1|<c'\}$. Furthermore, this function satisfies

$$K(z)=K(1+\tfrac{1}{5}(z-1)). \tag{3.6}$$

The last relation can be used to give an analytic continuation of $K(z)$ to $\mathbb{C}\backslash[1,\infty)$. Since K is a periodic function in $\log(1-z)/\log 5$ which is holomorphic in $|\arg(1-z)|<\pi$, it has a Fourier expansion

$$K(z)=\sum_{k=-\infty}^{\infty}a_k\exp\left(-2k\pi i\frac{\log(1-z)}{\log 5}\right), \tag{3.7}$$

where the coefficients satisfy

$$a_k=\mathcal{O}_\varepsilon\left(\exp\left(-\left(\frac{2\pi^2}{\log 5}-\varepsilon\right)|k|\right)\right)\quad\text{for all } \varepsilon>0$$

(this explains the exponential decay of the Fourier coefficients of the function $\phi(q)$ in Friedberg and Martin, 1986, (2.8)). The coefficients are given by the integral

$$a_k=\int_0^1 K(1-5^{-t})\exp(-2k\pi i t)\,dt. \tag{3.8}$$

In order to get the proposed error term, we prove

$$K(z)-H(z)=\mathcal{O}_{\alpha,\varepsilon}(|z-1|^{1-\varepsilon})\quad\text{for } z\to 1\quad\text{and}\quad|\arg(1-z)|\leqslant\alpha. \tag{3.9}$$

Fix $\delta > 0$ and define the sets $\mathscr{C}_\delta = \{z \in \mathbb{C} : |\phi^{(-1)\prime}(z)| > 1/(5+\delta), |\arg(1-z)| \leqslant \alpha\}$ and $\mathscr{D}_\delta = \mathscr{C}_\delta \backslash \phi^{(-1)}(\mathscr{C}_\delta)$, which is contained in some annulus around the point $z = 1$. Since H is holomorphic in $\overline{\mathscr{D}_\delta}$ its derivative is bounded there. By using the equation $H(\phi^{(-1)}(z)) = H(z)$ we obtain

$$H'(z) = \mathcal{O}_{\alpha,\delta}(|z-1|^{-\log(5+\delta)/\log 5}). \tag{3.10}$$

Combining $\xi^{(-1)}(z) = z + \mathcal{O}(|z-1|^2)$ and (3.10) we obtain

$$\begin{aligned} K(z) - H(z) &= H(\xi^{(-1)}(z)) - H(z) = H(z + \mathcal{O}(|z-1|^2)) - H(z) \\ &= \mathcal{O}_{\alpha,\delta}(|z-1|^{2-\log(5+\delta)/\log 5}). \end{aligned}$$

Combining (3.3), (3.5), (3.7) and (3.9), we derive the statement of the proposition. Finally, we want to prove that the periodic function K cannot be a constant. Suppose that it were constant, then H would be constant. As $\tilde{G}(z) = H(z)/G^*(z)$, this would give us an analytic continuation of $\tilde{G}$ to some neighbourhood of $z = 1$. Now $\phi^{(2)}(6 - 2\sqrt{5}) = 1$ and the preimages $\phi^{(-n)}(6 - 2\sqrt{5})$ tend to 1 if n tends to ∞. Thus we have a sequence of points $z_n \to 1$ such that $\phi^{(n)}(z_n) = 1$. Inserting those points into (3.4) yields a value for $G(z_n)$ as a product of factors > 1 times $G(1)$ which gives a contradiction to the continuity of G. □

The following theorem is now an immediate consequence of an application of the method of "singularity analysis" (cf. Flajolet and Odlyzko, 1990) to the results of Proposition 2. We use uniform convergence of the Fourier series of $K(s)$. We note here that a (real) Tauberian theorem would not exhibit the fluctuating nature of the transition probabilities.

Theorem 1. *The transition probabilities* $p^{(n)}(0,0)$ *satisfy the asymptotic relation*

$$p^{(n)}(0,0) = n^{-\log 3/\log 5} F\left(\frac{\log n}{\log 5}\right)(1 + \mathcal{O}_\varepsilon(n^{\varepsilon - 1})) \quad \textit{for all } \varepsilon > 0,$$

where F is a periodic C^∞-function of period 1 *whose Fourier series is given by*

$$F(x) = \sum_{k=-\infty}^{\infty} a_k \Gamma\left(1 - \frac{\log 3}{\log 5} + \frac{2k\pi \mathrm{i}}{\log 5}\right)^{-1} \exp(2k\pi \mathrm{i} x).$$

The numbers a_k are given as the Fourier coefficients of the function $K(z)$ in (3.7).

Remark. Notice that

$$|\Gamma(1 - \eta + it)| = \sqrt{2\pi}|t|^{\frac{1}{2}-\eta} \exp\left(-\frac{\pi}{2}|t|\right)(1 + o(1)).$$

This and (3.7) imply that

$$a_k \Gamma\left(1 - \eta + \frac{2k\pi i}{\log 5}\right)^{-1} = \mathcal{O}_\varepsilon\left(\exp\left(-\left(\frac{\pi^2}{\log 5} - \varepsilon\right)|k|\right)\right) \quad \text{for all } \varepsilon > 0.$$

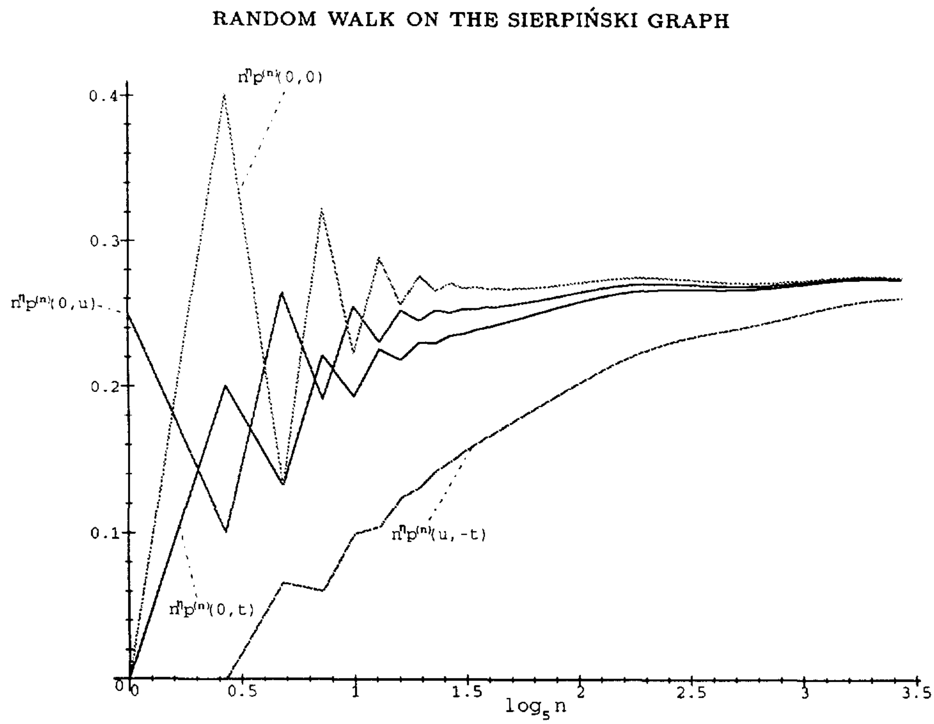

Fig. 4.

The integral representation of the a_k's makes it even possible to compute the first coefficients numerically (we used *Mathematica* to do this):

$$a_0 = 0.7740718\ldots, \qquad a_1 = 3.6437 \times 10^{-6} - 2.9988 \times 10^{-6}i.$$

Finally, we notice that the value of the first Fourier coefficient and the exponential decay of the others suggest that the fluctuation is quite small (see Fig. 4).

4. A ratio limit theorem

In order to complete the proof of (1.2), we now give a simple ratio limit theorem whose proof follows previous results which were restricted to groups, see Le Page (1974) and Gerl (1978).

Let X be an arbitrary countable set, and let $P = (p(x,y))_{x,y \in X}$ be the transition matrix of an irreducible Markov chain on X. We suppose that P has finite range, i.e., each row of P has only finitely many non-zero entries.

Irreducible means that for every x, y, there is n such that $p^{(n)}(x,y) > 0$. The "spectral radius" $\rho = \limsup_n p^{(n)}(x,y)^{1/n}$ is then independent of x and y.

A ρ-harmonic function is a function $h : X \to \mathbb{R}$ such that $Ph = \rho \cdot h$, where $Ph(x) = \sum_y p(x,y)h(y)$. Irreducibility yields that a non-negative, non-zero ρ-harmonic function is strictly positive in each point. If P is recurrent, that is, $G(x,y|1)=\infty$ for some ($\Leftrightarrow$ all) x,y then $\rho=1$ and all positive harmonic (1-harmonic) functions are constant.

Suppose that P is aperiodic, i.e., $\gcd\{n : p^{(n)}(x,x)>0\}=1$. Then P is called *strongly aperiodic*, if there is an n_0 such that

$$\inf\{p^{(n)}(x,x) : x\in X\}>0 \quad \text{for all } n\geqslant n_0.$$

Theorem 2. *Suppose that* (1) *P has finite range and is strongly aperiodic, and* (2) *up to multiplication with constants, there is a unique positive ρ-harmonic function h. Then, for all* $x,y\in X$,

$$\lim_{n\to\infty}\frac{p^{(n)}(x,y)}{p^{(n)}(y,y)}=\frac{h(x)}{h(y)}.$$

In particular, when P is strongly aperiodic and recurrent then the limit is equal to one.

Proof. Strong aperiodicity implies that

$$\lim_{n\to\infty}\frac{p^{(n+1)}(x,y)}{p^{(n)}(x,y)}=\rho, \tag{4.1}$$

see Gerl (1978) and Theorem 5.2(b) in Woess (1994). Now fix $y\in X$. By irreducibility, for each x there is n_x such that $p^{(n_x)}(y,x)>0$. Let $C_x=1/p^{(n_x)}(y,x)$. Then

$$\frac{p^{(n)}(x,y)}{p^{(n)}(y,y)}=C_x\frac{p^{(n_x)}(y,x)p^{(n)}(x,y)}{p^{(n)}(y,y)}\leqslant C_x\frac{p^{(n+n_x)}(y,y)}{p^{(n)}(y,y)}\to C_x\,\rho^{n_x}.$$

Thus, the sequence of functions $x\mapsto p^{(n)}(x,y)/p^{(n)}(y,y)$ is bounded pointwise in x, i.e., relatively compact with respect to pointwise convergence. Let (n') be a sequence of natural numbers such that the limit

$$\lim_{n'}\frac{p^{(n')}(x,y)}{p^{(n')}(y,y)}=g(x)$$

exists for all x. Using finite range and (4.1),

$$Pg(x)=\lim_{n'}\frac{\sum_w p(x,w)p^{(n')}(w,y)}{p^{(n')}(y,y)}=\lim_{n'}\frac{p^{(n'+1)}(x,y)}{p^{(n')}(y,y)}=\rho g(x). \tag{4.2}$$

We have $g(y)=1$, and g is ρ-harmonic. Therefore, it must be $g(x)=h(x)/h(y)$. Consequently, every convergent subsequence has the same limit, and relative compactness yields the result. □

Remark. When P is recurrent, one does not need the finite range assumption: one cannot exchange limits in the first identity of (4.2), but using Fatou's lemma, one gets $Pg(x)\leqslant g(x)$: the function g is superharmonic. In the recurrent case, all non-negative superharmonic functions are constant, whence $g\equiv 1$.

Corollary. For simple random walk on the Sierpiński gasket, one has

$$\lim_{n\to\infty}\frac{p^{(n)}(x,y)}{p^{(n)}(0,0)}=1 \quad \textit{for all } x,y\in\mathcal{G}.$$

Proof. On $\mathscr{G}$, we have $p^{(2)}(x,x) = \frac{1}{4}$ and, as each vertex is common to two triangles at least, $p^{(3)}(x,x) \geqslant \frac{1}{16}$. As every $n \geqslant 2$ can be written $n = 2k + 3l$ with $k, l \in \mathbb{N}_0$, strong aperiodicity follows. Using Theorem 2 and symmetry of P,

$$\frac{p^{(n)}(x,y)}{p^{(n)}(0,0)} = \frac{p^{(n)}(x,y)}{p^{(n)}(y,y)} \frac{p^{(n)}(y,y)}{p^{(n)}(0,y)} \frac{p^{(n)}(y,0)}{p^{(n)}(0,0)} \to 1. \qquad \square$$

5. Concluding remarks

1. The description of the Green function by means of functional iterations can also be used to calculate $p^{(n)}(x,y)$ numerically for moderate values of n. We used the equations

$$G(0,u|z) = \frac{z}{4-3z} G(0,u|\phi(z)) + \frac{2z}{8-2z-z^2} G(0,0|z),$$
$$G(0,t|z) = \frac{z}{2}(G(0,u|z) + G(0,u|\phi(z))),$$
$$G(u,-t|z) = \frac{G(0,u|z)G(0,t|z)}{G(0,0|z)}$$

(u, t as in Fig. 1), which can be derived by using the nearest neighbour relations for the Green function and Proposition 1. We note here that such functional equations relating $G(x,y|z)$ to $G(a,b|z)$ with $a, b \in \{0,u,v,t,2u,2v\}$ exist for any choice of x and y. This would yield an alternative though more tedious proof of (1.2).

The numerical experiments using *Maple* produced the picture in Fig. 4, where $n^{\eta} p^{(n)}(x,y)$ is plotted against $\log n / \log 5$ for some of the points indicated in Fig. 1.

2. Our method does not seem to lend itself to asymptotic evaluations of transition probabilities which are uniform in time and space variables. Conversely, it also seems that the method of Jones (1996) for obtaining uniform estimates via stopping time arguments will not be applicable for finding the periodic oscillations.

3. The methods used in this paper can immediately be applied to any higher dimensional version of the Sierpiński gasket (as defined in Kigami, 1989). In that case the properties of the corresponding function ϕ remain the same as in Lemma 1.

4. For general nested fractals as defined in Lindstrom (1990) and Elworthy and Ikeda (1993) the method has to be modified: for every edge having different transition probability in an elementary cell of the fractal one additional variable for the generating functions has to be introduced. Instead of iterations of one function one has to consider iterations of several multivariate functions. It seems plausible that the fluctuating behaviour of transition probabilities will occur also in those cases.

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