MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 8

“He stepped down, trying not to look long at her,
as if she were the sun, yet he saw her,
like the sun, even without looking.”
L. Tolstoj

Exercise 1. Let $\mathrm{Spec}(A)^{\mathrm{g}}$ be the prime spectrum of a ring A, endowed with the $\mathscr{G}$-topology, i.e. the topology such that the nonempty closed subsets are the sets $C \subseteq \mathrm{Spec}(A)$ satisfying the following condition: whenever D is a subset of C and $\bigcap\{\mathfrak{p} : \mathfrak{p} \in D\}$ is a prime ideal of A, then $\bigcap\{\mathfrak{p} : \mathfrak{p} \in D\} \in C$.

(1) Determine if any open subset of $\mathrm{Spec}(A)$, endowed with the Zariski topology, is clopen in $\mathrm{Spec}(A)^{\mathrm{g}}$.
(2) Prove that the constructible topology and the $\mathscr{G}$-topology on $\mathrm{Spec}(A)$ are the same topology if and only if $\mathrm{Spec}(A)$, with the Zariski topology, is a Noetherian space.

Exercise 2. Let A be a ring and let $\mathfrak{p} \in \mathrm{Spec}(A)$. Show that the following conditions are equivalent.

(1) $\mathfrak{p}$ is a G-ideal of A.
(2) $V(\mathfrak{p}) - \{\mathfrak{p}\}$ is closed, with respect to the constructible topology.

Exercise 3. Let A be a ring. Determine when $\mathrm{Spec}(A)$ is a Noetherian space, when it is endowed with the constructible topology.

Exercise 4. Determine if a closed and bounded interval of the real line $\mathbb{R}$ is a spectral space, when it is endowed with the subspace topology induced by the euclidean topology of $\mathbb{R}$.

Exercise 5. Let X be any set, endowed with the cofinite topology (i.e., the proper closed subsets of X are the finite subsets of X). Determine if X is spectral.

Exercise 6. Let A be a ring and let Y, Z be subsets of $\mathrm{Spec}(A)$ having the same closure, with respect to the constructible topology. Determine if $\bigcup\{\mathfrak{p} : \mathfrak{p} \in Y\} = \bigcup\{\mathfrak{q} : \mathfrak{q} \in Z\}$.

Exercise 7. Let $\{A_x : x \in X\}$ be a nonempty family of local rings, and let $A := \prod_{x\in X} A_x$. Determine if $\mathrm{Max}(A)$, equipped with the subspace topology induced by the Zariski topology of $\mathrm{Spec}(A)$, is homeomorphic to βX (endowed with the Stone-Cech topology). In such a case, find explicitly a homeomorphism $\beta X \longrightarrow \mathrm{Max}(A)$.