MAT.632 - Elective subjects Mathematics Topological Methods in Commutative Ring Theory

Homework 2

"Give sorrow words: the grief that does not speak
whispers the o'er-fraught heart, and bids it break."
W. Shakespeare

Exercise 1. Let D be a G-domain and E be an *overring of* D, that is, E is a subring of the quotient field of D containing D. Show that E is a G-domain.

Exercise 2. Let $f : A \longrightarrow B$ be a surjective ring homomorphism.
(1) Show that if A is a Hilbert ring, then B is a Hilbert ring.
(2) Show that if $\mathfrak{q}$ is a prime ideal of B such that $f^{-1}(\mathfrak{q})$ is a G-ideal of A, then $\mathfrak{q}$ is a G-ideal of B.

Exercise 3. Let D be an integral domain and let K be the quotient field of D. Show that the following conditions are equivalent.
(1) D is a G-domain.
(2) K is of finite type over D (i.e., there are elements $x_1, \dots x_n \in K$ such that $K = D[x_1, \dots, x_n]$).

Exercise 4. Let D be an integral domain such that $\mathrm{Spec}(D)$ is finite. Show that D is a G-domain.

Exercise 5. Let A be a Hilbert ring such that $\mathrm{Max}(A)$ is finite. Show that $\mathrm{Spec}(A) = \mathrm{Max}(A)$.

Exercise 6. Let D be an integral domain and let T be an indeterminate over D. Is there a positive integer n such that $D[T^n]$ is a G-domain?

Exercise 7. Let K be a field. Show that any maximal ideal of the polynomial ring $K[T_1, \dots, T_n]$ can be generated by n polynomials.

Exercise 8. Let A be a ring such that for any $f \in A$ there are an invertible element $u \in A$ and an idempotent $e \in A$ such that $f = ue$. Show that A is a Hilbert ring.

Exercise 9. Let $\mathcal{F}$ be a nonempty collection of fields. Show that $\prod_{K \in \mathcal{F}} K$ is a Hilbert ring.

Exercise 10. (Review on integral dependence) Let $A \subseteq B$ be an integral ring extension, and let $f : A \longrightarrow L$ be a ring homomorphism, where L is an algebraically closed field. Show that f extends to a ring homomorphism $B \longrightarrow L$.