

Uniqueness of currents in infinite resistive networks

Paolo M. Soardi and Wolfgang Woess

Dipartimento di Matematica, Università di Milano, via C. Saldini 50, 20133 Milano, Italy

Received 25 July 1988

Revised 13 June 1989

Abstract

Soardi, P.M. and W. Woess, Uniqueness of currents in infinite resistive networks, *Discrete Applied Mathematics* 31 (1991) 37–49.

If an infinite resistive network, whose edges have resistance 1 ohm, satisfies a certain graph theoretical condition, then the homogeneous Kirchhoff equations have no nonzero solutions vanishing at infinity. Every vertex transitive graph with polynomial growth satisfies such a condition. Furthermore uniqueness holds in Cartesian products of infinite regular graphs. Graphs with more than one end and satisfying an isoperimetric inequality provide a counterexample to uniqueness.

These results extend partially also to networks with nonconstant resistances.

Keywords: Infinite resistive networks, currents vanishing at infinity, Kirchhoff laws, harmonic functions on graphs, bounded automorphisms, polynomial growth, ends of a graph, Cartesian product of graphs.

1. Introduction

An infinite resistive network consists of an infinite, connected, locally finite graph Γ with the assignment of a nonnegative resistance to every edge. The network is energized by a number of voltage and current generators. A current in the network is defined as a 1-chain on Γ satisfying Kirchhoff's laws.

As noted by Zemanian [22] such laws, coupled with the relationships between currents and voltages given by the element of the network, are in general too weak to yield a unique current distribution on Γ . A natural restriction on the class of admissible solutions is therefore the class of all currents having finite energy.

In the papers by Flanders [3] and Zemanian [21] it was proved that, if the total power available in the network is finite, then there exists a current with finite energy flowing in the network and such a current is unique provided that it satisfies an extra

condition: in [3] the current must be a limit of currents in finite subnetworks, while in [21] the voltage drop must be zero also along infinite cycles. There are examples where the two current flows do not coincide, so that Kirchhoff's laws alone are not even sufficient to determine a unique solution with finite energy (take for instance a homogeneous tree of degree greater than 2, with all edges of resistance 1 ohm; see [17] and the discussion in [3, p. 326] and in [21]; see also Section 4 below).

In Sections 2–4 we assume that all the edges of Γ have resistance 1 ohm. Suppose that there is a bounded automorphism of the graph with no periodic points. Under these conditions we are able to prove that there are no currents vanishing at infinity (and, a fortiori, no currents with finite energy) satisfying the *homogeneous* Kirchhoff equations. This amounts to proving that there are no nonconstant harmonic functions on Γ with “first differences” vanishing at infinity.

Therefore for such networks, on account of the results by Flanders and Zemanian mentioned above, there is a unique current with finite energy flowing in the network energized as in [3] or [21].

While there are obvious examples of graphs satisfying the above stated condition (e.g. the infinite 2-dimensional grid studied in [4] or, more generally, the infinite n -dimensional grid), we are able to prove, using results by Trofimov and by Sabidussi [19,15], that every vertex transitive graph with polynomial growth has that property (Theorem 3.2). Moreover we prove the uniqueness result by other methods for Cartesian products of infinite regular graphs (Theorem 3.3). We notice here that a uniqueness result was proved by Thomassen [17], by other methods, for a class of planar graphs of square growth (but not necessarily vertex transitive).

In Section 4 we exhibit a class of graphs for which uniqueness fails. They are the graphs which satisfy the isoperimetric inequality of Dodziuk [2] and have at least two ends. This class of graphs includes in particular all vertex transitive graphs with more than two ends and trees which do not have arbitrarily long unbranched paths.

In Section 5 we discuss extensions of the previous results to the case where the resistances assigned to the edges do not necessarily have constant value one.

2. Preliminaries

Throughout this paper Γ will denote an infinite, connected, locally finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The vertex and edge set are given the discrete topology. The edges (bonds) are unoriented *a priori* and there are no self-loops. Every edge is assigned resistance 1 ohm and a fixed orientation (the case of more general resistances will be discussed in Section 5). None of the results presented here depends on the orientation; in general we shall write B for an *oriented* edge, while $[x, y]$ will not refer to orientation.

We say that two vertices x and y are neighbours, $x \sim y$, if they are joined by an edge. The degree of x , $\deg(x)$, is the number of neighbours of x . The distance $d(x, y)$ of two arbitrary vertices x and y is the minimal length (number of nonoriented

edges) in a path connecting x and y . An automorphism of Γ is an isometry of $V(\Gamma)$ onto itself with respect to this metric, and $\text{Aut}(\Gamma)$ denotes the group of all automorphisms of Γ .

In Flanders' setup currents are represented by 1-chains, voltage generators by 1-cochains, voltage distributions by coboundaries etc. We refer to Flanders' paper [3] for definitions and properties of these and other related notions. We will also keep the notations of [3] unless otherwise stated. Let us only recall that a 1-chain I is a *formal infinite* linear combination, with real coefficients i_B , of directed edges, i.e.,

$$I = \sum_{B \in E(\Gamma)} i_B B. \quad (1)$$

In particular a chain $Z = \sum_{B \in E(\Gamma)} z_B B$ is called a finite cycle if only finitely many coefficients z_B are different from 0 and if $\partial Z \equiv 0$, where ∂ is the boundary operator which maps 1-chains to 0-chains (see [3]).

Flows with finite energy are represented by square summable 1-chains I , i.e., chains of the form (1) such that $\sum_{B \in E(\Gamma)} i_B^2 < \infty$. We shall also consider the more general class of all 1-chains vanishing at infinity.

Definition. We say that a chain I of the form (1) *vanishes at infinity* if

$$i_B \rightarrow 0 \quad \text{as } B \rightarrow \infty \text{ in } E(\Gamma).$$

Let \mathcal{R} be the resistance operator (which maps 1-chains to 1-cochains) and $\langle \cdot, \cdot \rangle$ the pairing between 1-cochains and 1-chains.

Definition. Let E be a 1-cochain. We say that a 1-chain I *satisfies the Kirchhoff equations* if

$$\begin{aligned} \partial I &\equiv 0, \\ \langle \mathcal{R}(I) - E, Z \rangle &= 0 \quad \text{for all finite cycles } Z. \end{aligned}$$

In this case we will say that I is a current (relative to E).

In particular, we say that I satisfies the *homogeneous* Kirchhoff equations if

$$\begin{aligned} \partial I &\equiv 0, \\ \langle \mathcal{R}(I), Z \rangle &= 0 \quad \text{for all finite cycles } Z. \end{aligned} \quad (2) \quad (3)$$

By [21, Theorem 3.3], given E such that $\mathcal{R}^{-1}(E)$ has finite energy, if the homogeneous Kirchhoff equations have no nonzero solution vanishing at infinity, then there exists a unique current I with finite energy. Moreover, if E is finite, I is the limit (in the l^2 norm given by the energy) of finite cycles.

Vice versa, if (2) and (3) admit a nontrivial solution with finite energy, then there are infinitely many currents with finite energy. These considerations can be restated in terms of harmonic functions.

Definition. Suppose u is a function on $V(\Gamma)$. We say that u is *harmonic* on Γ if $P(u) = u$, where the operator P , corresponding to the simple random walk on Γ (see e.g. Gerl [7]), is given by

$$P(u)(x) = \frac{1}{\deg(x)} \sum_{x \sim y} u(y) \quad (4)$$

for all $x \in V(\Gamma)$.

By the Theorem in [3, p. 328], a 1-chain satisfies the homogeneous Kirchhoff equations if and only if there is a harmonic function u on Γ (the potential) such that

$$u(y) - u(x) = i_B, \quad (5)$$

whenever B is an edge and x, y are its origin and endpoint, respectively. In particular there is at most one electric current vanishing at infinity or having finite energy, if and only if the constants are the only harmonic functions whose first differences (5) have the required behaviour at infinity.

3. Uniqueness

We already noted that there are infinite networks such that there exist nonzero solutions vanishing at infinity to (2) and (3). We will show that this is not the case if the graph Γ satisfies a certain condition.

Definition. Let ϕ be an automorphism of the graph Γ . We say that ϕ is *bounded* if

$$\sup_{x \in V(\Gamma)} d(x, \phi(x)) < \infty.$$

We say that a point $x \in V(\Gamma)$ is *periodic* if $\phi^n(x) = x$ for some integer $n \geq 1$, where ϕ^n is the n th iterate of ϕ , i.e., $\phi^n(x) = \phi(\phi^{n-1}(x))$ (compare with Halin [10, Proposition 12]).

Theorem 3.1. *Suppose that there exists a bounded automorphism ϕ of Γ with no periodic points. Let I be a solution to the homogeneous Kirchhoff equations (2) and (3). If I vanishes at infinity, then $I \equiv 0$.*

Proof. Let u be a harmonic function on Γ whose first differences (5) vanish at infinity. We define a new function $f(x) = u(\phi(x)) - u(x)$ on $V(\Gamma)$. Clearly f is harmonic. Since Γ is connected and ϕ is bounded, there exists a number $K > 0$ independent of x such that $\phi(x)$ and x are joined by an unoriented path π_x whose length does not exceed K . Therefore $\phi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Now, f can be expressed as a sum, with bounded number of terms, of first differences of u along π_x . By assumption the latter tend to 0 as x tends to infinity. Hence $f(x) \rightarrow 0$ as $x \rightarrow \infty$. If $\varepsilon > 0$, there is a finite set U of vertices such that $|f(x)| < \varepsilon$

if $x \notin U$. By the maximum (minimum) principle for harmonic functions, we must have $|f(x)| < \varepsilon$ for all $x \in V(\Gamma)$. We infer $f(x) \equiv 0$, so that $u(\phi(x)) = u(x)$ for all x .

Let now x be fixed and let y be a neighbour of x . Then, for every positive integer n

$$u(x) - u(y) = u(\phi(x)) - u(\phi(y)) = \dots = u(\phi^n(x)) - u(\phi^n(y)). \quad (6)$$

Since ϕ has no periodic points $\phi^n(x) \rightarrow \infty$ and $\phi^n(y) \rightarrow \infty$ as $n \rightarrow \infty$. Using again the fact that I vanishes at infinity and (4) we obtain $u(x) = u(y)$ and, by connectedness, $u(x) = u(z)$ for all $z \in V(\Gamma)$. Therefore u is constant and $I \equiv 0$. \square

The problem now arises of investigating which graphs fall within the scope of Theorem 3.1. A first simple example is provided by the graphs which can be imbedded into R^n in such a way that the imbedding is periodic in some direction. By this we mean that there exists $v \in R^n$ such that the map $x \mapsto x + v$ ($x \in V(\Gamma)$) induces a graph automorphism. In particular, infinite or semi-infinite n -dimensional grids (compare with [23]) satisfy this condition as well as the periodic graphs considered in percolation theory.

In general every Cartesian product $Z \times \Gamma$ (where Z is the two-way infinite path, represented by the integers) has clearly a bounded automorphism with no periodic points (see also Theorem 3.3 below).

We now show that another important example is provided by the class of all vertex transitive graphs with polynomial growth, whose combinatorial structure has been studied in detail by Trofimov [19].

Definition. We will say that Γ has *polynomial growth* if the number of vertices of Γ at a distance not more than n from some fixed vertex is bounded above by a polynomial in n .

This class of graphs includes all lattices (in the sense of Trofimov [18]) and the Cayley graphs of infinite, finitely generated nilpotent-by-finite groups.

Before stating the next result, let us recall the notion of the quotient graph with respect to an imprimitivity system.

An imprimitivity system of a vertex transitive group $G \leq \text{Aut}(\Gamma)$ is a partition σ of $V(\Gamma)$ into subsets called blocks, such that every element of G is a permutation of the blocks of σ . The quotient graph Γ^σ has vertex and edge set given respectively by

$$\begin{aligned} V(\Gamma^\sigma) &= V(X)/\sigma \quad (\text{the blocks of } \sigma), \\ E(\Gamma^\sigma) &= \{[x^\sigma, y^\sigma]: x^\sigma \neq y^\sigma, [x, y] \in E(X)\}, \end{aligned}$$

where x^σ is the block containing x . There is a natural homomorphism from G into $\text{Aut}(\Gamma^\sigma)$. The corresponding image of $\gamma \in G$ is denoted by γ^σ , and $G^\sigma = \{\gamma^\sigma: \gamma \in G\}$.

Theorem 3.2. *If Γ is vertex transitive and has polynomial growth, then there exists a bounded automorphism of Γ with no periodic points.*

Proof. By [19, Theorem 2], there exists an imprimitivity system σ of $\text{Aut}(\Gamma)$ with finite blocks such that $H = \text{Aut}(\Gamma)^\sigma$ is a finitely generated nilpotent-by-finite group and the stabilizer in H of a vertex in Γ^σ is finite. Moreover, $\gamma \in \text{Aut}(\Gamma)$ is bounded on Γ if and only if γ^σ is bounded on Γ^σ .

For a given set A of generators of H , denote by $X_A = X(H, A)$ the Cayley graph of H with respect to A . Fix a vertex of Γ^σ and let J denote its stabilizer in H . By [15, Theorem 2] (and its proof) there is a finite set of generators A of H such that $A \cap J = \emptyset$ and Γ^σ is isomorphic with X_A^τ , where τ denotes the imprimitivity system induced by the left cosets of J in $H = V(X_A)$.

By [12, Theorem 17.2.2] we know that there exists an infinite, torsion free, nilpotent subgroup $N \leq H$ with finite index in H . In particular, N has nontrivial center.

Let ξ be an element in the center of N different from the identity e . Then $\{g^{-1}\xi g : g \in H\}$ is finite, and

$$\sup_{x^\sigma \in V(\Gamma^\sigma)} d(x^\sigma, \xi x^\sigma) \leq \sup_{g \in H} d(g, \xi g) = \sup_{g \in H} d(e, g^{-1}\xi g) < \infty$$

(in the first term d refers to the metric of Γ^σ , while it refers to the metric of X_A in the other two terms).

In other words, ξ is bounded on Γ^σ . Lifting ξ back to an automorphism ϕ of Γ , we have that ϕ is bounded.

Suppose now that $\phi^n(x) = x$ for some integer $n > 1$ and $x \in V(\Gamma)$. Then $\xi^n x^\sigma = x^\sigma$ and, by the above, x^σ arises by isomorphism as a coset gJ , for some $g \in H$. Therefore $\xi^n gJ = gJ$, so that $\xi^{nk} g = g$ for $k = |J|!$. Hence $\xi^{nk} = e$, a contradiction. This concludes the proof. \square

Next, we turn our attention to Cartesian products of graphs. Suppose that Γ_1 and Γ_2 are (infinite, connected) graphs. The Cartesian product $\Gamma_1 \times \Gamma_2$ is defined as the graph Γ such that

$$\begin{aligned} V(\Gamma) &= V(\Gamma_1) \times V(\Gamma_2), \\ E(\Gamma) &= \{[(x_1, x_2), (y_1, y_2)] : \text{either } x_2 = y_2 \text{ and } [x_1, y_1] \in E(\Gamma_1) \\ &\quad \text{or } x_1 = y_1 \text{ and } [x_2, y_2] \in E(\Gamma_2)\}. \end{aligned}$$

The Cartesian product of n graphs is now defined recursively.

We will show that in the case of Cartesian products of $n \geq 2$ infinite regular graphs, the homogeneous Kirchhoff equations have no nontrivial solutions vanishing at infinity (regular means that all vertices have the same finite degree). The proof, although similar to the proof of Theorem 3.1, does not rest on the existence of bounded automorphisms with no periodic points.

Theorem 3.3. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ where each Γ_j ($j = 1, \dots, n$) is an infinite, connected regular graph. If I satisfies equations (2) and (3) in Γ and vanishes at infinity, then $I \equiv 0$.*

Proof. We may assume $n=2$. For $j=1,2$ let d_j denote the common vertex degree in Γ_j . Let P_j denote the random walk operators as defined in (4) for functions on $V(\Gamma_j)$. We lift each of these to an operator defined for functions on $V(\Gamma)$:

$$P_1(f)(x_1, x_2) = \frac{1}{d_1} \sum_{\substack{x_1 \sim y_1 \\ y_1 \in V(\Gamma_1)}} f(y_1, x_2),$$

$$P_2(f)(x_1, x_2) = \frac{1}{d_2} \sum_{\substack{x_2 \sim y_2 \\ y_2 \in V(\Gamma_2)}} f(x_1, y_2).$$

Set $d = d_1 + d_2$. Then the linear operator P associated with the simple random walk on Γ is given by:

$$P = \frac{d_1}{d} P_1 + \frac{d_2}{d} P_2. \quad (7)$$

Obviously P commutes with P_1 and P_2 (compare e.g. with Cartwright and Soardi [1]). Let now I be a solution of (2) and (3) vanishing at infinity and let u denote a harmonic function on Γ whose first differences satisfy (5). Set $v = P_1(u)$. Then

$$v = P_1(P(u)) = P(P_1(u)) = P(v).$$

Hence v is harmonic on Γ and so is the difference $u - v$. Since I vanishes at infinity we clearly have $(u - v)(x_1, x_2) \rightarrow 0$ as $(x_1, x_2) \rightarrow \infty$. As in the proof of Theorem 3.1, connectedness of Γ and the maximum principle now yield $u = v$.

By the definition of v we obtain $u = P_1(u)$. In other words, for each fixed $x_2 \in \Gamma_2$, $u(\cdot, x_2)$ is a harmonic function on Γ_1 . Fix x_2 and let y_2 be any neighbour of x_2 in Γ_2 . Then the difference $u(\cdot, x_2) - u(\cdot, y_2)$ is harmonic on Γ_1 and $u(x_1, x_2) - u(x_1, y_2) \rightarrow 0$ as $x_1 \rightarrow \infty$. By the maximum principle, this time applied to harmonic functions on Γ_1 , it follows that $u(x_1, x_2) = u(x_1, y_2)$ for every $(x_1, x_2) \in V(\Gamma)$ and every neighbour y_2 of x_2 in $V(\Gamma_2)$.

Since by (7) we have also $u = P_2(u)$, repeating the above argument we obtain also $u(x_1, x_2) = u(y_1, x_2)$ for every $(x_1, x_2) \in V(\Gamma)$ and every neighbour y_1 of x_1 in $V(\Gamma_1)$. Hence u is constant and $I \equiv 0$. \square

Remark. An instance of the previous theorem is provided by the Cartesian product Γ of $n \geq 2$ homogeneous trees (Bethe lattices) of degree q . Note that, if $q > 2$, uniqueness fails for each factor, as shown by Theorem 4.1 below.

It is not difficult to write the effective resistance between two neighbouring vertices a and b of Γ when $q > 2$. In fact (thinking of Γ as a Cayley graph) if $G(x)$ denotes the expected number of visits to x in the simple random walk starting from the identity e , then G is harmonic at every point except e and

$$u(x) = (nq)^{-1}(G(xa^{-1}) - G(xb^{-1})) \in l^2(V(\Gamma))$$

(see e.g. [1,7]). Therefore, by the properties of G and Theorem 3.3, u represents the

potential generated by a dipole at a and b . It follows from the properties of G again that the value of the effective resistance is $2/nq$.

4. Nonuniqueness

In this section we show that for a large class of graphs there are nontrivial solutions with finite energy to equations (2) and (3). This class includes for instance the homogeneous trees of degree greater than 2.

Suppose that L is a subgraph of Γ . In this section we will denote by $\beta(L)$ the boundary of L , defined as the subset of all vertices in $V(L)$ which have a neighbour $x \in V(\Gamma)$ not in $V(L)$. Finally, we will denote by $\Gamma \setminus L$ the subgraph of Γ induced by all vertices in $V(\Gamma)$ which are not in $V(L)$.

Definition [2,7]. Γ is said to satisfy a *strong isoperimetric inequality* if there is a constant $\kappa > 0$ such that, for every finite subgraph L

$$|\beta(L)| \geq \kappa |L|, \quad (8)$$

where $|\cdot|$ denotes cardinality.

Note that if Γ satisfies a strong isoperimetric inequality, then Γ has exponential growth: the number of vertices in a ball of radius n around some fixed vertex $x \in V(\Gamma)$ tends to infinity exponentially with n .

If L is a finite subgraph, then by the connectedness and local finiteness of Γ , there is a finite number $\nu(L)$ of infinite connected components in $\Gamma \setminus L$. The number of ends of Γ is defined by

$$\sup\{\nu(L): L \text{ a finite subgraph of } \Gamma\}. \quad (9)$$

One can go more into detail and introduce the space of all ends of a locally finite, connected graph. This concept is due to Freudenthal [5] and Halin [9]. Indeed, ends are of some interest in the study of harmonic functions, see Picardello and Woess [14].

Theorem 4.1. *Suppose that Γ satisfies a strong isoperimetric inequality (8) and that*

$$\sup_{x \in V(\Gamma)} \deg(x) < \infty. \quad (10)$$

If Γ has at least two ends, then the homogeneous Kirchhoff equations (2),(3) have a nonzero solution with finite energy.

Proof. It suffices to show that there exists a nonconstant harmonic function u on $V(\Gamma)$ such that

$$\sum_{\substack{x, y \in V(\Gamma), \\ x \sim y}} |u(x) - u(y)|^2 < \infty. \quad (11)$$

First of all there exists a finite subgraph L of Γ such that $\Gamma \setminus L$ splits into $v \geq 2$ infinite connected components $\Gamma_1, \dots, \Gamma_v$.

Let c_1, \dots, c_v be pairwise different real numbers. Define a function u_1 on $V(\Gamma)$ in the following way

$$u_1(x) = \begin{cases} c_j, & \text{if } x \in V(\Gamma_j), j=1, \dots, v, \\ 0, & \text{if } x \in V(L). \end{cases}$$

Then u_1 is a well-defined function on $V(\Gamma)$ which is harmonic in every point in the interior of Γ_j for all j . Moreover (11) holds with u_1 in place of u .

Consider now the operator P defined as in (4) for all functions on $V(\Gamma)$. By (8), (10) and [7], P has norm smaller than 1 as an operator on $l^2_{\#}(V(\Gamma))$, the Hilbert space of all functions f on $V(\Gamma)$ such that

$$\|f\|_{\#}^2 = \sum_{x \in V(\Gamma)} |f(x)|^2 \deg(x) < \infty.$$

Hence, the Green operator

$$G = (\text{Id} - P)^{-1} = \sum_{k=0}^{\infty} P^k,$$

where Id denotes the identity operator, is a well-defined bounded operator on $l^2_{\#}(V(\Gamma))$.

Set now $f(x) = (\text{Id} - P)(u_1)(x)$. Then f belongs to $l^2_{\#}(V(\Gamma))$ (it has finite support). It follows that $u_2 = G(f)$ belongs to $l^2_{\#}(V(\Gamma))$ too. Therefore the function $u = u_1 - u_2$ is harmonic. By (10), u_2 and hence also u satisfy (11). Finally, u_2 vanishes at infinity, so that $u(x)$ tends to c_j as x tends to infinity within Γ_j . Thus u is non-constant. \square

Remark. Actually the above quoted theorem of Gerl implies that $\sum_{x \sim y} |u(x) - u(y)|^r < \infty$ for all $r \in (1, +\infty)$. Moreover (8) is equivalent to the fact that the spectral radius of the simple random walk on Γ is smaller than 1, if (10) holds.

Once more we turn to the question to determine particular classes of graphs which satisfy the assumptions of Theorem 4.1. It is well known and not hard to prove that an infinite vertex transitive graph has one, two or infinitely many ends (see e.g. [5], where this is stated for Cayley graphs, and [10]). Vertex transitive graphs with two ends have linear growth, see Imrich and Seifert [11], so that the uniqueness result of Theorem 3.2 applies. On the other hand it is proved in Soardi and Woess [16] that every vertex transitive, locally finite, connected graph with infinitely many ends satisfies a strong isoperimetric inequality.

Corollary 4.2. *If Γ is a vertex transitive, connected, locally finite graph with infinitely many ends, then there is a nontrivial solution of the homogeneous Kirchhoff equations with finite energy.*

Remark. The homogeneous trees of degree at least three fall into this class, but there are also more complicated examples, such as free products of two (or more) vertex transitive graphs (see Mohar and Woess [13] for a definition). Note that, in accordance with our results, every bounded automorphism of a vertex transitive graph with infinitely many ends has finite order (see Godsil et al. [8] for the details).

We remark that a vertex transitive graph with only one end may (e.g. Cartesian products of homogeneous trees of degree greater than three), or may not satisfy a strong isoperimetric inequality (e.g. graphs with nonlinear polynomial growth, see Section 3).

Finally, we consider a class of not necessarily homogeneous trees. If Γ is an infinite tree, we say that a path in Γ is unbranched if all of its vertices have degree 2.

Corollary 4.3. *Let Γ denote an infinite tree satisfying (10), such that there are only finitely many vertices of degree 1 and there is a finite upper bound for the length of all unbranched paths of Γ . Then, the homogeneous Kirchhoff equations have a nontrivial solution with finite energy.*

Proof. Indeed, such a tree has infinitely many ends and satisfies the strong isoperimetric inequality (Gerl [6]), so that Theorem 4.1. applies. \square

5. Nonconstant resistances

In this final section we discuss how the results of the preceding sections extend to the case when the resistances assigned to the edges are not necessarily constant. While the graph Γ is as in Section 2, each edge B is assigned an arbitrary positive resistance r_B , giving rise to a resistance operator as described in [3]. We will refer to the pair $\mathcal{N} = (\Gamma, \mathcal{R})$ as a *network*. An automorphism of a network is defined as an automorphism of Γ which leaves invariant the resistances, or, in other words, commutes with \mathcal{R} .

While the homogeneous Kirchhoff equations remain formally the same as in (2), (3), now the energy of a flow I becomes $\sum_{B \in E(\Gamma)} r_B i_B^2$. Thus, in general, it is not true that a flow with finite energy vanishes at infinity. This is however the case if $\inf_{B \in E(\Gamma)} r_B > 0$.

The conductance of the edge $[x, y]$ is the number $c(x, y) = r_B^{-1}$, where B is the oriented version of $[x, y]$. The total conductance at $x \in V(\Gamma)$ is $c(x) = \sum_{x \sim y} c(x, y)$. We have now to consider harmonic functions u on \mathcal{N} with respect to the operator P given by

$$P(u)(x) = \frac{1}{c(x)} \sum_{x \sim y} c(x, y) u(y), \quad (4')$$

for all $x \in V(\Gamma)$.

Consequently, we will say that a function u on $V(\Gamma)$ is harmonic on \mathcal{N} if $P(u) = u$. The Theorem in [3, p. 328] now reads as follows.

Lemma 5.1. *A 1-chain I satisfies the homogeneous Kirchhoff equations if and only if there exists a harmonic function u on \mathcal{N} such that*

$$u(y) - u(x) = r_B i_B, \quad (5')$$

whenever B is an edge and x, y are its origin and endpoint respectively.

The proof of Theorem 3.1 remains valid in this setting.

Corollary 5.2. *Suppose that the resistances of \mathcal{N} have a positive lower bound and that there exists a bounded automorphism of \mathcal{N} with no periodic points. If I solves the homogeneous Kirchhoff equations and vanishes at infinity, or has finite energy, then $I \equiv 0$.*

If $\text{Aut}(\mathcal{N})$ acts transitively on $V(\Gamma)$, then there is only a finite number of distinct resistances in \mathcal{N} . Hence every flow vanishing at infinity has finite energy. Furthermore, if Γ has polynomial growth, we may apply [19, Theorem 2] to $\text{Aut}(\mathcal{N})$ in place of $\text{Aut}(\Gamma)$.

Corollary 5.3. *If \mathcal{N} is vertex transitive and Γ has polynomial growth, then there is a bounded automorphism of \mathcal{N} with no periodic points. In particular, the only solution of the homogeneous Kirchhoff equations vanishing at infinity is $I \equiv 0$.*

The Cartesian product of networks $\mathcal{N}_1, \dots, \mathcal{N}_n$ is defined in the obvious way. In order to obtain the analogue of Theorem 3.3 we must require, instead of the regularity of Γ_j , that the total conductance at every node of \mathcal{N}_j is a constant c_j .

Corollary 5.4. *Let $\mathcal{N}_j = (\Gamma_j, \mathcal{R}_j)$, $j = 1, \dots, n$ ($n \geq 2$), be infinite networks satisfying*

$$\begin{aligned} \inf_{B \in E(\Gamma_j)} r_B &> 0, \\ c_j(x) &= \sum_{\substack{x \sim y, \\ y \in V(\Gamma_j)}} c_j(x, y) = c_j \quad (\text{constant for all } x \in V(\Gamma_j)). \end{aligned}$$

If I is a 1-chain vanishing at infinity on $\mathcal{N} = \mathcal{N}_1 \times \dots \times \mathcal{N}_n$ satisfying equations (2) and (3), then $I \equiv 0$.

Proof. It suffices to note that (7) now must be replaced by

$$P = \frac{c_1}{c} P_1 + \frac{c_2}{c} P_2,$$

where P , P_1 and P_2 are the respective “random walk” operators and $c = c_1 + c_2$ in the Cartesian product of two networks. \square

Finally, we have the following extension of Theorem 4.1.

Corollary 5.5. *Suppose that $\mathcal{N} = (\Gamma, \mathcal{R})$ is a network such that there are constants m and M satisfying*

$$0 < m \leq r_B \leq M < \infty \quad \text{for all } B \in V(\Gamma). \quad (12)$$

If Γ satisfies conditions (8) and (10) above and has more than one end, then the homogeneous Kirchhoff equations have a nontrivial solution with finite energy.

Proof. By [7] the operator P given in (4') has norm smaller than 1 as an operator on $l^2(V(\Gamma), c(\cdot))$. The proof now is as in Theorem 4.1. \square

We see from Corollary 5.5 that Corollary 4.2 remains true with \mathcal{N} instead of Γ , while in Corollary 4.3 we have to add condition (12).

Finally we mention a remarkable uniqueness result which is essentially proved in Yamasaki [20, Lemma 1.3 and Lemma 3.1]: if the operator P defined in (4') defines a recurrent random walk (i.e., if the network \mathcal{N} is parabolic in Yamasaki's terminology), then there are no nontrivial harmonic functions with finite energy.

References

- [1] D.I. Cartwright and P.M. Soardi, A local limit theorem for random walks on the cartesian product of discrete groups, *Boll. Un. Mat. Ital. A* (1987) 107–115.
- [2] J. Dodziuk, Isoperimetric inequality and transience of certain random walks, *Trans. Amer. Math. Soc.* 284 (1984) 787–794.
- [3] H. Flanders, Infinite electrical networks: I – resistive networks, *IEEE Trans. Circuit Theory* 18 (1971) 326–331.
- [4] H. Flanders, Infinite electrical networks: II – resistance in an infinite grid, *J. Math. Anal. Appl.* 40 (1972) 30–34.
- [5] H. Freudenthal, Über die Enden diskreter Räume und Gruppen, *Comment. Math. Helv.* 17 (1944) 1–38.
- [6] P. Gerl, Eine isoperimetrische Eigenschaft von Bäumen, *Sitzungsber. Öst. Akad. Wiss. Math.-Natur. Kl.* 195 (1986) 49–52.
- [7] P. Gerl, Random walks on a graph with a strong isoperimetric property, *J. Theoret. Probab.* 1 (1988) 171–188.
- [8] C.D. Godsil, W. Imrich, N. Seifter, M.E. Watkins and W. Woess, A note on bounded automorphisms of infinite graphs, *Graphs Combin.* 5 (1989) 333–338.
- [9] R. Halin, Über unendliche Wege in Graphen, *Math. Ann.* 157 (1964) 125–137.
- [10] R. Halin, Endomorphisms and automorphisms of infinite locally finite graphs, *Abh. Math. Sem. Univ. Hamburg* 39 (1973) 251–283.
- [11] W. Imrich and N. Seifter, A note on the growth of transitive graphs, *Discrete Math.* 73 (1988/89) 111–117.

- [12] M.I. Kargapolov and Ju.I. Merzljakov, *Fundamentals of the Theory of Groups* (Springer, New York, 1979).
- [13] B. Mohar and W. Woess, A survey on spectra of infinite graphs, *Bull. London Math. Soc.* 21 (1989) 209–234.
- [14] M. Picardello and W. Woess, Harmonic functions and ends of graphs, *Proc. Edinburgh Math. Soc.* 31 (1988) 457–461.
- [15] G. Sabidussi, Vertex transitive graphs, *Monatsh. Math.* 68 (1964) 427–438.
- [16] P.M. Soardi and W. Woess, Amenability, unimodularity, and the spectral radius of random walks on infinite graphs, *Math. Z.*, to appear.
- [17] C. Thomassen, Resistance and currents in an infinite electrical network, *J. Combin. Theory Ser. B*, to appear.
- [18] V.I. Trofimov, Automorphisms of graphs and a characterization of lattices, *Math. USSR-Izv.* 22 (1984) 379–391.
- [19] V.I. Trofimov, Graphs with polynomial growth, *Math. USSR-Sb.* 51 (1985) 405–417.
- [20] M. Yamasaki, Discrete potentials on an infinite network, *Mem. Fac. Sci. Shimane Univ.* 13 (1979) 31–44.
- [21] A.H. Zemanian, Infinite electrical networks, *Proc. IEEE* 64 (1976) 6–17.
- [22] A.H. Zemanian, The complete behaviour of certain infinite networks under Kirchhoff's node and loop laws, *SIAM J. Appl. Math.* 30 (1976) 278–295.
- [23] A.H. Zemanian, The characteristic-resistance method for grounded semi-infinite grids, *SIAM J. Math. Anal.* 12 (1981) 115–138.