MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 4

"Discovery is the privilege of the child:
the child who has no fear of being once again wrong,
of looking like an idiot, of not being serious,
of not doing things like everyone else."
Alexander Grothendieck

Exercise 1. Let A be a ring and let $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ be a short exact sequence of A-modules (that is, f is injective, g is surjective and $f(M) = \mathrm{Ker}(g)$). Prove that N is an Artin (resp., Noetherian) module if and only if M and P are Artin (resp., Noetherian) modules.

Exercise 2. Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field K and let M be an A-module. Consider the A-submodule $\mathfrak{m}M$ of A generated by the set $\{mx : m \in \mathfrak{m}, x \in M\}$. Then $M/\mathfrak{m}M$ has a natural structure of K-vector space (define this natural structure). Prove that if M is finitely generated, then $M/\mathfrak{m}M$ is both a Noetherian and an Artin A-module.

Exercise 3. Let $f : X \longrightarrow Y$ be a continuous function of topological spaces, and let S be a subset of X.

(1) Show that $f(\overline{S}) \subseteq \overline{f(S)}$ and provide an example to note that the inclusion can be strict.
(2) Show that if f is continuous and closed, then $f(\overline{S}) = \overline{f(S)}$.

Exercise 4. Let D be an integral domain with a unique nonzero prime ideal $\mathfrak{p}$, and let K be the quotient field of D. Let $f : D \longrightarrow (D/\mathfrak{p}) \times K$ be the ring homomorphism defined by $f(d) := (d + \mathfrak{p}, d)$, for any $d \in D$. Prove that the canonical continuous function $f^\star : \mathrm{Spec}((D/\mathfrak{p}) \times K) \longrightarrow \mathrm{Spec}(D)$, induced by f, is bijective but it is not a homeomorphism.

Exercise 5. Let D be an integral domain which is not a field, let K be the quotient field of D and let T be an indeterminate over K. Is the subring $A := \{f \in K[T] : f(0) \in D\}$ of $K[T]$ a Noetherian ring?

Exercise 6. Let $f : A \longrightarrow C, g : B \longrightarrow C$ be ring homomorphisms, and let

$$D := \{(a, b) \in A \times B : f(a) = g(b)\}$$

be the fiber product of f and g. Assume that g is surjective. Let $p : D \longrightarrow A$ (resp., $q : D \longrightarrow B$) be the projection into A (resp., into B).

(1) Let $\mathfrak{h}$ be a prime ideal of B such that $\mathfrak{h} \not\supseteq \mathrm{Ker}(g)$. Prove that $D_{q^{-1}(\mathfrak{h})}$ is canonically isomorphic to $B_{\mathfrak{h}}$.
(2) Show that $\mathrm{Max}(D) = p^\star(\mathrm{Max}(A)) \cup q^\star(\mathrm{Max}(B) - V(\mathrm{Ker}(g))$ (where, as usual, $p^\star$ and $q^\star$ are the continuous maps induced by p and q, respectively).
(3) Deduce that D is local if and only if A is local and $\mathrm{Ker}(g)$ is contained in the Jacobson radical of B.

Exercise 7. Describe the prime spectrum of the ring $A := \{f \in \mathbb{Q}[T] : f(0) \in \mathbb{Z}\}$ and compute $\dim(A)$.

Exercise 8. Recall that a subset of a topological space is locally closed if it is intersection of an open set with a closed set. Let A be a ring and let $\mathfrak{p} \in \mathrm{Spec}(A)$. Show that the following conditions are equivalent.

(1) $\mathfrak{p}$ is a G-ideal of A.
(2) $V(\mathfrak{p}) - \{\mathfrak{p}\}$ closed in $\mathrm{Spec}(A)$.
(3) $\{\mathfrak{p}\}$ is locally closed in $\mathrm{Spec}(A)$.