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Transience and volumes of trees

By

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1. Introduction. Given a locally finite, connected graph G with countable sets of vertices V and (undirected) edges E, the *simple random walk* (SRW) on G is a Markov chain X_n ($n = 0, 1, 2, \ldots$) which is associated with the graph in a natural way: Starting at some vertex of G, we perform random steps along the edges; having arrived at a vertex we select with equal chance that one among the adjacent vertices to which our next step will take us. In other words, the state space of (X_n) is V and the one-step transition probabilities $Pr[X_{n+1} = v \mid X_n = u] = p_{u,v}$ ($u, v \in V$) are given by

$$(1.1)\qquad p_{u,v} = \begin{cases} 1/d(u) & \text{if } v \text{ is adjacent to } u,\\ 0 & \text{otherwise.}\end{cases}$$

$d(u)$ denotes the degree of vertex u. The transition matrix $P = (p_{u,v})_{u,v \in V}$ can be regarded as the adjacency matrix of G, normalized by dividing each element by the corresponding row-sum. The n-step transition probabilities $Pr[X_n = v \mid X_0 = u] = p^{(n)}_{u,v}$ are just the entries of the matrix power P^n,

$$(1.2)\qquad p^{(0)}_{u,v} = \delta_{u,v} \quad \text{and} \quad p^{(n)}_{u,v} = \sum_{w \in V} p^{(n-1)}_{u,w}\, p_{w,v} \qquad \text{for } n = 1, 2, \ldots.$$

By the study of the SRW, one can relate geometric properties of G with probabilistic ones. One of the most important concepts concerning Markov chains is that of recurrence-transience: The SRW – and, briefly, also the graph G – is called

$$(1.3)\qquad \begin{aligned} &\textit{recurrent}, && \text{if } \sum_{n=0}^{\infty} p^{(n)}_{u,v} = \infty,\\ &\textit{transient}, && \text{if } \sum_{n=0}^{\infty} p^{(n)}_{u,v} < \infty. \end{aligned}$$

This definition does not depend on the particular choice of $u, v \in V$ [2]. The infinite sum of (1.3) is the mean number of visits to vertex v during the SRW after starting in u.

In the present article we exhibit a simple criterion for transience, or recurrence respectively, of infinite *trees* T. To faciliate the exposition, we assume that $d(v) \geqq 2$ for all $v \in V$, and talking of *subtrees*, we implicitly mean those subtrees which have the same property. We shall use a natural notion of the *volume* of T, denoted by $\mathscr{V}_e(T)$, as introduced in [4],

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[1]) Partially supported by CNR, GNAFA (Italy).

and the family of *generalized volumes* $\mathscr{V}_e(\omega)$, where ω is a *weight function* on T with respect to a chosen reference vertex e (Section 2). With e as the starting point, the mean number of visits of the SRW to e,

$$G_e(T) = \sum_{n=0}^{\infty} p_{e,e}^{(n)} \quad \text{(also briefly denoted by } G_e) \tag{1.4}$$

can be directly related with the volume, resp. the generalized volumes of T (Sections 3 and 4): $G_e(T) \leqq d(e)\,\mathscr{V}_e(\omega)$ for every generalized volume of T (Theorem 1), and there is a generalized volume $\mathscr{V}_e(\hat{\omega})$ such that $G_e(T) = d(e)\,\mathscr{V}_e(\hat{\omega})$ (Theorem 2).

As a consequence, T is transient if and only if it admits a finite generalized volume: this result is related to the criterion of [6], where transience of a large class of Markov chains is set in relation with the *energy* of *flows* on the corresponding graph. Our Theorems 1 and 2 give direct and elementary relations between the probabilistic quantity "mean number of visits" and the geometrical quantities "generalized volumes".

2. Generalized volumes of a tree. Chosing a reference vertex e gives rise to an *orientation* of the tree: we write $u \xrightarrow{e} v$ if the vertex u lies on the shortest path connecting e with vertex v. Given $u \in V$, we denote by $U(u)$ (more exactly: $U_e(u)$) the set of all vertices v adjacent to u such that $u \xrightarrow{e} v$; in particular, $U(e)$ consists of all neighbours of e.

Definition. A *weight function* on T with respect to the reference vertex e is a nonnegative function ω defined on E, $[u, v] \mapsto \omega_{u,v}$, such that

$$\sum_{v \in U(u)} \omega_{u,v} = 1 \quad \text{for all } u, v \in V. \tag{2.1}$$

The *generalized volume* of T with respect to ω is the infinite sum

$$\begin{aligned} \mathscr{V}_e(\omega) = &\sum_{u \in U(e)} \omega_{e,u}^2 + \sum_{u \in U(e)} \sum_{v \in U(u)} \omega_{e,u}^2 \omega_{u,v}^2 \\ &+ \sum_{u \in U(e)} \sum_{v \in U(u)} \sum_{w \in U(v)} \omega_{e,u}^2 \omega_{u,v}^2 \omega_{v,w}^2 + \cdots. \end{aligned} \tag{2.2}$$

If ω is the "natural" weight function with respect to e, i.e. $\omega_{u,v} = 1/\#U(u)$, then the corresponding infinite sum is called the (ordinary) *volume* of T, denoted by $\mathscr{V}_e(T)$ (compare [4]). $\mathscr{V}_e(T)$ is a sum of areas of consecutive squares – each one corresponding to an edge of T – filled into the region $[0, \infty) \times [0, 1]$ of the real plane: We start with $d(e)$ squares of side $1/d(e)$ each; having already drawn the square of side $a_{u,v}$ corresponding to edge $[u, v]$, $u \xrightarrow{e} v$, then we attach on the right $d(v) - 1$ squares of side $a_{u,v}/(d(v) - 1)$, each one corresponding to an edge $[v, w]$, $w \in U(v)$. See Fig. 1, where edges and corresponding squares are labelled by equal numbers.

In the general case, the side $a_{u,v}$ of the square corresponding to edge $[u, v]$ is subdivided into the sides $a_{u,v}\omega_{v,w}$ of the squares corresponding to the edges $[v, w]$, $w \in U(v)$.

3. Volumes and simple random walks on branches. Given two adjacent vertices u and v, the *branch* $B_{u,v}$ of T is the subtree spanned by u, v and all vertices w such that $v \xrightarrow{u} w$. If $u \xrightarrow{e} v$ and ω is a weight function on T with respect to e, then we can define the

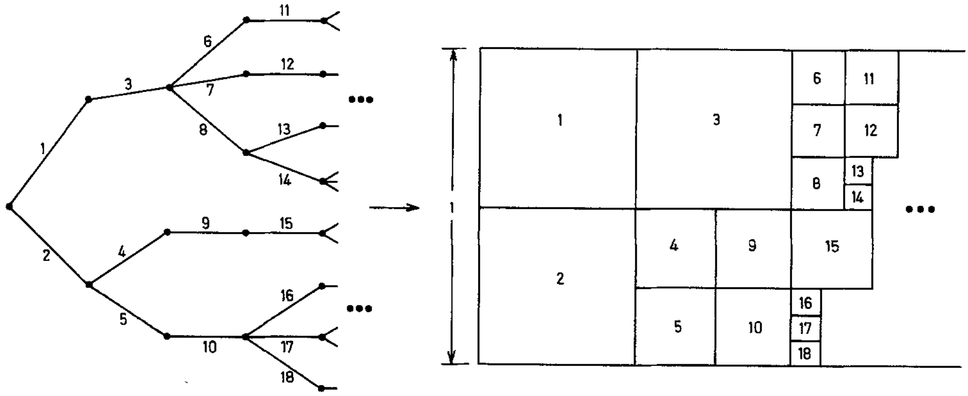

Fig. 1

generalized volume of $B_{u,v}$:

$$\begin{aligned}(3.1)\qquad \mathcal{V}_{u,v}(\omega) = 1 &+ \sum_{w\in U(v)} \omega^2_{v,w} + \sum_{w\in U(v)} \sum_{x\in U(w)} \omega^2_{v,w}\omega^2_{w,x} \\ &+ \sum_{w\in U(v)} \sum_{x\in U(w)} \sum_{y\in U(x)} \omega^2_{v,w}\omega^2_{w,x}\omega^2_{x,y} + \cdots .\end{aligned}$$

The partial sum of this infinite sum up to (and including) the $(k+1)$ *st* summand will be denoted by $\mathcal{V}_{u,v}(\omega, k)$. Thus we have:

Lemma 1. a) $\mathcal{V}_e(\omega) = \sum_{u\in U(e)} \omega^2_{e,u}\mathcal{V}_{e,u}(\omega)$.

b) $\mathcal{V}_{u,v}(\omega, 0) = 1$, $\mathcal{V}_{u,v}(\omega, k+1) = 1 + \sum_{w\in U(v)} \omega^2_{v,w}\mathcal{V}_{v,w}(\omega, k)$.

c) $\mathcal{V}_{u,v}(\omega, k)$ *tends to* $\mathcal{V}_{u,v}(\omega)$ *monotonically as* $k \to \infty$.

Clearly, the transition probabilities of the SRW $X^{u,v}_n$ $(n = 0, 1, 2, \ldots)$ on $B_{u,v}$ coincide with those of X_n on T at every vertex of the branch with the exception of u, where we have $p_{u,v} = 1$ instead of $p_{u,v} = 1/d(u)$. Now consider the SRW on $B_{u,v}$ with starting point u. Denote by $G_{u,v}$ the mean number of visits to u during this walk and by $G_{u,v}(k)$ the mean number of visits to u with the additional restriction that we count only those visits where the SRW does not exceed distance k from u before returning to u:

$$(3.2)\qquad G_{u,v} = \sum_{n=0}^{\infty} Pr[X^{u,v}_n = u \mid X^{u,v}_0 = u]$$

$$(3.3)\qquad G_{u,v}(k) = \sum_{n=0}^{\infty} Pr[X^{u,v}_n = u,\ \mathrm{dist}(X^{u,v}_m, u) \leqq k \text{ for } m \leqq n \mid X^{u,v}_0 = u].$$

Here, dist denotes the usual distance between vertices of the three, that is the number of edges of the shortest connecting path.

Proposition 1. a) $G_e = 1 \bigg/ \dfrac{1}{d(e)} \sum\limits_{u \in U(e)} \dfrac{1}{G_{e,u}}$.

b) $G_{u,v}(0) = 1$, $G_{u,v}(k+1) = 1 + 1 \bigg/ \sum\limits_{w \in U(v)} \dfrac{1}{G_{v,w}(k)}$.

c) $G_{u,v}(k)$ *tends to* $G_{u,v}$ *monotonically as* $k \to \infty$.

Proof. We use the particularities of the tree structure and apply methods of "taboo probabilities". See [2] for a general view, compare also [1], [5]. – Let u, v be two adjacent vertices, we introduce the quantities

$$\begin{aligned} f_{u,v}^{(n)} &= Pr[X_n^{u,v} = u,\, X_m^{u,v} \neq u \text{ for } m = 1, \ldots, n-1 \mid X_0^{u,v} = u], \qquad f_{u,v}^{(0)} = 0 \\ q_{u,v}^{(n)} &= Pr[X_n^{u,v} = u \mid X_0^{u,v} = u], \qquad q_{u,v}^{(0)} = 1. \end{aligned} \tag{3.4}$$

For $k \geqq 0$, $f_{u,v}^{(n)}(k)$ and $q_{u,v}^{(n)}(k)$ are defined in the same way with the additional restriction: $\operatorname{dist}(X_m^{u,v}, u) \leqq k$ for $m \leqq n$. Set

$$F_{u,v} = \sum_{n=0}^{\infty} f_{u,v}^{(n)} \quad \text{and} \quad F_{u,v}(k) = \sum_{n=0}^{\infty} f_{u,v}^{(n)}(k). \tag{3.5}$$

$F_{u,v}$ is the probability that the SRW on $B_{u,v}$ ever returns to u, and $F_{u,v}(k)$ is the probability that it returns before reaching a distance larger than k from the starting point u. Furthermore, for the SRW on T we define

$$\begin{aligned} f_e^{(n)} &= Pr[X_n = e,\, X_m \neq e \text{ for } m = 1, \ldots, n-1 \mid X_0 = e], \qquad f_e^{(0)} = 1 \\ F_e &= \sum_{n=0}^{\infty} f_e^{(n)}. \end{aligned} \tag{3.6}$$

Then we have $p_{e,e}^{(n)} = \sum\limits_{i=0}^{n} f_e^{(i)} p_{e,e}^{(n-i)}$ for $n \leqq 1$, hence

$$G_e = 1/(1 - F_e) \tag{3.7}$$

and, in the same way

$$G_{u,v} = 1/(1 - F_{u,v}). \tag{3.8}$$

Now, $f_e^{(n)} = \sum\limits_{u \in U(e)} \dfrac{1}{d(e)} f_{e,u}^{(n)}$, hence

$$F_e = \frac{1}{d(e)} \sum_{u \in U(e)} F_{u,v}. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) we obtain a).

To satisfy the conditions of the definition of $f_{u,v}^{(n)}(k+1)$, the first step of $X_m^{u,v}$ has to go from u to v (with probability 1) and the last step from v to u (with probability $1/d(v)$), and in the meantime $X_m^{u,v}$ remains in $B_{u,v} - \{u\}$ at distance $\leqq k$ from v. Similar arguments as above yield (if $u \xrightarrow{e} v$)

$$F_{u,v}(k+1) = \frac{1}{d(v)} \bigg/ \left(1 - \sum_{w \in U(v)} \frac{1}{d(v)} F_{v,w}(k)\right). \tag{3.10}$$

We have for $n \geqq 1$: $q_{u,v}^{(n)}(k) = \sum_{i=0}^{n} f_{u,v}^{(i)}(k)\, q_{u,v}^{(n-i)}(k)$, hence

$$G_{u,v}(k) = 1/(1 - F_{u,v}(k)). \tag{3.11}$$

This together with (3.10) yields b). c) is an immediate consequence of the monotone convergence theorem. □

Note that in the recursion formula of (3.10) there is a certain similarity with continued fractions, which has in fact inspired the above and the following proposition. Continued fraction methods can be used to deal with simple random walks on certain trees [3], [8]. The continued fraction – analogue of the next proposition (with $=$ instead of $\leqq$) is [7], Th. 11.1: compare with [3]. –

Proposition 2. $G_{u,v}(k) \leqq \mathscr{V}_{u,v}(\omega, k)$, $k = 0, 1, 2, \ldots,$.

Proof. We use induction k. For $k = 0$, both terms have value 1.

$k \to k + 1$. The function $x \mapsto 1/x$ is convex for $x > 0$. Hence we obtain from $\sum_{w \in U(v)} \omega_{v,w} = 1$ and *Jensen's* inequality:

$$1 \Big/ \sum_{w \in U(v)} \omega_{v,w}^2\, \mathscr{V}_{v,w}(\omega, k) \leqq \sum_{w \in U(v)} 1/\mathscr{V}_{v,w}(\omega, k).$$

Combining this with Lemma 1b), the induction hypothesis and Proposition 1b) yields

$$\begin{aligned} \mathscr{V}_{u,v}(\omega, k+1) &\geqq 1 + 1 \Big/ \sum_{w \in U(v)} \frac{1}{\mathscr{V}_{v,w}(\omega, k)} \\ &\geqq 1 + 1 \Big/ \sum_{w \in U(v)} \frac{1}{G_{v,w}(k)} = G_{u,v}(k+1). \qquad \square \end{aligned}$$

4. Main results.

Theorem 1. *Given a reference vertex e of T and a weight function ω with respect to e, we have*

$$G_e(T) \leqq d(e)\, \mathscr{V}_e(\omega).$$

Proof. Proposition 2, Lemma 1c) and Proposition 1c) imply

$$G_{e,u} \leqq d(e)\, \mathscr{V}_{e,u}(\omega) \quad \text{for all } u \in U(e).$$

Using Lemma 1a) and *Jensen's* inequality we get

$$\mathscr{V}_e(\omega) \geqq 1 \Big/ \sum_{u \in U(e)} \frac{1}{\mathscr{V}_{e,u}(\omega)} \geqq 1 \Big/ \sum_{u \in U(e)} \frac{1}{G_{e,u}}.$$

The last term is equal to $\dfrac{G_e}{d(e)}$ by Proposition 1a). □

Suppose that X_n is transient, $G_e < \infty$. Besides the "natural" weight function and the corresponding (ordinary) volume there is another weight function $\hat{\omega}$, that we call the "harmonic" weight function, which is also natural in a certain sense. We define $\hat{\omega}$ as follows:

$$\hat{\omega}_{e,u} = (1 - F_{e,u})\, G_e/d(e) \qquad \text{if } u \in U(e). \tag{4.1}$$

For $v \neq e$, there is a unique vertex u such that $v \in U(u)$; we define for $w \in U(v)$

$$\hat{\omega}_{v,w} = \begin{cases} F_{u,v}(1 - F_{v,w})/(1 - F_{u,v}) & \text{if } F_{u,v} < 1 \\ \text{arbitrary (e.g. } = 1/(d(v) - 1)) & \text{if } F_{u,v} = 1 \end{cases} \tag{4.2}$$

($F_{u,v}$ as defined in (3.5)). Note that $F_{u,v} = 1$ (< 1) means that the SRW on $B_{u,v}$ is recurrent (transient). The proof of Proposition 1 shows that $\sum_{u \in U(e)} (1 - F_{e,u}) = d(e)/G_e$, i.e. $\sum_{u \in U(e)} \hat{\omega}_{e,u} = 1$. If $v \neq e$, $v \in U(u)$ and $F_{u,v} = 1$ then $\sum_{w \in U(v)} \hat{\omega}_{v,w} = 1$ by definition. Suppose $F_{u,v} < 1$ and let $k \to \infty$ in (3.10): we get $\sum_{w \in U(v)} (1 - F_{v,w}) = (1 - F_{u,v})/F_{u,v}$, hence $\sum_{w \in U(v)} \hat{\omega}_{v,w} = 1$ and $\hat{\omega}$ is in fact a weight function.

For $u \in V$ denote by $[e = u_0, u_1, \ldots, u_{k-1}, u_k = u]$ the unique shortest path in T connecting e and u. Then we get

$$\mathscr{V}_e(\hat{\omega}) = (G_e/d(e))^2 \sum_{u \in V} \sum_{v \in U(u)} (F_{e,u_1} F_{u_1,u_2} \cdots F_{u_{k-1},u}(1 - F_{u,v}))^2. \tag{4.3}$$

Define

$$W_{u,v} = Pr[\exists n \geqq 0 \colon X_n = v \mid X_0 = u] \text{ for } u, v \in V, \text{ in particular } W_u = W_{u,e}. \tag{4.4}$$

It is well known [1] that by the tree structure,

$$W_u = W_{u,u_{k-1}} W_{u_{k-1},u_{k-2}} \cdots W_{u_1,e}.$$

On the other hand it is obvious that for $u \in V$, $v \in U(u)$,

$$F_{u,v} = W_{v,u}. \tag{4.5}$$

Hence we have

$$\mathscr{V}_e(\hat{\omega}) = (G_e/d(e))^2 \sum_{u \in V} \sum_{v \in U(u)} (W_u - W_v)^2, \tag{4.6}$$

and the term on the right is in fact half of the energy of a flow on T as defined in [6]. For the following theorem we refine the inequalities obtained in [6] to show that the minimum among all generalized volumes is in fact $G_e(T)/d(e)$.

Theorem 2. *Given the reference vertex e, there is a weight function $\hat{\omega}$ such that*

$$d(e)\, \mathscr{V}_e(\hat{\omega}) = G_e(T).$$

Proof. If X_n is recurrent we have nothing to prove. So assume that X_n is transient and take $\hat{\omega}$ to be the harmonic weight function. Define for $k \geqq 0$

$$U_k = \{u \in V \mid \operatorname{dist}(u, e) \leqq k\}, \qquad U_k^+ = U_k - \{e\}$$

and for $u \in V$

$$W_u(k) = Pr[\exists n \geqq 0 \colon X_n = e, X_m \in U_k \text{ for } m = 0, \ldots, n \mid X_0 = u].$$

Observe that $W_e(k) = 1$, $W_u(k) \geqq W_v(k)$ if $u \xrightarrow{e} v$ and $W_u(k) = 0$ if $\operatorname{dist}(u, e) > k$. Clearly, $W_u(k)$ tends to W_u monotonically as $k \to \infty$. As in [6], we have

$$\sum_{u \in U_k^+} \sum_{\operatorname{dist}(v,u)=1} (W_u(k) - W_v(k))^2 \leqq \sum_{u \in U(e)} (1 - W_u(k)^2) \leqq d(e).$$

Hence, letting $k \to \infty$, we get by the dominated convergence theorem

$$\sum_{u \neq e} \sum_{\operatorname{dist}(v,u)=1} (W_u - W_v)^2 \leqq \sum_{u \in U(e)} (1 - W_u^2),$$

and adding on the left the term where $u = e$,

$$\begin{aligned} \sum_{u \in V} \sum_{\operatorname{dist}(v,u)=1} (W_u - W_v)^2 &\leqq \sum_{u \in U(e)} ((1 - W_u^2) + (1 - W_u)^2) \\ &= 2 \sum_{u \in U(e)} (1 - W_u). \end{aligned} \tag{4.7}$$

In (4.7), each edge has been counted twice. If we introduce the orientation $\xrightarrow{e}$ and use (4.5) and (3.9) we obtain

$$\sum_{u \in V} \sum_{v \in U(u)} (W_u - W_v)^2 \leqq \sum_{u \in U(e)} (1 - F_{e,u}) = d(e)/G_e,$$

in other words, $\mathscr{V}_e(\hat{\omega}) \leqq G_e/d(e)$. □

Observe that in the proof we did not use the tree structure.

Corollary 1. *A tree is recurrent if and only if there is a reference vertex e and a weight function ω with respect to e such that $\mathscr{V}_e(\omega) < \infty$.*

An immediate consequence of Corollary 1 is the following.

Corollary 2. *If a tree is recurrent then every subtree is recurrent.*

Proof. If T' is a subtree of T and ω' is a weight function on T' with respect to reference vertex $e \in V(T')$, then we can define a weight function on T: If $[u, v] \in E(T)$, set

$$\omega_{u,v} = \begin{cases} \omega'_{u,v}, & \text{if } u, v \in V(T') \\ 0, & \text{if } u \in V(T') \text{ and } v \in V(T) - V(T') \\ 1/(d(u) - 1), & \text{if } u, v \in V(T) - V(T'). \end{cases} \tag{4.8}$$

Obviously $\mathscr{V}_e(\omega') = \mathscr{V}_e(\omega)$, the latter volume is infinite and thus every generalized volume of T' is infinite. □

On the other hand, every tree has recurrent subtrees (for example, copies of the positive integers $\mathbb{N}$ with edges $[i, i+1]$, $i \in \mathbb{N}$), and a transient tree may even have infinitely many recurrent branches: take a homogeneous tree of degree $\geqq 3$ (which is well known to be transient), and to each vertex attach a copy of the half-line $\mathbb{N}$. Each of these half-lines is a recurrent branch. However, the construction of $\hat{\omega}$ in Theorem 2 yields the following observation.

Theorem 3. *Every transient tree has a transient subtree which has no recurrent branch.*

Proof. Consider $\hat{\omega}$ as defined in (4.1) and (4.2). If $u \in V$ and $v \in U(u)$ then (3.10) yields

$$F_{u,v} = 1/(1 + \sum_{w \in U(v)} (1 - F_{v,w})). \tag{4.9}$$

Hence, if $F_{u,v} < 1$ then $F_{v,w} < 1$ for some $w \in U(v)$. Thus, if we take

$$V' = \{e\} \cup \{v \in V - \{e\} \mid F_{u,v} < 1 \text{ for the unique } u \in V \text{ s.t. } u \xrightarrow{e} v\}, \tag{4.10}$$

then V' spans a subtree T' of T, and the restriction $\hat{\omega}'$ of $\hat{\omega}$ to T' is a weight function on T' such that $\mathscr{V}_e(\hat{\omega}') = \mathscr{V}_e(\hat{\omega}) < \infty$ and $\hat{\omega}'_{u,v} > 0$ for all $u, v \in V'$ with $v \in U(u)$. It may happen that T' is not a subtree in our sense, namely, if $d'(e) = 1$, where d' denotes the degree in T'. However, T' is transient and cannot be a half-line, i.e. there are vertices in T' of degree $\geqq 3$. Among these, let e'' be the one that is closest to e. Now let T'' be what remains when cutting off the line segment that connects e and e'' in T'. The restriction $\hat{\omega}''$ of $\hat{\omega}'$ to T'' is a strictly positive weight function on T'' with reference vertex e'', and $\mathscr{V}_{e''}(\hat{\omega}'') \leqq \mathscr{V}_e(\hat{\omega}')$.

If $B''_{u,v}$ is a branch of T'' such that $u \xrightarrow{e''} v$ then clearly $\lambda^2 \mathscr{V}_{u,v}(\hat{\omega}'') \leqq \mathscr{V}_{e''}(\hat{\omega}'') < \infty$, where $\lambda > 0$, and $B''_{u,v}$ is transient by Proposition 2 (with $k \to \infty$). If $v \xrightarrow{e''} u$ then by (4.9), recurrence of $B''_{u,v}$ would imply recurrence of $B''_{v,v_{k-1}}, B''_{v_{k-1},v_{k-2}}, \ldots, B''_{v_1,e''}$ and finally of $B''_{e'',w}$, where $[e'', v_1, \ldots, v_{k-1}, v]$ is the shortest path in T'' connecting e'' and v, and $w \in U''(e'') - \{v_1\}$. However, $B''_{e'',w}$ cannot be recurrent by the above argument. □

Coming back to "ordinary" volumes, another consequence of Theorem 1 is the following corollary.

Corollary 3. *If T' is a subtree of T and e is a vertex of T', then*

$$G_e(T) \leqq d(e)\, \mathscr{V}_e(T'),$$

where $d(e)$ is the degree of e in T and $\mathscr{V}_e(T')$ is the volume of T' with respect to e.

Proof. Take the "natural" weight function of T' and extend it to a weight function on T as in the proof of Corollary 2. Now Theorem 1 yields the result. □

We conclude with a conjecture (Corollary 3 is the "if"-part, and Theorem 3 seems to be a vague affirmation of the "only if"-part).

Conjecture. A tree is transient if and only if it has a subtree of finite (ordinary) volume.

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