MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 6

"She walks in Beauty, like the night
of cloudless climes and starry skies;
and all that's best of dark and bright
meet in her aspect and her eyes."
George Gordon Byron

Exercise 1. Let A be a ring and let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals of A such that $\bigcap_{i=1}^n \mathfrak{a} = (0)$ and that $A/\mathfrak{a}_i$ is a Noetherian ring, for any $1 \leq i \leq n$. Show that A is a Noetherian ring.

Exercise 2. Let $f : A \longrightarrow C, g : B \longrightarrow C$ be ring homomorphisms and let

$$D := \{(a, b) \in A \times B : f(a) = g(b)\}$$

be the fiber product of f and g. Let $p : D \longrightarrow A$ (resp., $q : D \longrightarrow B$) be the restriction to D of the projection of $A \times B$ into A (resp., into B). Show that D is a Noetherian ring if and only if $p(D), q(D)$ are Noetherian rings.

Exercise 3. Let A_1, A_2 be rings. Show that, if $f : \mathrm{Spec}(A_1) \longrightarrow \mathrm{Spec}(A_2)$ is any homeomorphism, then f is order preserving (that is, for any $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{Spec}(A_1)$, then $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$ if and only if $f(\mathfrak{h}_1) \subseteq f(\mathfrak{h}_2)$).

Exercise 4. Using a construction based on fiber products, give an example of a 3-dimensional valuation domain.

Exercise 5. Consider the integral domain $D := \mathbb{R} + T\mathbb{C}[T]_{(T)}$.

(1) Find the quotient field of D.
(2) Determine if D is a Noetherian domain.
(3) Show that $\mathrm{Spec}(D)$ and $\mathrm{Spec}(\mathbb{C}[T]_{(T)})$ are canonically homeomorphic, and find a homeomorphism $\mathrm{Spec}(D) \longrightarrow \mathrm{Spec}(\mathbb{C}[T]_{(T)})$.
(4) Determine if D is a valuation domain.

Exercise 6. Let V be a local domain with residue field K, and let D be a subring of K. Let $\pi : V \longrightarrow K$ denote the canonical projection and set $E := \pi^{-1}(D)$. Show that, if V is not a field, then E is a G-domain if and only if V is a G-domain.

Exercise 7. Let A be a ring. Define a subset Y of $\mathrm{Spec}(A)$ to be *$\mathscr{G}$-closed* if, whenever Z is a subset of Y such that $\mathfrak{q} := \bigcap\{\mathfrak{p} : \mathfrak{p} \in Z\}$ is a prime ideal, then $\mathfrak{q} \in Y$.

(1) Show that the $\mathscr{G}$-closed subsets of $\mathrm{Spec}(A)$ form the collection of the closed sets for a topology $\mathcal{G}$ on $\mathrm{Spec}(A)$.
(2) Show that the topology $\mathcal{G}$ is finer than the Zariski topology.
(3) Determine if the topology $\mathcal{G}$ is Hausdorff.
(4) Find the connected components of $\mathrm{Spec}(A)$, with respect to the topology $\mathcal{G}$.
(5) Determine if the topology $\mathcal{G}$ is compact.