MAT.632 - Elective subjects Mathematics
Topological Methods in Commutative Ring Theory

Homework 9

"Always to me beloved was this lonely hillside
And the hedgerow creeping over and always hiding
The distances, the horizon's furthest reaches.
But as I sit and gaze, there is an endless
Space still beyond, there is a more than mortal
Silence spread out to the last depth of peace,
Which in my thought I shape until my heart
Scarcely can hide a fear. And as the wind
Comes through the copses sighing to my ears,
The infinite silence and the passing voice
I must compare: remembering the seasons,
Quiet in dead eternity, and the present,
Living and sounding still. And into this
Immensity my thought sinks ever drowning,
And it is sweet to shipwreck in such a sea."

Giacomo Leopardi

Exercise 1. Consider the set $X := \{0,1,2\}$, endowed with the topology whose open sets are $\emptyset, \{0\}, \{0,1\}, X$. Show that X is spectral and find a ring A such that $\mathrm{Spec}(A)$ is homeomorphic to X.

Exercise 2. Let X be a spectral space, $\mathcal{B}$ be a subbasis of open and compact subspaces of X, let Y be a subset of X and let $\mathscr{U}$ be an ultrafilter on Y. Show that the set

$$Y(\mathscr{U}) := \{x \in X : [\forall B \in \mathcal{B}, (x \in B \iff B \cap Y \in \mathscr{U})]\}$$

is a singleton.

Exercise 3. Let X be a spectral space and let C be a closed subset of the patch space X^{patch} of X (i.e., X endowed with the patch topology). Show that C is a spectral space, when it endowed with the subspace topology induced by that of X.

Exercise 4. Let $f : A \longrightarrow B$ be a ring homomorphism, and let $\mathfrak{p} \in \mathrm{Spec}(A)$. Show that $f^{\star -1}(\{\mathfrak{p}\})$ is a spectral subspace of $\mathrm{Spec}(B)$, where $f^{\star} : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is the canonical map induced by f.

Exercise 5. Let K be a field and let D be a subring of K. Let $\mathcal{R}(K|D)$ denote the set of all the subrings C of K such that D is a subring of C, endowed with the natural topology whose subbasic open sets are the sets of the form $U(x) := \{C \in \mathcal{R}(K|D) : x \in C\}$, where x is any element of K.

(1) Determine if $U(x)$ is spectral, with the subspace topology, for any $x \in K$.

(2) Set

$$\mathcal{L}(K|D) := \{A \in \mathcal{R}(K|D) : A \text{ is a local ring}\}$$
$$\mathcal{N}(K|D) := \{A \in \mathcal{R}(K|D) : A \text{ is a Noetherian ring}\}.$$

Determine if $\mathcal{L}(K|D), \mathcal{N}(K|D)$ are spectral subspaces of $\mathcal{R}(K|D)$.

Exercise 6. Let X be a spectral space and let $x, y \in X$ be such that, whenever U is a neighborhood of x and V is a neighborhood of y, then $U \cap V \neq \emptyset$. Determine if there exists a point $z \in X$ such that $x, y \in \overline{\{z\}}$.

Exercise 7. Let $\{X_i : i \in I\}$ be a nonempty family of spectral spaces and let $X := \prod_{i \in I} X_i$ be endowed with the product topology. Determine if X is spectral.