MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 7

"Mathematics reveals its secrets only to those
who approach it with pure love, for its own beauty."
Archimedes

Exercise 1. Let X be a set and let $\mathscr{U}$ be an ultrafilter on X. Show that, if $U \in \mathscr{U}$, then $\mathscr{U}|_U := \{V \subseteq U : V \in \mathscr{U}\}$ is an ultrafilter on U.

Exercise 2. Let $f : X \longrightarrow Y$ be a function and let $\mathscr{U}$ be an ultrafilter on X. Show that $\mathscr{U}_f := \{Z \subseteq Y : f^{-1}(Z) \in \mathscr{U}\}$ is an ultrafilter on Y.

Exercise 3. Let X be a set and let $\mathscr{U}$ be a collection of subsets of X. Show that the following conditions are equivalent.

(1) $\mathscr{U}$ is an ultrafilter on X.
(2) $\mathscr{U}$ is a filter on X and, for any subset Y of X satisfying the condition $Y \cap U \neq \emptyset$ for any $U \in \mathscr{U}$, then $Y \in \mathscr{U}$.

Exercise 4. Let X be a set and let βX denote the set of all ultrafilters on X, endowed with the Stone-Cech topology.

(1) Find all clopen subsets of βX.
(2) Show that the closure of any open subset of βX is open.

Exercise 5. Let $f : X \longrightarrow Y$ be a function, let $\iota_X : X \longrightarrow \beta X, \iota_Y : Y \longrightarrow \beta Y$ be the canonical topological embeddings (where X, Y are equipped with the discrete topology and $\beta X, \beta Y$ with the Stone-Cech topology).

(1) Show that there exists a unique continuous function $\widetilde{f} : \beta X \longrightarrow \beta Y$ such that $\widetilde{f} \circ \iota_X = \iota_Y \circ f$.
(2) Show that if f is injective (resp., surjective), then $\widetilde{f}$ is injective (resp., surjective).
(3) Show that if f is bijective, then $\widetilde{f}$ is a homeomorphism.

Exercise 6. Let A be a ring. Show that the following conditions are equivalent.

(1) A is reduced and any prime ideal of A is maximal.
(2) For any element $a \in A$ there is an element $\lambda \in A$ such that $a = a^2\lambda$.

[Hint: in order to show that (1)$\Longrightarrow$(2) it could be useful to consider, for any $a \in A$, the ideal $\mathfrak{a}_a := \mathrm{Ann}_A(a) + aA$ of A...]

Exercise 7. For any ring E let $N(E)$ (resp., $J(E)$) denote the nilradical (resp., Jacobson radical) of A. Let $\{A_x : x \in X\}$ be a nonempty collection of rings and let $A := \prod_{x \in X} A_x$. Solve the following questions.

(1) Show that $J(A) = \prod_{x \in X} J(A_x)$.
(2) Show that the following conditions are equivalent.
   (a) A is zero-dimensional.
   (b) A_x is zero-dimensional, for any $x \in X$, and $N(A) = \prod_{x \in X} N(A_x)$.

   [Hint: (b)$\Longrightarrow$(a). Consider the factor ring $A/N(A)$...]
(3) Let p be a prime number. Determine if the ring $\prod_{i \geq 1} (\mathbb{Z}/p^i\mathbb{Z})$ is zero-dimensional.