MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 10

"When you love someone, you love the person as they are,
and not as you'd like them to be."

L. Tolstoj

Exercise 1. Let $f : X \longrightarrow Y$ be a continuous map of T_0 spaces.

(1) Show that f is order preserving, with respect to the orders induced by the topologies of X and Y.
(2) Deduce that, if f is a homeomorphism, then f is an order isomorphism.

Exercise 2. Let X be a compact T_0 space and let $x_0 \in X$. Determine if there exists a point $x \in X$ such that $x_0 \leq x$ and $\{x\}$ is closed, where $\leq$ denotes the order induced by the topology of X.

Exercise 3. Let K be a field and let T, U be indeterminates over K. Consider the subring $D := K + UK(T)[U]_{(U)}$ of $K(T, U)$.

(1) Draw a picture of $\mathrm{Spec}(D)$.
(2) Determine if D is Noetherian.
(3) Determine if D is a valuation domain.
(4) Determine if D is a Prüfer domain.
(5) Determine if D is integrally closed.

Exercise 4. Let A be a ring and let $Y \subseteq \mathrm{Spec}(A)$. We say that Y is *locally finite* if, for any $a \in A - \{0\}$ the set $V(a) \cap Y$ is finite. Assume that there exists a subset Δ of $\mathrm{Spec}(A)$ which is locally finite and infinite.

(1) Determine if A is an integral domain.
(2) Find the closure of Δ, with respect to the constructible topology.

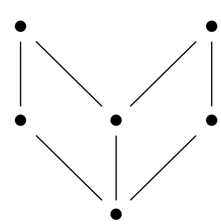
Exercise 5. Let A be a ring.

(1) Show that, if $\mathrm{Max}(A)$ is locally finite and $A_{\mathfrak{m}}$ is Noetherian, for any maximal ideal $\mathfrak{m}$ of A, then A is Noetherian.
(2) Determine if part (1) is true without the assumption that $\mathrm{Max}(A)$ is locally finite.

Exercise 6. Preserve the notation of Exercise 5 of Homework 9, let K be a field, let D be a subring of K and let $\mathcal{F}(K|D) := \{F \in \mathcal{R}(K|D) : F \text{ is a field}\}$ be endowed with the subspace topology induced by the topology of $\mathcal{R}(K|D)$.

(1) Show that $\mathcal{F}(K|D)$ is a spectral space.
(2) Let $\zeta \in \mathbb{C}$ be a primitive 7th root of 1. Sketch a procedure to find a ring whose prime spectrum is homeomorphic to $\mathcal{F}(\mathbb{Q}(\zeta)|\mathbb{Q})$.

Exercise 7. Let A be a ring whose prime spectrum has the following order structure



and assume that some field is a subring of A. Use the ring A to find a ring B whose prime spectrum has the following order structure