MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 11

"What's in a name? A rose by any name would smell as sweet."

W. Shakespeare

Exercise 1. Let $\mathbb{P}$ be the set of all prime numbers and let $\mathrm{Zar}(\mathbb{Q}|\mathbb{Z})$ be endowed with the inverse topology. Determine if $\{\mathbb{Z}_{(p)} : p \in \mathbb{P}\} \subseteq \mathrm{Zar}(\mathbb{Q}|\mathbb{Z})$ is a strongly irredundant representation of $\mathbb{Z}$.

Exercise 2. Let A be a ring and let $\mathfrak{a}$ be a radical ideal of A.

(1) Find all minimal closed representations $Z \subseteq \mathrm{Spec}(A)$, endowed with the Zariski topology, of $\mathfrak{a}$ and the critical elements in Z.
(2) Show that $\mathfrak{a}$ admits at most one strongly irredundant representation.

Exercise 3. Let D be a set. Find subsets $A \subsetneq C$ of D and a spectral C-representation X of A such that there is a set $B \in X$ that is irredundant but not strongly irredundant in X.

Exercise 4. Let D be a set and A be a subset of D.

(1) Show that $X := \{B \subseteq D : A \subseteq B\}$ is a spectral representation of A.
(2) Is there a minimal closed representation Y of A such that $A \in \mathrm{Min}(Y)$?

Exercise 5. Let $A := \{1\} \subseteq \mathbb{N}$ and let

$$X := \{B \subseteq \mathbb{N} : B \supseteq \{1, 2, 3\}\} \cup \{\{1, 2, 5\}, \{1, 2, 4, 5\}, \{1, 4\}\}.$$

(1) Show that X is a spectral representation of A.
(2) Find all the critical sets in X.
(3) Find a minimal closed representation of A in X.
(4) Is $\{B \in X : B \supseteq \{1, 2, 3\}\} \cup \{\{1, 4\}\}$ a minimal closed representation of A in X?
(5) Find, if there exists, a representation $Z \subseteq X$ of A such that there exists a set $B \in Z$ which is redundant in Z but isolated in Z (with respect to the subspace topology induced by the spectral topology of X).

Exercise 6. Consider the subset $X := \mathrm{Max}(\mathbb{Z}) - \{p\mathbb{Z}\}$ of $\mathrm{Spec}(\mathbb{Z})$, where p is a fixed prime number. Find the closure of X in $\mathrm{Spec}(\mathbb{Z})$, in each of the following cases:

- $\mathrm{Spec}(\mathbb{Z})$ is endowed with the Zariski topology.
- $\mathrm{Spec}(\mathbb{Z})$ is endowed with the constructible topology.
- $\mathrm{Spec}(\mathbb{Z})$ is endowed with the inverse topology.

Exercise 7. Let X be a T_0 space, let $\leq$ be the order on X induced by the topology and let Y be a subset of X satisfying the following condition:

($\star$) *whenever $x, y \in Y$, there exists an element $z \in Y$ such that $z \leq x, y$.*

(1) Show that $\overline{Y}$ is irreducible.
(2) Show that, if X is spectral and Y satisfies condition ($\star$), then Y has the infimum.