MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 5

"Respect was invented to cover
the empty place where love should be."
Lev Tolstoj

Exercise 1. Let A be a ring and let $\mathfrak{p}$ be a prime ideal of A.

(1) Show that the residue field of $A_\mathfrak{p}$ is isomorphic to the quotient field of $A/\mathfrak{p}$.
(2) Find the residue field of $A_\mathfrak{p}$ in the following cases:
- $A := K[T]$, $\mathfrak{p} := (T-q)A$, where $q \in K$ and K is any field.
- $A := K[T,U]$, $\mathfrak{p} := (T-q)A$, where $q \in K$ and K is any field.
- $A := \mathbb{Q}[T]$, $\mathfrak{p} := (T^2+T+1)A$.

Exercise 2. Let D be an integral domain with quotient field K and T be an indeterminate over K. Show that the quotient field of the ring $A := \{f \in K[T] : f(0) \in D\}$ is $K(T)$.

Exercise 3. Let $A \subseteq B$ be a ring extension and set

$$\mathfrak{c}_{B|A} := (A : B) := \{b \in B : bB \subseteq A\}$$

(1) Show that $\mathfrak{c}_{B|A} \subseteq A$ and that it is an ideal both of A and B.
(2) More precisely, show that $\mathfrak{c}_{B|A}$ is the largest common ideal of A and B.
(3) Show that if $\mathfrak{c}_{B|A}$ is a *regular ideal of* B (i.e., $\mathfrak{c}_{B|A}$ contains an element which is not a zero-divisor of B), then A and B have the same total ring of quotient.
(4) Show that if B is an integral domain and $\mathfrak{c}_{B|A} \neq (0)$, then A and B have the same quotient field.

Exercise 4. Let D be an integral domain and let $\mathfrak{q}$ be a prime ideal of D. Show that the ideal $(\mathfrak{q}^n D_\mathfrak{q}) \cap D$ is of D is $\mathfrak{q}$-primary, for any positive integer n.

Exercise 5. Let $f : A \longrightarrow C, g : B \longrightarrow C$ be ring homomorphisms, and let

$$D := \{(a,b) \in A \times B : f(a) = g(b)\}$$

be the fiber product of f and g. Assume that g is surjective. Let $q : D \longrightarrow B$ be the projection into B. Show that, if f is a finite (resp., of finite type, integral) ring homomorphism, then q is a finite (resp., of finite type, integral) ring homomorphism.

Exercise 6. Let $A_1, \ldots, A_n$ be rings.

(1) Find $\mathrm{Spec}(A_1 \times \ldots \times A_n)$.
(2) Draw pictures of $\mathrm{Spec}(K^n)$, where K is any field, and of $\mathrm{Spec}(\mathbb{Z} \times \mathbb{Z})$.
(3) Is $\mathrm{Spec}(A_1 \times \ldots \times A_n)$ a connected space?
(4) More precisely, show that $\mathrm{Spec}(A_1 \times \ldots \times A_n)$ is canonically homeomorphic to the disjoint union $\sqcup_{i=1}^n \mathrm{Spec}(A_i)$ (with its natural topology).

Exercise 7. Let p be a prime number. For any $x \in \mathbb{Z}$, let $\overline{x}$ denote the residue class of x modulo p. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, and consider the subring

$$D := \{(\frac{x}{s}, \frac{y}{t}) \in \mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)} : \overline{xt} = \overline{ys}, x, y, z, t \in \mathbb{Z}, s, t \notin p\mathbb{Z}\}.$$

of $\mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}$. Draw a picture of $\mathrm{Spec}(D)$.