MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 3

"None of us can ever express the exact measure of his needs or his thoughts or his sorrows;
and human speech is like a cracked kettle on which we tap crude rhythms for bears to dance to,
while we long to make music that will melt the stars."
G. Flaubert

Exercise 1. Let X be a topological space. Show that the following conditions are equivalent.

(1) X is irreducible.
(2) Any nonempty open subset of X is irreducible.
(3) Any nonempty open subset of X is connected.

Exercise 2. Let A be a ring.

(1) Show that if $\mathfrak{a}$ is a primary ideal of A, then $\mathfrak{p} := \sqrt{\mathfrak{a}}$ is a prime ideal of A. We will say that $\mathfrak{a}$ is *a* $\mathfrak{p}$*-primary ideal*.
(2) Show that if $\mathfrak{a}$ is an ideal of A whose radical is a maximal ideal, then $\mathfrak{a}$ is primary.
(3) Give an example of an ideal $\mathfrak{a}$ which is not primary and such that $\sqrt{\mathfrak{a}}$ is a prime ideal.
(4) Show that if $\mathfrak{a}, \mathfrak{b}$ are $\mathfrak{p}$-primary ideals, then $\mathfrak{a} \cap \mathfrak{b}$ is a $\mathfrak{p}$-primary ideal.
(5) Prove that if $\mathfrak{a}$ is a $\mathfrak{p}$-primary ideal and $x \in A - \mathfrak{a}$, then
$$(\mathfrak{a} :_A x) := \{a \in A : ax \in \mathfrak{a}\}$$
is a $\mathfrak{p}$-primary ideal of A.
(6) Find a ring A and an irreducible ideal of A which is not prime.

Exercise 3. Let K be a field let T, U be indeterminates over K, and consider the ideal $\mathfrak{a} := (T^2, TU)K[T, U]$. Set
$$\mathfrak{p} := TK[T, U], \mathfrak{m} := (T, U)K[T, U], \mathfrak{q} := (T^2, U)K[T, U].$$

(1) Prove that $\mathfrak{p}, \mathfrak{m}^2, \mathfrak{q}$ are primary ideals of $K[T, U]$.
(2) Prove that $\{\mathfrak{p}, \mathfrak{m}^2\}, \{\mathfrak{p}, \mathfrak{q}\}$ are distinct irredundant primary decompositions of $\mathfrak{a}$.

Exercise 4. Let X be a topological space and let $\mathcal{F}$ be a finite collection of closed irreducible subspaces of X such that $\bigcup \mathcal{F} = X$. Show that any irreducible component of X belongs to $\mathcal{F}$. In particular, X has only finitely many irreducible components. Moreover, prove that, if the sets in $\mathcal{F}$ are pairwise incomparable, then $\mathcal{F}$ is the set of the irreducible components of X.

Exercise 5. Let K be a field and let $S \subseteq \mathbb{A}^n_K$. Show that $\overline{S} = Z(I(S))$, where $\overline{S}$ denotes the closure of S, with respect to the Zariski topology.

Exercise 6. Let K be an infinite field, let T_1, T_2, T_3 be indeterminates over K, and let
$$C := Z(\{T_2 - T_1^2, T_3 - T_1T_2\}) \subseteq \mathbb{A}^3_K.$$

(1) Prove that $C = \{(t, t^2, t^3) : t \in K\}$.
(2) Find a system of generators of $I(C)$.
(3) Show that C is irreducible, with respect to the Zariski topology of $\mathbb{A}^3_K$.

Exercise 7. Let K be an algebraically closed field, and let $f_1, \dots, f_n \in K[T]$ be polynomials.

(1) Show that $\{(f_1(k), \dots, f_n(k)) : k \in K\}$ is a closed subset of $\mathbb{A}^n_K$, with respect to the Zariski topology.
(2) Is the statement of part (1) necessarily true, if K is not algebraically closed?