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A note on tilings and strong isoperimetric inequality

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Abstract

We show that the edge graph of a locally finite tiling in the plane satisfies a strong isoperimetric inequality if one among the lower averages of what we call the characteristic numbers of edges, vertices and tiles is positive.

1. *Introduction, statement of results*

In this note we extend and complete results of Dodziuk [**4**], Dodziuk and Kendall [**5**], Beardon and Stephenson [**1**], Soardi [**13**, **14**], Calogero [**3**], and He and Schramm [**9**] concerning sufficient conditions under which the edge graph of a plane tiling with bounded geometry must satisfy a strong isoperimetric inequality.

Let X be an infinite graph, thought of as a set of vertices connected by edges which are given by a symmetric neighbourhood relation, the edge set $E(X)$. Graphs considered here are always connected and locally finite, i.e. the degree (number of neighbours) $\deg(x)$ of any vertex x is finite. For finite $A \subset X$, we denote by ∂A the set of all edges going from A to $X \setminus A$, and define

$$\mathrm{Vol}\,(A) = \sum_{x \in A} \deg(x) \quad \text{and} \quad \mathrm{Area}\,(\partial A) = |\partial A|\,.$$

We say that the graph X satisfies a *strong isoperimetric inequality,* if there is a constant $\kappa > 0$ such that

$$\mathrm{Area}\,(\partial A) \geqslant \kappa\,.\,\mathrm{Vol}\,(A) \quad \text{for every finite } A \subset X\,. \tag{IS}$$

We are usually interested in the existence, not the numerical value, of κ. We remark

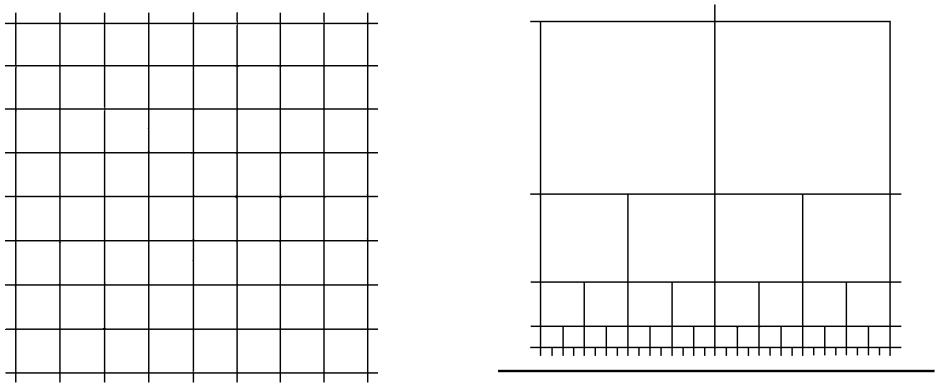

Fig. 1. The tilings $\mathscr{T}_1$ of the plane and $\mathscr{T}_2$ of the upper half plane.

that for connected graphs with bounded vertex degrees, one often replaces Vol (A) with the cardinality of A and Area (A) with $|dA|$, where dA is the set of vertices in A having a neighbour in $X \setminus A$. This makes no (qualitative) difference when the vertex degrees are bounded.

Strong isoperimetric inequalities are of interest in various contexts. If X is a Cayley graph of a finitely generated group, then (IS) holds if and only if the group is non-amenable (this is a variant of Følner's criterion [**6**]). In spectral theory on graphs, (IS) is used for the discrete analogue of Cheeger's inequality: besides [**4**], see, for example, Gerl [**7**], Biggs, Mohar and Shawe-Taylor [**2**] and the survey of Mohar and Woess [**11**]. For the simple random walk on X, the n-step transition probabilities decay exponentially if and only if (IS) holds (see, for example, Woess [**16**]) so that the random walk must be transient. (IS) is also useful in the study of harmonic functions on graphs, see, for example, Kaimanovich and Woess [**10**]. Here we shall consider a certain class of planar graphs.

Let $\mathscr{D}$ be an open simply connected subset of the Euclidean plane $\mathbb{R}^2$. A *tiling* of $\mathscr{D}$ is a family $\mathscr{T}$ of closed topological discs $T \subset \mathscr{D}$ (the tiles) with pairwise disjoint interiors and such that $\bigcup_{T \in \mathscr{T}} T = \mathscr{D}$. We shall always assume that the tiling is *locally finite,* that is, every compact subset of $\mathscr{D}$ meets only finitely many tiles. A connected component of the intersection of two or more tiles is called a *vertex* if it is a single point, and an *edge* if it is an arc (it must then come from the intersection of two tiles). By local finiteness, the boundary of each tile is the union of a finite number of edges, which we assume here to be at least three. The graph $X(\mathscr{T})$ with this set of vertices and edges is the *edge graph* of the tiling. By abuse of notation, we shall also think of a tile T as the subgraph induced by its edges and vertices, so that $x \in T$ will mean that x is a vertex, $|T|$ is the number of vertices and $E(T)$ is the set of edges of T (so that $|E(T)| = |T|$). We refer to Grünbaum and Shephard [**8**] for the theory of tilings and many pictures.

As key examples for illustrating our results, we shall consider the tilings $\mathscr{T}_1$ of the plane and $\mathscr{T}_2$ of the upper half plane depicted in Figure 1. The edge graph of $\mathscr{T}_1$ is the square grid, which does not satisfy (IS). The edge graph of $\mathscr{T}_2$ with the corresponding graph metric is a discrete approximation of the hyperbolic metric in the upper half plane, well known to satisfy (IS). (Our results lead to the simplest methods for verifying this.)

We return to the general setting of a tiling $\mathcal{T}$ and its edge graph $X = X(\mathcal{T})$. Unless mentioned otherwise, all subgraphs A of X will be induced subgraphs of X (i.e. if $x, y \in A$ are neighbours in X then they are also neighbours in A). For a finite, connected subgraph A of X, we set $\mathcal{P}(A) = \{T \in \mathcal{T}\colon E(T) \subset E(A)\}$. Consider its infinite face $F_\infty = F_\infty(A)$, that is, the closure of the component of ∞ in $\widehat{\mathbb{R}}^2 \setminus E(A)$. We say that A is *simply connected,* if every tile $T \in \mathcal{T}$ that is not contained in F_∞ belongs to $\mathcal{P}(A)$. Thus, if we contract each component of $\bigcup_{T\in\mathcal{P}(A)} T$ to a single point, then A becomes a finite tree.

We define the *characteristic numbers* of edges e, vertices x and tiles T of X by

$$\phi(e) = 1 - \sum_{x\in e} \frac{1}{\deg(x)} - \sum_{T:E(T)\ni e} \frac{1}{|T|}\,,$$

$$\psi(x) = \frac{\deg(x)}{2} - 1 - \sum_{T\ni x} \frac{1}{|T|}\,, \quad \text{and}$$

$$\chi(T) = \frac{|T|}{2} - 1 - \sum_{x\in T} \frac{1}{\deg(x)}\,.$$

We start by working with edges. (Every edge $e \in E(X)$ contains two vertices and is contained in two tiles.) For a finite subset E of $E(X)$, we consider the average $\overline{\phi}(E) = (1/|E|)\sum_{e\in E} \phi(e)$. The *lower average* of ϕ on $E(X)$ is

$$\overline{\phi}\big(E(X)\big) = \liminf_{|E(A)|\to\infty} \overline{\phi}\big(E(A)\big)\,,$$

where the lim inf is over all simply connected, finite subgraphs A of X.

Theorem 1. *If* $\overline{\phi}\big(E(X)\big) > 0$ *then the edge graph of* $\mathcal{T}$ *satisfies a strong isoperimetric inequality.*

In the tiling $\mathcal{T}_1$ (square grid), we see that $\phi(e) = 0$ for every edge. As we know, the grid does not satisfy (IS). In $\mathcal{T}_2$ (upper half plane), $|T| = 5$ for each tile. For an edge e, either both endpoints have degree 4, in which case $\phi(e) = \frac{1}{10}$, or one has degree 4 and the other degree 3, so that $\phi(e) = \frac{1}{60}$. Consequently, $\overline{\phi}\big(E(X)\big) \geqslant \frac{1}{60}$, and $X(\mathcal{T}_2)$ satisfies (IS).

From Theorem 1, we get

Corollary 1. *If* $\deg(x) \geqslant k$ *for all* $x \in X$ *and* $|T| \geqslant m$ *for all* $T \in \mathcal{T}$, *with* $km > 2(k+m)$, *then* X *satisfies* (IS).

Indeed, we then get $\phi(e) \geqslant 1 - (2/k) - (2/m) > 0$ for all edges. In particular, if $\deg(x) \geqslant 7$ for all vertices, or $|T| \geqslant 7$ for all tiles, then $X(\mathcal{T})$ satisfies (IS): we always have $k, m \geqslant 3$, so that $\phi(e) \geqslant 1 - \frac{2}{7} - \frac{2}{3} = \frac{1}{21}$ for all edges. This applies, in particular, to triangulations (i.e. each tile is a triangle), compare with [**5**]. Similarly, the edge graph of a square tiling satisfies (IS), if all vertices have degree 5 or more, and so on. Below, we shall see how to replace this with a condition on the average degree. Also, observe that $X(\mathcal{T}_2)$ has $k = 3$, $m = 5$, so that Corollary 1 does not apply, although (IS) holds.

As another example where Theorem 1 applies, suppose that $\mathcal{T}$ arises from a circle packing, where the tiles are disks and the triangular interstices between three mutually tangent disks (compare, for example, with Rodin and Sullivan [**12**] and the figures there). Every vertex has degree 4. Each edge lies on a disk and a triangular interstice. Thus, if each disk touches seven or more disks, we have $\phi(e) \geqslant 1-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}=\frac{1}{42}$, and the edge graph satisfies (IS). Compare with Beardon and Stephenson [**1**].

In the same way as above, we consider the averages $\overline{\psi}(A) = (1/|A|)\sum_{x\in A}\psi(x)$ and $\overline{\chi}(\mathscr{P}(A)) = 1/|\mathscr{P}(A)|\sum_{T\in\mathscr{P}(A)}\chi(T)$ for finite $A \subset X$. The *lower averages* of ψ and χ on X and $\mathscr{T}$, respectively, are

$$\overline{\psi}(X) = \liminf_{|A|\to\infty} \overline{\psi}(A) \quad \text{and} \quad \overline{\chi}(\mathscr{T}) = \liminf_{|\mathscr{P}(A)|\to\infty} \overline{\chi}(\mathscr{P}(A)),$$

where the two lim inf are again over all simply connected, finite subgraphs A of X.

Theorem 2. *Each of the conditions*

$$(a) \quad \overline{\psi}(X) > 0, \qquad (b) \quad \overline{\chi}(\mathscr{T}) > 0$$

implies that the edge graph of $\mathscr{T}$ *satisfies a strong isoperimetric inequality.*

Corollary 2. (a) *Suppose that* $|T| \geqslant m \geqslant 3$ *for all tiles, and that we have the lower average*

$$\liminf_{|A|\to\infty} \frac{1}{|A|}\sum_{x\in A}\deg(x) > \frac{2m}{m-2}$$

over simply connected subgraphs A *of* X. *Then* $X(\mathscr{T})$ *satisfies* (IS).

(b) *Suppose that* $\deg(x) \geqslant k \geqslant 3$ *for all vertices, and that we have the lower average*

$$\liminf_{|A|\to\infty} \frac{1}{|\mathscr{P}(A)|}\sum_{T\in\mathscr{P}(A)}|T| > \frac{2k}{k-2}$$

over simply connected subgraphs A *of* X. *Then* $X(\mathscr{T})$ *satisfies* (IS).

Compare (a) with the last theorem in [**9**]. Theorem 2(b) is of course linked with 2(a) by duality. We prove each of them directly. It seems to require at least similar efforts to pass from one to the other by showing that the dual graph satisfies (IS) when X has this property. Note that we did not require any upper bound on the (finite) vertex degrees or sizes (number of vertices) of the tiles. In the presence of such bounds it is easy to prove that $X(\mathscr{T})$ satisfies (IS) if and only if the dual graph satisfies (IS): one either shows that mapping each tile to one of its vertices is a so-called rough isometry from the dual graph to $X(\mathscr{T})$ (compare e.g. with [**16**] and the references given there), or – more specifically – adapts the second part of the proof of theorem 6·33 in [**17**].

Returning to our two examples, we see that for $\mathscr{T}_1$ (grid), $\psi(x) = \chi(T) = 0$ for all vertices and tiles. For $\mathscr{T}_2$ (upper half plane), we have $\psi(x) = \frac{4}{2} - 1 - \frac{2}{5} = \frac{3}{5}$ when $\deg(x) = 4$ and $\psi(x) = \frac{1}{10}$ when $\deg(x) = 3$, so that $\overline{\psi}(X(\mathscr{T}_2)) \geqslant \frac{1}{10}$. Also, every tile of $\mathscr{T}_2$ has two vertices with degree 3 and three with degree 4, so that $\chi(T) = \frac{1}{12}$. Furthermore, Corollary 2(b) does not apply to $\mathscr{T}_2$, although (IS) holds.

The proofs of the two theorems (Section 2) will be careful applications of Euler's formula $|\mathscr{F}(A)|-|E(A)|+|A| = 1$ for any finite, connected planar graph, where $\mathscr{F}(A)$ is the set of faces of X (not including the infinite face). In Section 3, we discuss a few variations of the results.

In conclusion, we remark that the theorems generalize, for example, to tilings of surfaces with finite genus. The details will be straightforward and are left to the reader.

2. *Proofs*

Reduction Lemma 1. *It is enough to prove* (IS) *for simply connected induced subgraphs A of X.*

Proof. Suppose that $\mathrm{Area}\,(\partial A) \geqslant \kappa\,.\,\mathrm{Vol}\,(A)$ for every simply connected subgraph of X.

1. If A is connected, but not simply connected, then we can fill in the 'holes', that is, all tiles which do not belong to the infinite face of A. We obtain a new graph $\bar{A}$, which is simply connected. Also, $\bar{A} \supset A$, while $\partial\bar{A} \subset \partial A$. Hence $\mathrm{Area}\,(\partial A) \geqslant \mathrm{Area}\,(\partial\bar{A}) \geqslant \kappa\,.\,\mathrm{Vol}\,(\bar{A}) \geqslant \kappa\,.\,\mathrm{Vol}\,(A)$.

2. It is immediate that when (IS) holds for all *connected* finite subgraphs of X then it holds for *all* finite subgraphs (with the same κ). □

If A is a subgraph of X, then $\deg_A(x)$ denotes the number of neighbours in A of vertex $x \in A$. The A in consideration will be induced subgraphs, so that $\deg_A(x) < \deg(x) \iff x \in dA$.

Proof of Theorem 1. By assumption, there are $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $\overline{\phi}\big(E(A)\big) \geqslant \varepsilon$ for every simply connected, finite subgraph A of X with $|E(A)| > n_0$.

First, let A be of this type, but with $|E(A)| \leqslant n_0$. Observe that in any case, $\mathrm{Vol}\,(A) = 2|E(A)| + \mathrm{Area}\,(\partial A)$, so that $\mathrm{Area}\,(\partial A) \geqslant 1 \geqslant 1/n_0|E(A)| = 1/2n_0\big(\mathrm{Vol}\,(A) - \mathrm{Area}\,(\partial A)\big)$, and

$$\mathrm{Area}\,(\partial A) \geqslant \frac{1}{2n_0 + 1}\,\mathrm{Vol}\,(A)\,.$$

So now suppose $|E(A)| > n_0$. Set $\mathscr{P} = \mathscr{P}(A)$. As A is simply connected, the faces of A are $\mathscr{F}(A) = \mathscr{P}$. We compute

$$\begin{aligned}
\sum_{e\in E(A)} \phi(e) &= |E(A)| - \sum_{x\in A}\frac{\deg_A(x)}{\deg(x)} - \sum_{T\in\mathscr{T}}\frac{|E(A)\cap E(T)|}{|T|} \\
&= |E(A)| - |A| - |\mathscr{P}| + \sum_{x\in dA}\left(1 - \frac{\deg_A(x)}{\deg(x)}\right) - \sum_{T\in\mathscr{T}\setminus\mathscr{P}}\frac{|E(A)\cap E(T)|}{|T|} \\
&\leqslant -1 + |dA| \leqslant \mathrm{Area}\,(\partial A)\,.
\end{aligned}$$

Hence, $\mathrm{Area}\,(\partial A) \geqslant \varepsilon\,.\,|E(A)| = \varepsilon/2\big(\mathrm{Vol}\,(A) - \mathrm{Area}\,(\partial A)\big)$, and

$$\mathrm{Area}\,(\partial A) \geqslant \frac{\varepsilon}{\varepsilon + 2}\,\mathrm{Vol}\,(A)\,. \quad \square$$

The proofs of Theorem 2(a) and (b) are quite similar, but we have to be careful with several details that vary.

Proof of Theorem 2(a). Suppose that $\overline{\psi}(A) \geqslant \varepsilon > 0$ for every simply connected, finite subgraph A of X with $|A| > n_0$.

Since $|E(T)| \geqslant 3$ for every $T \in \mathscr{P}(A)$ and each $e \in E(A)$ lies on no more than two $T \in \mathscr{P}(A)$, we have $3|\mathscr{P}(A)| \leqslant 2|E(A)|$, so that Euler's formula yields $|A| \geqslant |E(A)|/3$.

First, let $|A| \leqslant n_0$. Then Area $(\partial A) \geqslant 1 \geqslant |A|/n_0 \geqslant |E(A)|/3n_0$, and, as above,

$$\text{Area}\,(\partial A) \geqslant \frac{1}{6n_0+1}\,\text{Vol}\,(A)\,.$$

Now let $|A| > n_0$, and set $\mathscr{P} = \mathscr{P}(A) = \mathscr{F}(A)$. As above, we compute

$$\sum_{x\in A} \psi(x) = \sum_{x\in A} \frac{\deg(x) - \deg_A(x)}{2} + |E(A)| - |A| - |\mathscr{P}| - \sum_{T\in\mathscr{T}\setminus\mathscr{P}} \frac{|T\cap A|}{|T|}$$
$$\leqslant \tfrac{1}{2}\,\text{Area}\,(\partial A)\,.$$

Hence, Area $(\partial A) \geqslant 2\varepsilon\,.\,|A| \geqslant 2\varepsilon/3\cdot|E(A)|$, and

$$\text{Area}\,(\partial A) \geqslant \frac{\varepsilon}{\varepsilon+3}\,\text{Vol}\,(A)\,. \qquad \square$$

In the next proof we have to be particularly careful.

Proof of Theorem 2(*b*). Let $\overline{\chi}\big(\mathscr{P}(A)\big) \geqslant \varepsilon > 0$ for every simply connected, finite subgraph A of X with $|\mathscr{P}(A)| > n_0$, and set $\mathscr{P} = \mathscr{P}(A)$.

Each vertex $x \in A\setminus dA$ satisfies $\deg_A(x) \geqslant 3$, while $\deg_A(x) \geqslant 1$ for $x \in dA$. Thus $2|E(A)| = \sum_{x\in A}\deg_A(x) \geqslant 3|A\setminus dA| + |dA|$, and $|A| \leqslant \frac{2}{3}\big(|E(A)| + |dA|\big)$. Euler's formula gives

$$|\mathscr{P}| \geqslant \tfrac{1}{3}|E(A)| - \tfrac{2}{3}\,\text{Area}\,(\partial A) \geqslant \tfrac{1}{6}\,\text{Vol}\,(A) - \tfrac{5}{6}\,\text{Area}\,(\partial A)\,.$$

Now, if $|\mathscr{P}| \leqslant n_0$, then proceeding as above, Area $(\partial A) \geqslant |\mathscr{P}|/n_0$ and

$$\text{Area}\,(\partial A) \geqslant \frac{1}{6n_0+5}\,\text{Vol}\,(A)\,.$$

Next, define $E_i = E_i(A)$ to be the set of all edges $e \in E(A)$ such that $|\{T \in \mathscr{P} : T \ni e\}| = i$, $i = 0,1,2$. Then $E_0 \cup E_1$ is the set of all edges on the infinite face of A. For $x \in A$, let $r_i(x) = |\{e \in E_i\colon e \ni x\}|$, $i = 0,1,2$. We have

$$\sum_{x\in A} r_i(x) = 2|E_i|\,.$$

Given $T \in \mathscr{T}$ containing x, let $e_j(T) = e_j(T,x)$ $(j = 1,2)$ be the first and second edge of T incident with x, in clockwise order. Then

$$2|\{T \in \mathscr{P} : T \ni x\}| = \sum_{j=1}^{2} |\{e_j(t) : x \in T \in \mathscr{P}\}|$$
$$= \sum_{e\ni x} |\{T \in \mathscr{P} : T \ni e\}| = r_1(x) + 2r_2(x)\,.$$

Furthermore, if $\in A\setminus dA$ then $r_0(x) + r_1(x) + r_2(x) = \deg(x)$.

With all these preliminaries, suppose now that $|\mathscr{P}| > n_0$ and compute

$$\begin{aligned}
\sum_{T\in\mathscr{P}} \chi(T) &= \frac{1}{2}\sum_{e\in E(A)} |\{T\in\mathscr{P} : T\ni e\}| - |\mathscr{P}| - \sum_{x\in A}\frac{|\{T\in\mathscr{P} : T\ni x\}|}{\deg(x)}\\
&= |E(A)| - |E_0| - \frac{|E_1|}{2} - |\mathscr{P}| - |A| + \sum_{x\in A}\frac{\deg(x) - |\{T\in\mathscr{P} : T\ni x\}|}{\deg(x)}\\
&= -1 - |E_0| - \frac{|E_1|}{2} + \sum_{x\in dA}\frac{\deg(x) - |\{T\in\mathscr{P} : T\ni x\}|}{\deg(x)}\\
&\quad + \sum_{x\in A\setminus dA}\frac{2r_0(x)+r_1(x)}{2\deg(x)}\\
&\leqslant -|E_0| - \frac{|E_1|}{2} + |dA| + \frac{1}{6}\sum_{x\in A}\big(2r_0(x)+r_1(x)\big)\\
&= -|E_0| - \frac{|E_1|}{2} + |dA| + \frac{2|E_0|}{3} + \frac{|E_1|}{3} \leqslant |dA| \leqslant \mathrm{Area}\,(\partial A)\,.
\end{aligned}$$

By assumption the first term is $\geqslant \varepsilon\,.\,|\mathscr{P}|$, so that

$$\mathrm{Area}\,(\partial A) \geqslant \frac{\varepsilon}{5\varepsilon+6}\,\mathrm{Vol}\,(A)\,.$$

3. *Variations*

The averages taken in the definitions of $\overline{\phi}\big(E(X)\big)$, $\overline{\psi}(X)$ and $\overline{\chi}(\mathscr{T})$ are over sufficiently large, but arbitrary simply connected subgraphs, which might be 'long and slim', so as to include too many 'bad' edges (vertices) with too small characteristic numbers. We now show that one can restrict to 'fatter' subgraphs in the case when the tiling $\mathscr{T}$ has *bounded geometry,* i.e. there are bounds M, N such that $\deg(x) \leqslant M$ for all $x \in X(\mathscr{T})$ and $|T| \leqslant N$ for all $T \in \mathscr{T}$.

We define a *part* $\mathscr{P}$ of $\mathscr{T}$ as a finite subfamily of $\mathscr{T}$. We say that $\mathscr{P}$ is (*simply*) *connected,* if $\bigcup_{T\in\mathscr{P}} T \subset \mathbb{R}^2$ has this property. The subgraph $X(\mathscr{P})$ of $X(\mathscr{T})$ is defined in the obvious way. When it is an induced subgraph (which is not always true), then $\mathscr{P}$ is called *complete.* Note that $X(\mathscr{P})$ is then a simply connected subgraph of X in the above sense, while conversely, it is not necessarily true that for a simply connected subgraph A one has that $\mathscr{P}(A)$ is simply connected (as $\mathscr{P}(A)$ can have several connected components even if A is connected).

Reduction Lemma 2. *If $\mathscr{T}$ has bounded geometry, then it is enough to prove* (IS) *for subgraphs of the form $B = X(\mathscr{P})$, where $\mathscr{P}$ is a simply connected, complete part of $\mathscr{T}$.*

Proof. Suppose that $\mathrm{Area}\,(\partial B) \geqslant \kappa\,.\,\mathrm{Vol}\,(B)$ for all B as stated.

1. Let $A = X(\mathscr{P})$, where $\mathscr{P}$ is a connected part of $\mathscr{T}$, but not necessarily simply connected and complete. Let $\bar{A}$ be the subgraph of $X(\mathscr{T})$ induced by A (it is obtained by adding edges between vertices of A only). This is a finite, planar graph in $\mathbb{R}^2$. All its vertices have degree $\geqslant 2$. The vertices and edges of $\bar{A}$ lying on its infinite face form a simple closed curve $\mathscr{C}$. Now consider the family $\bar{\mathscr{P}}$ ($= \mathscr{P}(\bar{A})$ in the notation of Section 1) of all tiles $T \in \mathscr{T}$ lying inside that curve. $\bar{\mathscr{P}}$ is finite by local finiteness of $\mathscr{T}$. It is clear that $\bar{\mathscr{P}}$ is simply connected. Outside of $\mathscr{C}$, there is no edge between any two vertices on $\mathscr{C}$. Thus, $\bar{\mathscr{P}}$ is complete, and $\bar{\mathscr{P}} \supset \mathscr{P}$ (the 'completion' of $\mathscr{P}$). Let

$B = X(\bar{\mathscr{P}})$. Then $B \supset A$, while $\partial B \subset \partial A$ (if $e \in \partial B$ then it must have one endpoint in $\mathscr{C}$ and the other outside). But then, as in Reduction Lemma 1, $\text{Area}\,(\partial A) \geqslant \text{Area}\,(\partial B) \geqslant \kappa\,.\,\text{Vol}\,(B) \geqslant \kappa\,.\,\text{Vol}\,(A)$.

2. Now let A be an arbitrary finite, connected subgraph of $X(\mathscr{T})$. Define $\mathscr{P} = \{T \in \mathscr{T} : T \cap A \neq \emptyset\}$ (a connected part of $\mathscr{T}$), and set $\bar{A} = X(\mathscr{P})$. Then every vertex $x \in d\bar{A} \setminus dA$ must lie on a tile $T \in \mathscr{P}$ which also contains a vertex y in dA. Each such vertex is surrounded by at most M tiles, and on each such tile there are at most $N-1$ vertices of $d\bar{A} \setminus dA$. This means that

$$|d\bar{A}| \leqslant \big(M(N-1)+1\big)|dA|\,, \quad \text{and} \quad |\bar{A}| \geqslant |A| \geqslant \text{Vol}\,(A)/M\,,$$

so that $\text{Area}\,(\partial A) \geqslant |dA| \geqslant \kappa'\,.\,\text{Vol}\,(A)$ with $\kappa' = \kappa/(M^2N - M^2 + M)$.

3. Finally, we know that we can pass from connected finite subgraphs of X to all finite subgraphs (with the same κ'). □

Thus we define the 'better' averages $\overline{\phi}_1\big(E(X)\big)$, $\overline{\psi}_1(X)$ and $\overline{\chi}_1(\mathscr{T})$, where we take the lim inf over simply connected parts and the corresponding subgraphs only.

Corollary 3. *If $\mathscr{T}$ has bounded geometry, and one of $\overline{\phi}_1\big(E(X)\big)$, $\overline{\psi}_1(X)$ or $\overline{\chi}_1(\mathscr{T})$ is (strictly) positive, then $X(\mathscr{T})$ satisfies* (IS).

Returning to general locally finite tilings, another variation regards transience of the simple random walk on $X(\mathscr{T})$. Recall that this is the Markov chain with state space X and transition probabilities $p(x,y) = 1/\deg(x)$, if x and y are neighbours, and $p(x,y) = 0$, otherwise. Transience means that the probability to return to the starting point infinitely often is zero, and in this case, the graph itself is called transient. If X has a transient subgraph, then X is transient as well. We refer to [**16**] for more details. The following extends the last theorem in [**9**].

Proposition. *Suppose that are a finite subgraph A_0 and $\varepsilon > 0$ such that $\overline{\phi}\big(E(A)\big) \geqslant \varepsilon$ for every finite, simply connected subgraph A of X containing A_0. Then $X(\mathscr{T})$ contains a transient subtree.*

The same holds when when $\overline{\phi}\big(E(A)\big)$ is replaced with $\overline{\psi}(A)$ or with $\overline{\chi}\big(\mathscr{P}(A)\big)$.

Proof (*outline*). By passing to a larger subgraph (if necessary), we may suppose that A_0 is simply connected. Rereading the proofs of Theorems 1 and 2, we see that (IS) holds with some constant $\kappa > 0$ for every connected subgraph containing A_0. If we contract A_0 to a single point o, then we obtain a new graph X'. We get that in X', every finite connected subgraph containing o satisfies (IS) with a new but still positive constant κ': this can be seen directly, or else, by observing that the contraction mapping is a so-called rough isometry, compare e.g. with [**16**]. Now X' has a transient subtree T' by a beautiful construction of Thomassen [**15**]. We now replace vertex o with a (finite) spanning tree of A_0 to obtain a subtree of X which is again transient.

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