Homework 1

Exercise 1. Let $f : A \longrightarrow B$ be a ring homomorphism and let $\mathfrak{q}$ be a prime ideal of B. Show that $f^\star(\mathfrak{q}) := f^{-1}(\mathfrak{q})$ is a prime ideal of A. Thus f induces a natural map $f^\star : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$, $\mathfrak{q} \mapsto f^\star(\mathfrak{q})$.

Exercise 2. Preserve the notation given in Exercise 1. Is it true that if $\mathfrak{m}$ is a maximal ideal of B, then $f^\star(\mathfrak{m})$ is a maximal ideal of A?

Exercise 3. Let $f : A \longrightarrow B$ be a surjective ring homomorphism, and let $f^\star : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ be the natural map defined in Exercise 1.

(1) Show that $f^\star(\mathrm{Max}(B)) \subseteq \mathrm{Max}(A)$.
(2) Give an example of when the inclusion $\subseteq$ of part (1) is strict.
(3) Show that, if f is an isomorphism, then $f^\star(\mathrm{Max}(B)) = \mathrm{Max}(A)$.

Exercise 4. Let K be a field and let $A := K[T_1, \dots, T_n]$ be the polynomial ring over K in the indeterminates $T_1, \dots, T_n$. Let $\mathbf{a} := (\lambda_1, \dots, \lambda_n) \in K^n$.

(1) Let $\tau_{\mathbf{a}} : A \longrightarrow A$ be the ring homomorphism defined by setting $\tau_{\mathbf{a}}(f(T_1, \dots, T_n)) := f(T_1 - \lambda_1, \dots, T_n - \lambda_n)$. Show that τ is a ring automorphism.
(2) Show that the ideal $(T_1, \dots, T_n)A$ is a maximal ideal of A.
(3) More generally, show that $\mathfrak{m}_{\mathbf{a}} := (T_1 - \lambda_1, \dots, T_n - \lambda_n)A$ is a maximal ideal of A.
(4) Find a field K and a positive integer n such that the inclusion

$$\{\mathfrak{m}_{\mathbf{a}} : \mathbf{a} \in K^n\} \subseteq \mathrm{Max}(A)$$

is strict.

Exercise 5. Let A be an integral domain, K be the quotient field of A, T be an indeterminate over A and let λ be a nonzero element of A. Consider the ring homomorphism

$$\tau : A[T] \longrightarrow K, \qquad f(T) \mapsto f(\lambda^{-1})$$

(1) Show that $\mathfrak{p} := \mathrm{Ker}(\tau)$ is a prime ideal of $A[T]$.
(2) Show that, if $f(T) \in \mathfrak{p}$ and $f(T)$ is linear, then $f(T) \in (\lambda T - 1)A[T]$.
(3) More generally, show that $\mathfrak{p} = (\lambda T - 1)A[T]$. [Hint: take a polynomial $f(T) \in \mathfrak{p}$. Argue by induction on the degree h of $f(T)$. Note that, if $f(T) = a_h T^h + \dots a_1 T + a_0$, then there is a polynomial $g(T) \in A[T]$ such that $f(T) + a_0(\lambda T - 1) = Tg(T)$...]