Computing Norms of Free Operators with Matrix Coefficients

Franz Lehner*

Abstract. Let $g_1, g_2, \dots, g_n$ be the generators of the free group $\mathbf{F}_n$ and denote by λ its left regular representation. A formula for norms of operators of the form $\sum a_i \otimes \lambda(g_i)$ where a_i are complex matrices or operators is derived similar to the formula of Akemann and Ostrand for the scalar case. This formula can be used to compute norms of arbitrary finitely supported convolution operators on the free group. Essentially the same technique yields a formula for the norms of sums of matrix valued creation and annihilation operators on full Fock space. Finally some results about the numerical evaluation of the formula are proved.

1. Introduction and main results

1.1. **Free group.** Let $\Gamma_n = \mathbf{Z}_2 * \mathbf{Z}_2 * \cdots * \mathbf{Z}_2$ be the free product of n copies of $\mathbf{Z}_2$ and denote its free generators by $g_1, g_2, \dots, g_n$. C. Akemann and P. Ostrand proved in their classical paper [AO76] a formula for the norm of free operators, that is operators of the form $\sum \alpha_i \lambda(g_i)$. This formula improved estimates of M. Leinert [Lei74], M. Bożejko [Boż75] and M. Flory [Flo76]. It was previously known in the case of equal coefficients [Kes59] and simpler proofs were found in [Woe86b], [Woe86a] and [PP88]. In this paper we continue the investigation started in [Leh97] and improve the method to obtain a formula similar to the Akemann-Ostrand formula in the matrix case. The main result about the free group, proved in section 3, is the following theorem.

Theorem 1.1. *Let $n \geq 2$ and $g_1, \dots, g_n$ be the generators of the free product group Γ_n, let $a_0, a_1, a_2, \dots, a_n$ be hermitian $m \times m$ matrices and assume moreover that a_0 is positive semidefinite. Then*

(1.1)
$$\left\| a_0 \otimes I + \sum_{i=1}^{n} a_i \otimes \lambda(g_i) \right\|_{\min} = \inf_{z>0} \left\| 2\,z + a_0 + \sum_{i=1}^{n} z^{\frac{1}{2}} \left(\left(I_m + \left(z^{-\frac{1}{2}} a_i z^{-\frac{1}{2}}\right)^2 \right)^{\frac{1}{2}} - I_m \right) z^{\frac{1}{2}} \right\|,$$

where the infimum is taken over all positive definite invertible $m \times m$ matrices z. Moreover, the infimum can be restricted to those z, for which the expression inside the norm signs equals a positive scalar multiple of the identity matrix I_m.

A similar formula holds for operators on a general Hilbert space (see remark 4.3) and an elementary matrix trick allows us to deduce a formula for free operators with non-selfadjoint coefficient matrices. Consider first the case $a_0 = 0$. Here the relevant property

Date: December 14, 1998.

1991 *Mathematics Subject Classification.* Primary 22D25; Secondary 60J15.

Key words and phrases. Norm of convolution operator, free group, Leinert property, random walk, semicircular system, Fock space.

*This work was supported by the EU-network "Non-commutative geometry" ERB FMRX CT960073

of $g_1, g_2, \dots, g_n$ is the so called *Leinert property* (see [AO76] for a definition) and formula (1.1) holds for any set with this property. Other examples of sets with the Leinert property are the set $\{x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$ of generators of the free group together with their inverses or any translate of the subset $\{e, x_1, x_2, \dots, x_n\}$ of the free group. The latter actually is the generic example of a set with the Leinert property. The set of generators of Γ_n (which are sometimes called *free Rademacher variables*) is chosen here mainly because the induced unitaries are selfadjoint; the free group $\mathbf{F}_n$ can be embedded into Γ_{n+1} in a natural way by mapping x_i to $g_0 g_i$ and the remarks in section 5 can be applied to the free group as well. Given non-selfadjoint coefficient matrices $a_1, a_2, \dots, a_n$ we have

$$\begin{aligned}\left\|\sum a_i \otimes \lambda(g_i)\right\| &= \left\|\sum a_i^* \otimes \lambda(g_i)\right\| \\ &= \left\|\sum \begin{bmatrix} a_i & 0 \\ 0 & a_i^* \end{bmatrix} \otimes \lambda(g_i)\right\| \\ &= \left\|\sum \begin{bmatrix} 0 & a_i^* \\ a_i & 0 \end{bmatrix} \otimes \lambda(g_i)\right\|.\end{aligned}$$

The latter has selfadjoint coefficient matrices and can be fed into formula (1.1). The last assertion in theorem 1.1 allows us to restrict the infimum to matrices of the form $z = \left[\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right]$ and we get the following formula.

Corollary 1.2. *Let Γ be a discrete group and suppose that the subset $\{t_1, t_2, \dots, t_n\} \subset \Gamma$ satisfies the Leinert property. Then for any set of $m \times m$ matrices $a_1, a_2, \dots, a_n$ we have*

$$\begin{aligned}&\left\|\sum_{i=1}^n a_i \otimes \lambda(t_i)\right\|_{\min} = \\ &\quad \inf_{z_1, z_2 > 0} \max\left\{ \left\| 2\, z_1 + \sum_{i=1}^n z_1^{\frac12} \left(\left(I_m + z_1^{-\frac12} a_i^* z_2^{-1} a_i z_1^{-\frac12} \right)^{\frac12} - I_m \right) z_1^{\frac12} \right\|, \right. \\ &\qquad\qquad \left. \left\| 2\, z_2 + \sum_{i=1}^n z_2^{\frac12} \left(\left(I_m + z_2^{-\frac12} a_i z_1^{-1} a_i^* z_2^{-\frac12} \right)^{\frac12} - I_m \right) z_2^{\frac12} \right\| \right\}.\end{aligned}$$

Restricting the infimum to matrices of the form $z_1 = sI_m$ and $z_2 = tI_m$ yields the inequality proved in [Leh97].

The case where a_0 is not positive definite can also be handled, using identity (5.1) below. The proof of theorem 1.1 splits into two parts. We show first that the norm of the operator in question is dominated by the expression on the right hand side of (1.1), using a technique which one might call a matrix-valued Gabber's Lemma. A formulation and proof of Gabber's lemma can be found in [GdlH97] and it was implicitly applied to the free group in [PP88]. In [Leh97] we observed how this technique can be adapted to the matrix valued case to get an inequality, which is sharp enough to derive the optimal constant for a Khinchin-type inequality on the free group which was proved in [HP93] with a different method. However the range of the infimum was not exhausting and examples showed that strict inequality can occur. In the present paper we overcome this limitation by extending the infimum over the whole range of positive definite invertible operators. There are two ways to show that the infimum of (1.1) is equal to the norm which we want to compute: the hard way exploits the fact that the derivative of the function on the right hand side of (1.1) becomes singular when approaching the infimum and gives rise to a family of "asymptotic eigenvectors" giving a lower bound for the norm, approaching the upper bound from below. These eigenvectors are also useful to give error estimates when evaluating the infimum numerically, as explained section 6. Alternatively, relying on the fact that free operators with matrix coefficients are free with amalgamation over the matrix algebra, one can use results about operator-valued free convolution from [Voi95], see lemma 2.3. Although not indispensable, this section sheds some light on how such formulas can be found and also exhibits the analogy to the scalar case.

1.2. **Fock space.** Another "canonical" example of free operators, which plays a central role in free probability theory and also has some relevance to random matrix theory, is given by semicircular operators. Let us recall the definition of the *full Fock space* over a Hilbert space, see e.g. [VDN92]. For a Hilbert space H with orthonormal basis $\{e_i : i \in I\}$ the full Fock space is defined as the Hilbert space

$$\mathcal{T}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}$$

with orthonormal basis

$$\{e_{i_1,i_2,\dots,i_k} = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k} : k \in \mathbf{N}, i_j \in I\} \cup \{\Omega\}. \tag{1.2}$$

For $h \in H$ define the *creation operator* on $\mathcal{T}(H)$ by

$$l(h)\,\Omega = h \qquad l(h)\, h_1 \otimes h_2 \otimes \cdots \otimes h_k = h \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_k;$$

its adjoint is the *annihilation operator*

$$l^*(h)\,\Omega = 0 \qquad l^*(h)\, h_1 \otimes h_2 \otimes \cdots \otimes h_k = \langle h_1, h\rangle\, h_2 \otimes h_3 \otimes \cdots \otimes h_k.$$

We shall abbreviate the notation by writing $l_i = l(e_i)$. It is not difficult to show that the family of algebras $\mathcal{A}_i$ generated by $\{l_i, l_i^*\}$, $i \in I$ is *free* (see definition 2.1) with respect to the *vacuum state* $\langle\, .\,\Omega, \Omega\rangle$.

Full Fock space is the natural framework for *free semicircular families* (sometimes called *semicircular systems*), that is free families of self-adjoint operators x_i with distribution

$$\tau(x_i^n) = \frac{1}{2\pi}\int_{-2}^{2} t^n \sqrt{4-t^2}\, dt \tag{1.3}$$

which appears as the limit distribution of the free central limit theorem and is therefore sometimes called the *free Gaussian* distribution. It can be shown that the family $\{l_i + l_i^* : i \in I\}$ forms a semicircular system.

The basis of $\mathcal{T}(H)$ given by (1.2) can be viewed as the vertex set of a tree with origin Ω and with edges $[e_{i_1,i_2,\dots,i_k}, e_{i_1,i_2,\dots,i_{k+1}}]$ and it is not surprising that a formula similar to (1.1) holds in the context creation and annihilation operators on Fock space.

Theorem 1.3. *For $n \geq 2$ let $l_1, l_2, \dots, l_n$ be the canonical set of creation operators on $\mathcal{T}(\mathbf{C}^n)$ and let $a_0, a_1, \dots, a_n$ be $m\times m$ matrices such that a_0 is positive semidefinite. Then*

$$\left\| a_0 \otimes I + \sum_{i=1}^{n} a_i \otimes l_i + a_i^* \otimes l_i^* \right\|_{\min} = \inf_{z>0} \left\| z + a_0 + \sum_{i=1}^{n} a_i^* z^{-1} a_i \right\|, \tag{1.4}$$

where the infimum is taken over all positive definite invertible $m\times m$ matrices z. Moreover, the infimum can be restricted to those z, for which the expression inside the norm signs equals a multiple of the identity matrix I_m.

In the case of the more general operator $\sum a_i \otimes l_i + \sum b_i \otimes l_i^*$ we can replace the coefficients according to the rule

$$\tilde{a}_i = \begin{bmatrix} 0 & a_i \\ b_i^* & 0 \end{bmatrix} \qquad \tilde{b}_i = \tilde{a}_i^*$$

without changing the norm and we get the following formula.

Corollary 1.4. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be arbitrary $m \times m$ matrices. Then*

$$\left\| \sum_{i=1}^n a_i \otimes l_i + b_i \otimes l_i^* \right\|_{\min} = \inf_{z_1, z_2 > 0} \max \left\{ \left\| z_1 + \sum_{i=1}^n a_i^* z_2^{-1} a_i \right\|, \left\| z_2 + \sum_{i=1}^n b_i z_1^{-1} b_i^* \right\| \right\}.$$

As a special case, whenever $\sum a_i^* a_i = \alpha^2 I$ and $\sum b_i b_i^* = \beta^2 I$, then

$$\left\| \sum_{i=1}^n a_i \otimes l_i + b_i \otimes l_i^* \right\|_{\min} = \alpha + \beta;$$

in particular, the Banach space spanned by $l_1, l_2, \dots, l_n$ and their adjoints is isometric to the ℓ_1-direct sum $\ell_2^n \oplus_1 \ell_2^n$.

As another special case, when the coefficient matrices a_i are selfadjoint, we get the following formula for semicircular operators.

Corollary 1.5. *For $n \geq 2$ let $x_1, x_2, \dots, x_n$ be a free semicircular family and let $a_0, a_1, \dots, a_n$ be hermitian $m \times m$ matrices such that a_0 is positive semidefinite. Then*

$$\left\| a_0 \otimes I + \sum_{i=1}^n a_i \otimes x_i \right\|_{\min} = \inf_{z>0} \left\| z + a_0 + \sum_{i=1}^n a_i z^{-1} a_i \right\|, \tag{1.5}$$

where the infimum is taken over all positive definite invertible $m \times m$ matrices z. Moreover, the infimum can be restricted to those z, for which the expression inside the norm signs equals a multiple of the identity matrix I_m.

Again remark 4.3 shows that a similar formula holds for coefficients taken from $B(H)$ for an arbitrary Hilbert space H, and the same matrix trick as above can be used to get rid of the limitation to self-adjoint matrices. Also note that in the case where $a_0 = 0$, formula (1.5) remains unchanged if we replace the semicircular system by a circular system, i.e. operators of the form $\frac{1}{\sqrt{2}}(x_i + i x_i')$, where $x_1, x_1', x_2, x_2', \dots, x_n, x_n'$ form a free semicircular family. Such operators can be modeled on Fock space by free copies of $l_1 + l_2^*$. Following the lines before corollary 1.2 we get a formula for non-selfadjoint coefficient matrices.

Corollary 1.6. *Let $x_1, x_2, \dots, x_n$ be a free circular or semicircular family and let $a_1, a_2, \dots, a_n$ be arbitrary $m \times m$ matrices. Then*

$$\left\| \sum_{i=1}^n a_i \otimes x_i \right\|_{\min} = \inf_{z_1, z_2 > 0} \max \left\{ \left\| z_1 + \sum_{i=1}^n a_i^* z_2^{-1} a_i \right\|, \left\| z_2 + \sum_{i=1}^n a_i z_1^{-1} a_i^* \right\| \right\}$$

The upper bound in theorem 1.3 is again obtained by the use of a Gabber-type function. The proof of the lower bound is complicated by the fact that the vacuum state is neither tracial nor faithful, so that lemma 2.3 does not apply to give a lower bound for the norm. The alternative method of exhibiting a norming set of vectors is still possible and gives the lower bound.
It is interesting to note here the following consequence of the formulae (1.1) and (1.5) together with the operator inequality (3.10).

Corollary 1.7. *The map $x_i \mapsto \lambda(g_i)$ is a complete contraction, i.e. for any set of matrices $a_1, \dots, a_n$ the inequality*

$$\left\| \sum a_i \otimes \lambda(g_i) \right\| \leq \left\| \sum a_i \otimes x_i \right\|$$

holds.

In order to apply the formula to more general operators, we have to include the one-dimensional projection p_Ω. A slight modification of the proof then yields the following formula.

Theorem 1.8. *Let $a_0, a_1, \dots, a_n$ and b be $m \times m$ matrices such that a_0 and $a_0 + b$ are positive semidefinite. Then*

(1.6)

$$\left\| a_0 \otimes I + b \otimes p_\Omega + \sum_{i=1}^n a_i \otimes l_i + a_i^* \otimes l_i^* \right\|_{\min} = \inf_{z>0, z+b>0} \left\| z + a_0 + b + \sum_{i=1}^n a_i^* (z+b)^{-1} a_i \right\|,$$

where the infimum is taken over all positive definite invertible $m \times m$ matrices z such that $b + z$ is also positive definite and invertible. Moreover, the infimum can be restricted to those z, for which the expression inside the norm signs equals a multiple of the identity matrix I_m.

The paper is organized as follows.
In section 2 we review some relevant facts about free probability.
Sections 3 and 4 are devoted to the proofs of the main theorems on the free group and Fock space respectively. For the case of the free group only the upper inequality needs to be shown, the lower inequality follows from the general considerations in remark 2.4. However, we also include an alternative, more technical proof of the lower bound which involves asymptotic eigenvectors. This is also of some relevance for the numerical computations considered in section 6. This proof can be literally transferred to operators on Fock space, where the state is neither tracial nor faithful and remark 2.4 does not apply. Since the formulae are somewhat less complicated in the latter case, the reader may want to choose to read section 4 first.
In section 5 we show how the matrix-valued formulae can be used to compute the norm of finitely supported convolution operators on free groups and on Fock space.
Section 6 gives some theoretical background on how to perform numerical computations on a computer.
Acknowledgements. I am grateful to Ken Dykema and Uffe Haagerup for some useful discussions.

2. Preliminaries about $\mathcal{B}$-valued free convolution

In this section we give an overview about some results from [Voi95] which are needed in the proofs. Although the proofs can be done without explicitly relying on this theory, it enlightens the concepts involved. First of all let us recall the definition of freeness and freeness with amalgamation.

Definition 2.1 ([VDN92])**.** Let $\mathcal{A}$ be a unital algebra over $\mathbf{C}$ and let $\mathcal{A}_i \subset \mathcal{A}$, $i \in I$ be unital subalgebras containing a common unital subalgebra $\mathcal{B}$. Let further $\varphi : \mathcal{A} \to \mathcal{B}$ be a conditional expectation, i.e. a linear map such that $\varphi(b) = b$ and $\varphi(b'ab'') = b'\varphi(a)b''$ for every $b, b', b'' \in \mathcal{B}$ and $a \in \mathcal{A}$. The elements of $\mathcal{A}$ will also be called *$\mathcal{B}$-valued random variables* and $(\mathcal{A}, \varphi)$ is called a *$\mathcal{B}$-valued noncommutative probability space.* The family $\mathcal{A}_i$ is called *$\mathcal{B}$-free* with respect to φ if

$$\varphi(a_1 a_2 \dots a_k) = 0$$

whenever $a_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_k$ are such that $\varphi(a_j) = 0$. A family of subsets $\Omega_i \subset \mathcal{A}$ (respectively elements $a_i \in \mathcal{A}$) is called *$\mathcal{B}$-free* if the family of subalgebras generated by $\Omega_i \cup \mathcal{B}$ (respectively, $\{a_i\} \cup \mathcal{B}$) is $\mathcal{B}$-free. In the case where $\mathcal{B} = \mathbf{C}$ and φ is a linear functional, the term *free* is used instead of $\mathbf{C}$-free. The term *C^*-probability space* is used when the involved algebras are C^*-algebras and the conditional expectation is positive.

Typical examples of $\mathcal{B}$-free families arise in amalgamated free products of C^*-algebras. In what follows we shall be concerned mainly with the case of C^*-algebras and tracial states, in particular the following situation. Suppose that $\mathcal{A}_i \subset \mathcal{A}$, $i \in I$ are $\mathbf{C}$-free with respect to some trace state τ on $\mathcal{A}$. Let $\mathcal{B}$ be an arbitrary C^*-algebra and form the minimal C^* tensor product $\mathcal{B} \otimes \mathcal{A}$. Then the algebras $\mathcal{B} \otimes \mathcal{A}_i$ are $(\mathcal{B} \otimes I_{\mathcal{A}})$-free with respect to the conditional expectation $Id_{\mathcal{B}} \otimes \tau$. We shall identify $\mathcal{B}$ with the subalgebra $\mathcal{B} \otimes I$ of $\mathcal{B} \otimes \mathcal{A}$. We shall also need some facts about $\mathcal{B}$-valued additive free convolution. In the setting of definition 2.1 above, for any element $a \in \mathcal{A}$ we use the term *G-transform* to denote the function

$$G_a(b) = \varphi\left((b-a)^{-1}\right)$$

and the same term is used for the formal power series over $\mathcal{B}$ defined by

$$\begin{aligned} \tilde{G}_a(b) &= G(b^{-1}) \\ &= \sum_{n=0}^{\infty} \varphi\left(b(ab)^n\right) \\ &= \varphi\left(b(I-ab)^{-1}\right). \end{aligned}$$

This function is invertible for $b \in \mathcal{B}$ invertible and $\|b\|$ sufficiently small, i.e. there exists some function $L_a : \mathcal{B} \to \mathcal{B}$ and some neighbourhood $\mathcal{V}$ of 0 such that

$$\tilde{G}_a(L_a(b)) = b = L_a(\tilde{G}_a(b))$$

for $b \in GL(\mathcal{B}) \cap \mathcal{V}$. Moreover, the values of this function are invertible and have the form

$$K_a(b) = L_a(b)^{-1} = b^{-1} + R_a(b) \tag{2.1}$$

so that

$$G(K(b)) = b$$

The functions K_a and R_a are called the K- and R-transforms of a. We will be working with slightly modified functions, namely $\tilde{K}_a(z) = K_a(\frac{z^{-1}}{2})$ on the free group and $\tilde{K}_a(z) = K_a(z^{-1})$ on Fock space. The R-transform linearizes free convolution as shown in the following theorem.

Theorem 2.2 ([Voi95]). *Let $(\mathcal{A}, \varphi)$ be $\mathcal{B}$-valued noncommutative probability space and suppose $a', a'' \in \mathcal{A}$ are $\mathcal{B}$-freely independent random variables with R-transforms R' and R'' respectively. Then the R-transform of $a = a' + a''$ is given by*

$$R(b) = R'(b) + R''(b).$$

We will need some more information about the range of invertibility of the G-transform in our special case, compare [Woe86b, Prop. 1] for the scalar case.

Lemma 2.3. *Let $(\mathcal{A}, \tau)$ be a C^*-probability space with faithful trace state τ. Let further $\mathcal{B} \subset \mathcal{A}$ be a finite dimensional unital $*$-subalgebra and denote by $\tau(\,.\,|\mathcal{B})$ the (faithful) conditional expectation from $\mathcal{A}$ to $\mathcal{B}$. Let $a \in \mathcal{A}$ be a self-adjoint $\mathcal{B}$-valued random variable in $(\mathcal{A}, \tau(\,.\,|\mathcal{B}))$ with Cauchy transform $\tilde{G}(b)$. Then $\tilde{G}$ as a function defined on some subset of the positive cone of $\mathcal{B}$ is invertible in an open set including the open line segment $]0, \frac{1}{\|a\|}[$.*

Proof. It suffices to show that the Fréchet derivative of $\tilde{G}$ is invertible along this line segment. Since $\mathcal{B}$ is finite dimensional, this is equivalent to it being injective. For $h, b \in \mathcal{B}$ the directional derivative of $\tilde{G}$ at b with respect to h can be computed as follows. It follows from the product rule that $D_h b^{-1} = -b^{-1}hb^{-1}$ and hence

$$\begin{aligned} D_h\tilde{G}(b) &= \lim_{t\to 0}\frac{1}{t}(\tilde{G}(b+th)-\tilde{G}(b)) \\ &= \tau\left(h(I-ab)^{-1} + b(I-ab)^{-1}ah(I-ab)^{-1}|\mathcal{B}\right) \\ &= \tau\left((I-ba)^{-1}h(I-ab)^{-1}|\mathcal{B}\right) \end{aligned}$$

Now for $\beta \in]-\frac{1}{\|a\|},\frac{1}{\|a\|}[$ the inverse of $I-\beta a$ is positive definite and since the trace τ is faithful, if follows that

$$\begin{aligned} \tau\left(D_h\tilde{G}(\beta I)h^*\right) &= \tau\left(\tau\left((I-\beta a)^{-1}h(I-\beta a)^{-1}|\mathcal{B}\right)h^*\right) \\ &= \tau\left(\left((I-\beta a)^{-\frac{1}{2}}h(I-\beta a)^{-\frac{1}{2}}\right)\left((I-\beta a)^{-\frac{1}{2}}h(I-\beta a)^{-\frac{1}{2}}\right)^*\right) \end{aligned}$$

is different from zero, unless $h = 0$. In particular, $D_h\tilde{G}(\beta I) \neq 0$. □

Remark 2.4. From this lemma, one can deduce the following fact, analogous to the scalar case. Let $(\mathcal{A},\tau)$ be a C^*-probability space with faithful finite trace state τ. Consider the $\boldsymbol{M}_m$-valued noncommutative probability space $M_m(\mathcal{A})$ with conditional expectation $\varphi = Id_{\boldsymbol{M}_m}\otimes\tau$. Let $a \in M_m(\mathcal{A})$ be a self-adjoint $\boldsymbol{M}_m$-valued random variable. Then its Cauchy transform G is invertible in some open set containing $]-\infty,\min\sigma(a)[\,\cup\,]\max\sigma(a),+\infty[$ and if K is an univalent inverse of G on some maximal open set Ω, then the norm of a can be computed from the following data:

$$\begin{aligned} \min\sigma(a) &= \sup\{s : \exists z < 0 : K(z) = sI_m\} \\ \max\sigma(a) &= \inf\{s : \exists z > 0 : K(z) = sI_m\} \end{aligned}$$

The aim of the following two sections is to make this formula more explicit on the free group and to establish it also on Fock space, where the vacuum expectation is neither faithful nor tracial.
We finish this section with an elementary Cauchy-Schwarz type lemma for Hilbert-Schmidt operators which was already used in [Leh97].

Lemma 2.5. *Let H be a Hilbert space. Then for any sequence $x_1, x_2, \dots, x_n \in B(H)$ and $y_1, y_2, \dots, y_n \in HS(H)$ we have the inequality*

$$\left\|\sum_{i=1}^n x_iy_i\right\|_{HS} \leq \left\|\sum_{i=1}^n x_ix_i^*\right\|^{\frac{1}{2}} \operatorname{tr}\left(\sum_{i=1}^n y_i^*y_i\right)^{\frac{1}{2}}. \tag{2.2}$$

3. Matrix valued operators on the free group

Throughout this section let T denote the operator

$$T = a_0\otimes I + \sum a_i\otimes\lambda(g_i). \tag{3.1}$$

Its K-transform can be computed using theorem 2.2.

Proposition 3.1. *The K-transform of T at a positive invertible matrix z is given by*

$$K(z) = z^{-1} + a_0 + \frac{1}{2}\sum_{i=1}^{n} z^{-\frac{1}{2}} \left(\left(I + 4(z^{\frac{1}{2}}a_i z^{\frac{1}{2}})^2\right)^{\frac{1}{2}} - I\right) z^{-\frac{1}{2}}. \tag{3.2}$$

Proof. Let g be the generator of $\mathbf{Z}_2$ and τ be the canonical trace on $\mathcal{A} = vN(\mathbf{Z}_2)$. Then for $a \in \mathcal{B}$ the $\mathcal{B}$-valued random variable $a \otimes \lambda(g) \in \mathcal{B} \otimes \mathcal{A}$ has Cauchy-transform

$$\begin{aligned}
\tilde{G}(b) &= b(Id_{\mathcal{B}} \otimes \tau)\left((I_{\mathcal{B}} \otimes I - ab \otimes \lambda(g))^{-1}\right) \\
&= b\sum_{k=0}^{\infty}(ab)^k \tau(\lambda(g)^k) \\
&= b(I - (ab)^2)^{-1}
\end{aligned}$$

Finding the inverse of this function is accomplished by solving the operator equation $\tilde{G}(b) = z$ i.e.

$$b = z(I - (ab)^2). \tag{3.3}$$

In order to find the right formula we will assume that all the operators involved are invertible; once the formula is found, it can easily be verified for non-invertible arguments by a density argument. Assuming that a, b, z satisfy equation (3.3), it follows that $az = ab(I - (ab)^2)^{-1}$ commutes with ab and we can apply commutative algebra to solve the equivalent equation

$$(ab)^2 az + ab = az;$$

multiplying by az and completing the square on the left hand side yields

$$azab = \pm\left(\frac{1}{4}I + (az)^2\right)^{\frac{1}{2}} - \frac{1}{2}I.$$

It follows that $ab = f^{\pm}(az)$ where $f^{\pm}(x) = \frac{\pm\sqrt{1+4x^2}-1}{2x}$ satisfies

$$\frac{1}{f^{\pm}(x)} = \frac{2x}{\pm\sqrt{1+4x^2}-1}\,\frac{\mp\sqrt{1+4x^2}-1}{\mp\sqrt{1+4x^2}-1} = -f^{\mp}(x) \tag{3.4}$$

and the K-transform maps z to one of

$$b^{-1} = -f^{\mp}(az)a = -\frac{1}{2}\left(\mp\left(I + 4(az)^2\right)^{\frac{1}{2}} - I\right) z^{-1}$$

and since $K(z)$ must have the form (2.1), we must choose the f^+-branch of the solution, i.e.

$$\begin{aligned}
L(z) &= a^{-1}f^{+}(az) \\
K(z) &= -f^{-}(az)a = \frac{1}{2}\left((I + 4(az)^2)^{\frac{1}{2}} + I\right) z^{-1}.
\end{aligned}$$

For z positive definite we get

$$\begin{aligned}
R(z) &= K(z) - z^{-1} \\
&= \frac{1}{2}\left((I + 4(az)^2)^{\frac{1}{2}} - I\right) z^{-1} \\
&= \frac{1}{2}z^{-\frac{1}{2}}\left(\left(I + 4(z^{\frac{1}{2}}az^{\frac{1}{2}})^2\right)^{\frac{1}{2}} - I\right) z^{-\frac{1}{2}}
\end{aligned} \tag{3.5}$$

It follows by theorem 2.2 that the K-transform of the operator (3.1) at $z > 0$ is given by (3.2). ∎

It will be convenient to work with

$$\tilde{K}(z) = K(\tfrac{z^{-1}}{2}) = 2z + a_0 + \sum \tilde{R}_i(z) \tag{3.6}$$

instead; the modified R-transform $\tilde{R}_i(z) = R_i(\frac{z^{-1}}{2})$ satisfies the following relations. Let

$$S_i^{\pm}(z) = \left(\left(I + (a_i z^{-1})^2\right)^{\frac{1}{2}} \pm I\right) z = z^{\frac{1}{2}} \left(\left(I + (z^{-\frac{1}{2}} a_i z^{-\frac{1}{2}})^2\right)^{\frac{1}{2}} \pm I\right) z^{\frac{1}{2}}; \tag{3.7}$$

then it follows from (3.4) that

$$\tilde{R}_i(z) = R_i(\tfrac{z^{-1}}{2}) = S_i^-(z) = a_i S_i^+(z)^{-1} a_i \tag{3.8}$$

and

$$S_i^+ - S_i^- = 2z. \tag{3.9}$$

In particular, since $S_i^-(z)$ is positive this implies $S_i^+(z) \geq 2z$ and from (3.8) we get that

$$\tilde{R}_i(z) \leq \frac{1}{2} a_i z^{-1} a_i \tag{3.10}$$

which is needed for corollary 1.7.

Proof of Theorem 1.1. The lower inequality follows from lemma 2.3 which ensures that the open line segment $]\|T\|, \infty[\, I_m$ is in the range of K and hence in the range of $\tilde{K}$.
To prove the upper inequality we proceed as in [Leh97] with the new Gabber-function and the inclusion of the identity operator with coefficient a_0. Let us first assume that the matrices a_i are invertible. We represent $\boldsymbol{M}_m \otimes vN(\Gamma_n)$ on the Hilbert space $L_2(\boldsymbol{M}_m, \mathrm{tr}_m) \otimes_2 l_2(\Gamma_n)$, where tr_m is the usual trace on $\boldsymbol{M}_m$ and the noncommutative L_2-norm on $\boldsymbol{M}_2$ coincides with the Hilbert-Schmidt norm. In the first part of the proof we show that the right hand side of (1.1) provides an upper bound for the norm of T. Given any finitely supported $h = \sum h(g) \otimes \delta_g \in l_2(\Gamma_n; HS(\boldsymbol{M}_m))$ we have to estimate $\|Th\|_2^2 = \sum_{g \in \Gamma_n} \|Th(g)\|_{HS}^2$. Let $z \in \boldsymbol{M}_m$ be a positive definite invertible matrix and denote by $\tilde{R}_i$ the $\boldsymbol{M}_m$-valued R-transform given by formula (3.8). Define for $i \in \{1, 2, \ldots, n\}$ the functions

$$\tilde{P}_i(z, g)) = \begin{cases} a_i^{-1} \tilde{R}_i(z) = a_i^{-1} S_i^-(z) & \text{if } |g_i g| = |g| + 1 \\ \tilde{R}_i(z)^{-1} a_i = a_i^{-1} S_i^+(z) & \text{if } |g_i g| = |g| - 1 \end{cases}$$

For fixed $g \in \Gamma_n$ we have

$$\|Th(g)\|_{HS}^2 = \left\| \sum_{i=1}^n a_0 h(g) + a_i\, \tilde{P}_i(z, g)\, \tilde{P}_i(z, g)^{-1}\, h(g_i g) \right\|_{HS}^2$$

and since the operator $a_i \tilde{P}_i(z, g) = S_i^{\pm}(z)$ is positive, we can apply lemma 2.5 to the sequences

$$x_0 = a_0^{\frac{1}{2}} \qquad x_i = (a_i \tilde{P}_i(z, g))^{\frac{1}{2}}$$

and

$$y_0 = a_0^{\frac{1}{2}} h(g) \qquad y_i = (a_i \tilde{P}_i(z, g))^{\frac{1}{2}}\, \tilde{P}_i(z, g)^{-1}\, h(g_i g)$$

to obtain

$$\|Th(g)\|_{HS}^2 \leq \left\| a_0 + \sum_{i=1}^n a_i \tilde{P}_i(z, g) \right\| \, \mathrm{tr} \left(h(g)^* a_0 h(g) + \sum_{i=1}^n h(g_i g)^* \tilde{P}_i(z, g)^{*^{-1}} a_i h(g_i g) \right).$$

Note that among the words $\{ g_1g, g_2g, \dots, g_ng \}$ there is at most one cancellation so that there is always the "$-$" sign in $\tilde{P}_i(z,g)$ except maybe one case and since all the summands $a_i\,\tilde{P}_i(z,g)$ are positive operators, we get from (3.9) the operator inequality

$$a_0 + \sum_{i=1}^n a_i\,\tilde{P}_i(z,g) \leq 2z + a_0 + \sum \tilde{R}_i(z) = \tilde{K}(z)$$

This upper bound does not depend on the group element g and we can estimate the norm of Th as follows. Noting that $\tilde{P}_i(z,g)^{*^{-1}} a_i = S_i^{\pm}(z)$ is positive we get as before

$$\begin{aligned}
\|Th\|_2^2 &\leq \|\tilde{K}(z)\| \sum_{g\in\Gamma_n} \operatorname{tr}\left(h(g)^*a_0h(g) + \sum_{i=1}^n h(g)^*\tilde{P}_i(z,g_ig)^{*^{-1}} a_i\,h(g)\right) \\
&\leq \|\tilde{K}(z)\| \sum_{g\in\Gamma_n} \left\| a_0 + \sum_{i=1}^n \tilde{P}_i(z,g_ig)^{*^{-1}} a_i \right\| \|h(g)\|_{HS}^2;
\end{aligned}$$

now $\tilde{P}_i(z,g_ig)$ has always the "$+$" sign with at most one exception possible in the case when there is already cancellation in g_ig. Thus $\tilde{P}_i(z,g_ig)^{-1}$ has the "$-$" signs and with the formula for the inverse we get

$$a_0 + \sum_{i=1}^n \tilde{P}_i(z,g_ig)^{*^{-1}} a_i \leq \tilde{K}(z)$$

Again the bound is independent of g and finally we get the inequality we wanted:

$$\|Th\|_2^2 \leq \|\tilde{K}(z)\|^2\|h\|^2.$$

This finishes the proof for invertible coefficients a_i. If they are not, they can be approximated by such and the inequality still holds. □

To give an alternative proof of the lower inequality we need to compute the derivative of the K-transform. This will involve the following variant of $\tilde{R}_i$, namely

$$\tilde{P}_i(z) = a_i^{-1}S_i^-(z) = S_i^+(z)^{-1}a_i. \tag{3.11}$$

Denote for $x \in \boldsymbol{M}_m$ by L_x and R_x the representations of x on $L_2(\boldsymbol{M}_m, \operatorname{tr})$ by left and right multiplication.

Proposition 3.2. *The derivative of the modified K-transform* (3.6) *is given by*

$$D_h\tilde{K}(z) = 2\left(Id - \sum_{i=1}^n \left(Id - \left(Id + L_{\tilde{P}_i(z)^*}R_{\tilde{P}_i(z)}\right)^{-1}\right)\right)[h] \tag{3.12}$$

Proof. As a consequence of (3.8) we have $\tilde{R}_i(z)^{-1}a_i = a_i^{-1}\tilde{R}_i(z) + 2a_i^{-1}z$ and taking the derivative of this equation we get

$$-\tilde{R}_i(z)^{-1}(D_h\tilde{R}_i(z))\tilde{R}_i(z)^{-1}a_i = a_i^{-1}D_h\tilde{R}_i(z) + 2a_i^{-1}h$$

which after some elementary manipulations leads to

$$D_h\tilde{R}_i(z) + \tilde{P}_i(z)^*(D_h\tilde{R}_i(z) + 2h)\tilde{P}_i(z) = 0.$$

In terms of left and right multiplication operators we can rewrite this as

$$2h = (Id + L_{\tilde{P}_i(z)^*}R_{\tilde{P}_i(z)})[D_h\tilde{R}_i(z) + 2h]$$

and hence

$$D_h\tilde{R}_i(z) = 2\big((Id + L_{\tilde{P}_i(z)^*}R_{\tilde{P}_i(z)})^{-1}[h] - h\big), \tag{3.13}$$

which implies (3.12). □

In order to relate formula (3.12) to asymptotic eigenvectors as in [PP88], we need the following noncommutative version of [FTS94, Ch. 2, Lemma 1.5]

Proposition 3.3. *Let $\varphi : \Gamma_n \to B(H)$ be a function such that $\varphi(e) = I$ and $\varphi(xy) = \varphi(x)\varphi(y)$ whenever $|xy| = |x| + |y|$, i.e. whenever there is no cancellation in the product xy. Then*

$$\sum_{x\in\Gamma} \varphi(x) = \left(I - \sum_{i=1}^{n} \varphi(g_i)(I + \varphi(g_i))^{-1} \right)^{-1}$$

if the sum on the left hand side converges absolutely.

Proof. The proof is done by induction on the number n of generators of the group Γ_n. To simplify notation we shall denote $a_i = \varphi(g_i)$. This determines the function completely and for any reduced word $g = g_{i_1} g_{i_2} \cdots g_{i_k}$ we have $\varphi(g) = a_{i_1} a_{i_2} \cdots a_{i_k}$. For $n = 1$ the sum is finite and we have

$$\begin{aligned} \varphi(e) + \varphi(g_1) &= I + a_1 \\ &= (I - a_1(I + a_1)^{-1})^{-1}. \end{aligned}$$

Assuming that the claim is true for $1, 2, \ldots, n-1$, we split off the last free factor and set $A = a_n$ and $B = \sum_{x\in\Gamma_{n-1}\setminus\{e\}} \varphi(x)$. Then any element of Γ_n has one of the forms

$$\begin{gathered} (g_n x_1)(g_n x_2)\cdots(g_n x_m) \\ (g_n x_1)(g_n x_2)\cdots(g_n x_m) g_n \\ (x_1 g_n)(x_2 g_n)\cdots(x_m g_n) \\ (x_1 g_n)(x_2 g_n)\cdots(x_m g_n) x_{m+1} \end{gathered}$$

with $x_i \in \Gamma_{n-1} \setminus \{e\}$. Hence we can write

$$\begin{aligned} \sum_{x\in\Gamma_n} \varphi(x) &= \sum_{k=0}^{\infty} (AB)^k + \sum_{k=0}^{\infty} (AB)^k A + \sum_{k=1}^{\infty} (BA)^k + \sum_{k=0}^{\infty} (BA)^k B \\ &= (I + B)(I - AB)^{-1}(I + A) \\ &= \left((I + A)^{-1}(I - AB)(I + B)^{-1}\right)^{-1} \\ &= \left((I + A)^{-1}((I + A)(I + B) - A - B - 2AB)(I + B)^{-1}\right)^{-1} \\ &= \left(I - (I + A)^{-1}(A(I + B) + (I + A)B)(I + B)^{-1}\right)^{-1} \\ &= \left(I - A(I + A)^{-1} - B(I + B)^{-1}\right)^{-1} \end{aligned}$$

By assumption we have

$$B = \left(I - \sum_{i=1}^{n-1} a_i (I + a_i)^{-1} \right)^{-1} - I$$

and hence

$$B(I + B)^{-1} = \sum_{i=1}^{n-1} a_i (I + a_i)^{-1}$$

which yields the claim. □

Proposition 3.4. *The multiplicative function*

$$\begin{aligned} \omega_{z,y} : \Gamma_n &\to \boldsymbol{M}_m \\ g_{i_1} g_{i_2} \cdots g_{i_k} &\mapsto \tilde{P}_{i_1}(z) \tilde{P}_{i_2}(z) \cdots \tilde{P}_{i_k}(z) y. \end{aligned}$$

satisfies

$$T\omega_{z,y} = \tilde{K}(z)\omega_{z,y} - 2zy \otimes \delta_e \tag{3.14}$$

and the norm of $\tilde{\omega}_z = \omega_{z,I_m}$ *is given by*

$$\|\tilde{\omega}_z\|^2 = 2\operatorname{tr}\left(D\tilde{K}(z)^{-1}[I_m]\right) \tag{3.15}$$

Proof. The proof is straightforward calculation, namely

$$\begin{aligned} T\omega_{z,y}(e) &= a_0 y + \sum_{i=1}^{n} a_i \omega_{z,y}(g_i) \\ &= \left(a_0 + \sum_{i=1}^{n} a_i \tilde{P}_i(z)\right) y \\ &= \left(\tilde{K}(z) - 2z\right) y \end{aligned}$$

and

$$\begin{aligned} T\omega_{z,y}(g_{i_1} g_{i_2} \cdots g_{i_k}) &= a_{i_1}\omega_{z,y}(g_{i_2} \cdots g_{i_k}) + \sum_{i \neq i_1} a_i \omega_{z,y}(g_i g_{i_1} \cdots g_{i_k}) \\ &= \left(a_{i_1}\tilde{P}_{i_1}(z)^{-1} + \sum_{i \neq i_1} a_i \tilde{P}_i(z)\right) \omega_{z,y}(g_{i_1} g_{i_2} \cdots g_{i_k}) \\ &= \tilde{K}(z)\omega_{z,y}(g_{i_1} g_{i_2} \cdots g_{i_k}) \end{aligned}$$

which implies (3.14). We can compute the norm of $\tilde{\omega}_z = \omega_{z,I_m}$ as element of $l_2(\Gamma; L_2(\boldsymbol{M}_m))$ using proposition 3.3,

$$\begin{aligned} \|\tilde{\omega}_z\|^2 &= \sum_{x \in \Gamma} \operatorname{tr}(\tilde{\omega}_z(x)^* \tilde{\omega}_z(x)) \\ &= \sum_{i_1, i_2, \dots, i_k} \operatorname{tr}(\tilde{P}_{i_1}(z)^* \tilde{P}_{i_2}(z)^* \cdots \tilde{P}_{i_k}(z)^* \tilde{P}_{i_k}(z) \tilde{P}_{i_{k-1}}(z) \cdots \tilde{P}_{i_1}(z)) \\ &= \operatorname{tr}\left(\left(Id - \sum_{i=1}^{n} L_{\tilde{P}_i(z)^*} R_{\tilde{P}_i(z)} (Id + L_{\tilde{P}_i(z)^*} R_{\tilde{P}_i(z)})^{-1}\right)^{-1} [I_m]\right) \end{aligned}$$

and this together with (3.12) implies (3.15). □

The following fix point iteration gives a method to compute numerically the matrix valued Cauchy transform of T, i.e. the inverse of the K-transform.

Proposition 3.5. *Let* a_i, $i \in \{0, 1, \dots, n\}$ *be matrices as in theorem 1.1, let* $c > \frac{3}{2}\sqrt{2}$ *and let* $\mu > \frac{1}{\sqrt{2}}$ *be such that* $2\mu + \frac{1}{2\mu} = c$. *Then for any selfadjoint* $b \in \boldsymbol{M}_m$ *such that*

$b - a_0 \geq c\|\sum_{i=1}^n a_i^2\|^{\frac{1}{2}} I_m$ the sequence defined recursively by

$$z_0 \in M_\mu = \left\{ z \in \boldsymbol{M}_m : z \geq \mu \left\| \sum_{i=1}^n a_i^2 \right\|^{\frac{1}{2}} I_m \right\}$$

$$z_{k+1} = \frac{1}{2}\left(b - a_0 - \sum \tilde{R}_i(z_k)\right)$$

converges to the unique solution $\bar{z} \in \boldsymbol{M}_\mu$ of the equation $\tilde{K}(z) = b$ i.e. $G(b) = \frac{\bar{z}^{-1}}{2}$.

Proof. This is an application of the well known Banach fix point theorem. We have to show that the function

$$F(z) = \frac{1}{2}\left(b - a_0 - \sum \tilde{R}_i(z)\right)$$

maps the set M_μ contractively into itself. Then the unique fixed point of F is the claimed solution of the equation. From (3.10) we get that the image of any $z \in M_\mu$ satisfies

$$\begin{aligned} F(z) &\geq \frac{1}{2}\left(b - a_0 - \frac{1}{2}\sum a_i z^{-1} a_i\right) \\ &\geq \frac{1}{2}\left(c - \frac{1}{2\mu}\right)\left\|\sum a_i^2\right\|^{\frac{1}{2}} \\ &= \mu\left\|\sum a_i^2\right\|^{\frac{1}{2}}. \end{aligned}$$

This shows that $F(M_\mu) \subseteq M_\mu$. Assume now $z', z'' \in M_\mu$. We have to estimate the norm of the difference $F(z') - F(z'')$. From (3.8) we get that

$$\begin{aligned} \|F(z') - F(z'')\| &= \frac{1}{2}\left\|\sum a_i \left(S_i^+(z')^{-1} - S_i^+(z'')^{-1}\right) a_i\right\| \\ &\leq \frac{1}{2}\left\|\sum a_i^2\right\| \max_i \left\|S_i^+(z')^{-1} - S_i^+(z'')^{-1}\right\| \end{aligned}$$

and it suffices to show that the derivative of $S_i^+(z)^{-1}$ is bounded above uniformly by some constant strictly smaller than $\frac{2}{\|\sum a_i^2\|}$ for $z \in M_\mu$. From (3.8) and (3.13) we get

$$\begin{aligned} D_h S_i^+(z)^{-1} &= -S_i^+(z)^{-1}\left(D_h S_i^+(z)\right) S_i^+(z)^{-1} \\ &= -2S_i^+(z)^{-1}\left(Id + L_{\tilde{P}_i(z)^*} R_{\tilde{P}_i(z)}\right)^{-1} [h] S_i^+(z)^{-1}. \end{aligned}$$

Now for $\mu > \frac{1}{2}$ it follows from (3.11) that

$$\begin{aligned} \|\tilde{P}_i(z)\| &= \left\|S_i^+(z)^{-1} a_i\right\| \\ &\leq \frac{1}{2}\|z^{-1}\|\|a_i\| \\ &\leq \frac{\|a_i\|}{2\mu\left\|\sum a_i^2\right\|^{\frac{1}{2}}} \\ &< 1 \end{aligned}$$

and hence

$$\left\|\left(Id + L_{\tilde{P}_i(z)^*} R_{\tilde{P}_i(z)}\right)^{-1}\right\| \leq \frac{1}{1 - \|\tilde{P}_i(z)\|^2} \leq \frac{4}{4 - \|a_i\|^2\|z^{-1}\|^2}.$$

For $\mu > \frac{1}{\sqrt{2}}$ this implies

$$\begin{aligned}\left\|DS_i^+(z)^{-1}\right\| &\leq \|z^{-1}\|^2 \frac{2}{4 - \|a_i\|^2 \|z^{-1}\|^2} \\ &\leq \frac{2}{\|\sum a_i^2\|} \frac{1}{4\mu^2 - \frac{\|a_i\|^2}{\|\sum a_i^2\|}} \\ &< \frac{2}{\|\sum a_i^2\|}\end{aligned}$$

and hence F is contractive. □

Alternative proof of the lower inequality of theorem 1.1. This proof essentially is the same on the free group and on Fock space, and it relies on the following facts. We are given a differentiable function $\tilde{K}(z)$ - the inverse of the Cauchy transform mapping the set of positive matrices into itself. The inverse of its derivative at $y = G(z)^{-1}$ is given by

$$D\tilde{K}(y)^{-1} = -L_y R_y DG(z)$$

and maps the positive matrices into itself as well. Moreover there is a point z_0 such that

$$\tilde{K}(z_0) = \kappa_0 I_m \tag{3.16}$$

and such that the derivative $D\tilde{K}(z_0)$ is invertible and its inverse can be written as

$$D\tilde{K}(z_0)^{-1} = \sum_{s \in S} L_{x_s^*} R_{x_s} \tag{3.17}$$

so that it also maps positive matrices to positive matrices. Let $z(t)$ be the solution of the differential equation

$$\begin{aligned} z(0) &= z_0 \\ \frac{dz}{dt}(t) &= -D\tilde{K}(z(t))^{-1}[I_m]. \end{aligned}$$

The curve $z(t)$ satisfies $\tilde{K}(z(t)) = \kappa(t) I_m$ with $\kappa(t) = \kappa_0 - t$, since $z(0) = \kappa_0 I_m$ and

$$\frac{d}{dt}\tilde{K}(z(t)) = D\tilde{K}(z(t)) \left[\tfrac{dz}{dt}(t)\right] = -I_m.$$

Since $\tilde{K}(z_0)$ is bounded, $\tilde{K}(z(t))$ cannot decrease ad infinitum, thus the solution exists in some maximal finite interval $[0, t_1]$. First observe that since $\dot{z}(t)$ is negative definite, $z(t)$ is bounded. There are two possible reasons which can make the solution stop at t_1: Either $z(t)$ approaches the boundary of the cone of positive matrices which means that it tends to a non-invertible matrix, or the derivative $D\tilde{K}(z)$ becomes singular.
In the first case we shall exhibit a family of vectors ξ_z such that

$$\begin{aligned} \|\xi_z\|^2 &\geq \operatorname{tr}(z^{-2}) \\ T\xi_z &= \tilde{K}(z)\xi_z + \xi_z' \end{aligned} \tag{3.18}$$

where $\|\xi_z'\|$ is uniformly bounded. From this it follows that $\bar{\xi}_z = \frac{\xi_z}{\|\xi_z\|}$ is an asymptotic set of eigenvectors, i.e.

$$\lim_{t \to t_1} T\bar{\xi}_{z(t)} - \kappa(t)\bar{\xi}_{z(t)} = 0$$

which shows that $\|T\| \geq \inf_{t \to t_1} \|\tilde{K}(z(t))\|$.

In the second case by the positivity of the inverse of $D\tilde{K}(z)$ it follows that actually $D\tilde{K}^{-1}[I_m]$ becomes unbounded and it is enough to exhibit a family of vectors η_z satisfying

$$\begin{aligned} \|\eta_z\|^2 &= \operatorname{tr}(D\tilde{K}(z)^{-1}[I_m]) \\ T\eta_z &= \tilde{K}(z)\eta_z + \eta'_z \end{aligned} \tag{3.19}$$

and such that η'_z isuniformly bounded. Then we can conclude as before that $\bar{\eta}_z = \frac{\eta_z}{\|\eta_z\|}$ contains a family of asymptotic eigenvectors.
Concerning the details on the free group, the iteration procedure of proposition 3.5 provides us with a matrix z_0 which satisfies the requirements (3.16) and (3.17). We only have to choose a starting point large enough such that series of the inverse of (3.12) converges. For the vectors ξ_z and η_z we can choose $\omega_{z,z^{-1}}$ and $\tilde{\omega}_z$ respectively from proposition 3.4. □

4. Matrix valued operators on Fock space

In this section we will prove theorem 1.3 and compute the norm of the operator

$$T = a_0 \otimes I + \sum a_i \otimes l_i + a_i^* \otimes l_i^*. \tag{4.1}$$

First we have to compute the K-transform. Let x be a semicircular element in some noncommutative probability space $(\mathcal{A}, \tau)$ and consider a multiple αx i.e. its moments are given by formula (1.3) multiplied by appropriate powers of α. The R-transform of αx takes the particularly simple form [VDN92]

$$R_\alpha(z) = \alpha^2 z.$$

The matrix valued analogue (i.e. the matrix valued R-transform of $a \otimes x$) can be obtained as in [Haa97]. Then we can use theorem 2.2 to get the R-transform of the free sum. But since we will include the projection p_Ω too, which makes the sum non-free, we derive the K-transform directly.

Proposition 4.1. *(i) The K-transform of the operator T in (4.1) is given by*

$$K(z) = z^{-1} + a_0 + \sum a_i^* z a_i. \tag{4.2}$$

The modified K-transform

$$\tilde{K}(z) = K(z^{-1}) = z + a_0 + \sum a_i^* z^{-1} a_i$$

has derivative

$$D_h\tilde{K}(z) = h - \sum a_i^* z^{-1} h z^{-1} a_i$$

and the vector $\tilde{\omega}_z = (I - \sum z^{-1} a_i \otimes l_i)^{-1}\, I_m \otimes \Omega$ has norm

$$\|\tilde{\omega}_z\|^2 = \operatorname{tr}\left(D\tilde{K}(z)^{-1}[I_m]\right) \tag{4.3}$$

and satisfies

$$T\tilde{\omega}_z = \tilde{K}(z) \otimes I\, \tilde{\omega}_z - z \otimes \Omega \tag{4.4}$$

(ii) Adding the projection $p_\Omega = I - \sum l_i l_i^*$ to T has the following effect. The modified K-transform of

$$T_0 = a_0 \otimes I + b \otimes p_\Omega + \sum a_i \otimes l_i + a_i^* \otimes l_i^* \tag{4.5}$$

is given by

$$\tilde{K}_0(z) = K_0(z^{-1}) = K((b+z)^{-1}) = a_0 + b + z + \sum a_i^*(b+z)^{-1}a_i. \tag{4.6}$$

Proof. (i) In order to make the upcoming formulae more compact and readable, we will treat operators of the form $a \otimes I$ as "noncommutative scalars" and denote them by a instead. Define a vector $\omega_{z,y}$ in $L_2(\boldsymbol{M}_m) \otimes_2 \mathcal{T}(H)$ as follows

$$\begin{aligned}\omega_{z,y} &= \Big(I - \sum z a_i \otimes l_i\Big)^{-1} y \otimes \Omega \\ &= \sum_{k=0}^{\infty} \sum_{i_1,\dots i_k} z a_{i_1} z a_{i_2} \cdots z a_{i_k} y \otimes e_{i_1,i_2,\dots,i_k}.\end{aligned} \tag{4.7}$$

Then

$$\begin{aligned}T\omega_{z,y} &= a_0\omega_{z,y} + \sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k} \sum_i a_i z a_{i_1} z a_{i_2} \cdots z a_{i_k} y \otimes e_{ii_1,i_2,\dots,i_k} \\ &\qquad + \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k} a_{i_1}^* z a_{i_1} z a_{i_2} \cdots z a_{i_k} y \otimes e_{i_2,\dots,i_k} \\ &= a_0\omega_{z,y} + z^{-1} \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k} z a_{i_1} z a_{i_2} \cdots z a_{i_k} y \otimes e_{i_1,i_2,\dots,i_k} \\ &\qquad + \sum_{i=1}^{\infty} a_i^* z a_i \Big(y \otimes \Omega + \sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k} z a_{i_1} z a_{i_2} \cdots z a_{i_k} y \otimes e_{i_2,\dots,i_k} \Big) \\ &= \Big(z^{-1} + a_0 + \sum a_i^* z a_i\Big)\omega_{z,y} - z^{-1} y \otimes \Omega.\end{aligned}$$

It follows that

$$\Big(z^{-1} + a_0 + \sum a_i^* z a_i - T\Big)\,\omega_{z,z} = I_m \otimes \Omega$$

and hence

$$\begin{aligned}G\Big(z^{-1} + a_0 + \sum a_i^* z a_i\Big) &= \Big\langle \Big(z^{-1} + \sum a_i^* z a_i - T\Big)^{-1} (I_m \otimes \Omega), \Omega \Big\rangle \\ &= \langle \omega_{z,z}, \Omega \rangle \\ &= z\end{aligned}$$

i.e. the K-transform is given by (4.2).
The norm of $\omega_{z,y}$ can be calculated as follows.

$$\begin{aligned}\|\omega_{z,y}\|^2 &= \sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k} \|z a_{i_1} z a_{i_2} \cdots z a_{i_k} y\|_{HS}^2 \\ &= \operatorname{tr}\Big(\sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k} y^* a_{i_1}^* z a_{i_2}^* z \cdots a_{i_k}^* z^2 a_{i_k} z a_{i_{k-1}} \cdots z a_{i_1} y \Big) \\ &= \operatorname{tr}\Big(L_{y^*} R_y \sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k} L_{a_{i_1}^* z a_{i_2}^* z \cdots a_{i_k}^* z} R_{z a_{i_k} z a_{i_{k-1}} \cdots z a_{i_1}} [I_m] \Big) \\ &= \operatorname{tr}\Big(L_{y^*} R_y \Big(Id - \sum L_{a_i^* z} R_{z a_i} \Big)^{-1} [I_m] \Big)\end{aligned}$$

and the norm of $\tilde{\omega}_z = \omega_{z^{-1}, I_m}$ can be identified as (4.3).

(ii) The operator (4.5) is not a sum of freely independent operators anymore. However, its K-transform can still be computed by the above method. Indeed, it is easy to see that we get

$$T_0\omega_{z,y} = K(z)\omega_{z,y} + (b - z^{-1})y \otimes \Omega$$

and hence, whenever $b + z^{-1}$ is positive definite,

$$T_0\omega_{(b+z^{-1})^{-1},z} = \left(a_0 + b + z^{-1} + \sum a_i^*(b + z^{-1})^{-1}a_i\right)\omega_{(b+z^{-1})^{-1},z} - I_m \otimes \Omega.$$

As before we deduce that the modified K-transform is given by (4.6). □

The following iteration based on the Banach fix point theorem allows to compute $G(b)$ for big enough values.

Proposition 4.2. *Let $a_0, a_1, \dots, a_n$ be $m \times m$ matrices with a_0 selfadjoint and let $s > 2\|\sum_{i=1}^n a_i^* a_i\|^{\frac{1}{2}}$ be a given real number. Assume further that b is an $m \times m$ matrix such that $b - a_0 \geq sI_m$. Then for any $z_0 \in M_s = \{z \in \boldsymbol{M}_m : z \geq \frac{s}{2}I_m\}$ the iteration defined recursively by*

$$z_{k+1} = b - a_0 - \sum a_i^* z_k^{-1} a_i \tag{4.8}$$

converges to the unique solution $\bar{z} \in M_s$ of the equation

$$z + a_0 + \sum a_i^* z^{-1} a_i = b.$$

In other words, the Cauchy transform of T satisfies $G(b) = \bar{z}^{-1}$.

Proof. In order to apply the Banach fix point theorem, we have to show that the function $F : z \mapsto b - a_0 - \sum a_i^* z^{-1} a_i$ maps M_s into itself and that it is Lipschitz continuous there with Lipschitz constant strictly smaller than one. For the first part, assume that $z \geq \frac{s}{2}I_m$. Then

$$\sum a_i^* z^{-1} a_i \leq \frac{2}{s}\sum a_i^* a_i \leq \left\|\sum a_i^* a_i\right\|^{\frac{1}{2}} I_m$$

and hence

$$F(z) \geq \left(s - \left\|\sum a_i^* a_i\right\|^{\frac{1}{2}}\right) I_m \geq \frac{s}{2}I_m.$$

To show Lipschitz continuity, assume that z' and z'' are matrices in M_s. Then we estimate the norm of the difference as follows.

$$\begin{aligned}\|F(z') - F(z'')\| &= \left\|\sum a_i^*\left(z'^{-1} - z''^{-1}\right)a_i\right\| \\ &\leq \left\|\sum a_i^* a_i\right\| \left\|z'^{-1}(z'' - z')z''^{-1}\right\| \\ &\leq \frac{4}{s^2}\left\|\sum a_i^* a_i\right\| \|z'' - z'\|\end{aligned}$$

and by assumption the Lipschitz constant appearing in the last line is strictly smaller than one. □

Proof of the upper bound in Theorem 1.3. We shall use the full Fock space framework introduced in section 1. We will consider $T = a_0 \otimes I + \sum_{i=1}^n a_i \otimes l_i + a_i^* \otimes l_i^*$ acting on the Hilbert space $HS(\boldsymbol{M}_m) \otimes_2 \mathcal{T}(H)$ by left multiplication. It suffices to estimate the action of T on finitely supported vectors of the form $h = \sum_{k=0}^\infty \sum h_{i_1,i_2,\dots,i_k} \otimes e_{i_1,i_2,\dots,i_k}$. Here and later on, a sum over vectors $e_{i_1,\dots,i_k}$ with $k = 0$ is understood as the vector Ω. We have

$$\|Th\|^2 = \sum_{p=0}^\infty \sum_{j_1,j_2,\dots,j_p} \left\|\langle Th, e_{j_1,j_2,\dots,j_p}\rangle\right\|_{HS}^2;$$

for fixed $j_1, j_2, \dots, j_p$, $p \geq 0$ and $z > 0$ the estimate is as follows.

$$\begin{aligned}
\left\|\langle Th, e_{j_1,j_2,\dots,j_p}\rangle\right\|_{HS}^2
&= \Bigg\| a_0 h_{j_1,j_2,\dots,j_p} + \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_k} \sum_i a_i h_{i_1,i_2,\dots,i_k} \langle l_i e_{i_1,i_2,\dots,i_k}, e_{j_1,j_2,\dots,j_p}\rangle \\
&\qquad\qquad + a_i^* h_{i_1,i_2,\dots,i_k} \langle l_i^* e_{i_1,i_2,\dots,i_k}, e_{j_1,j_2,\dots,j_p}\rangle \Bigg\|_{HS}^2 \\
&= \Big\| a_0 h_{j_1,j_2,\dots,j_p} + a_{j_1} h_{j_2,j_3,\dots,j_p} + \sum_i a_i^* h_{i,j_1,j_2,\dots,j_p} \Big\|_{HS}^2 \\
&= \Big\| a_0 h_{j_1,j_2,\dots,j_p} + z^{\frac{1}{2}} z^{-\frac{1}{2}} a_{j_1} h_{j_2,j_3,\dots,j_p} + \sum_i a_i^* z^{-1} a_i (z^{-1} a_i)^{-1} h_{i,j_1,j_2,\dots,j_p} \Big\|_{HS}^2 .
\end{aligned}$$

At this point we can apply lemma 2.5 to the sequences

$$a_0^{\frac{1}{2}}, \quad z^{\frac{1}{2}}, \quad (a_1^* z^{-1} a_1)^{\frac{1}{2}}, \quad \dots, \quad (a_n^* z^{-1} a_n)^{\frac{1}{2}}$$

and

$$\begin{aligned}
&a_0^{\frac{1}{2}} h_{j_1,j_2,\dots,j_p}, \quad z^{-\frac{1}{2}} a_{j_1} h_{j_2,j_3,\dots,j_p}, \quad (a_1^* z^{-1} a_1)^{\frac{1}{2}} a_1^{-1} z h_{1,j_1,j_2,\dots,j_p}, \quad \dots \\
&\qquad\qquad \dots, \quad (a_n^* z^{-1} a_n)^{\frac{1}{2}} a_n^{-1} z h_{n,j_1,j_2,\dots,j_p}
\end{aligned}$$

and we get the inequality

$$\begin{aligned}
\left\|\langle Th, e_{j_1,j_2,\dots,j_p}\rangle\right\|_{HS}^2 \leq \Big\| a_0 + z + \sum a_i^* z^{-1} a_i \Big\| \operatorname{tr}\Big(& h_{j_1,j_2,\dots,j_p}^* a_0 h_{j_1,j_2,\dots,j_p} + \\
& h_{j_2,j_3,\dots,j_p}^* a_{j_1}^* z^{-1} a_{j_1} h_{j_2,j_3,\dots,j_p} + \sum_i h_{i,j_1,j_2,\dots,j_p}^* z h_{i,j_1,j_2,\dots,j_p} \Big).
\end{aligned}$$

Again the modified K-transform $\tilde{K}(z) = z + a_0 + \sum a_i^* z^{-1} a_i$ appears and after reordering the following sums appropriately we get

$$\begin{aligned}
\|Th\|_{HS}^2 &= \sum_{p=0}^{\infty} \sum_{j_1,j_2,\dots,j_p} \left\|\langle Th, e_{j_1,j_2,\dots,j_p}\rangle\right\|_{HS}^2 \\
&\leq \|\tilde{K}(z)\| \operatorname{tr}\Big(\sum_{p=0}^{\infty} \sum_{j_1,j_2,\dots,j_p} h_{j_1,j_2,\dots,j_p}^* a_0 h_{j_1,j_2,\dots,j_p} \\
&\qquad\qquad + \sum_{j_1,j_2,\dots,j_p} h_{j_2,j_3,\dots,j_p}^* a_{j_1}^* z^{-1} a_{j_1} h_{j_2,j_3,\dots,j_p} \\
&\qquad\qquad + \sum_{i,j_1,j_2,\dots,j_p} h_{i,j_1,j_2,\dots,j_p}^* z h_{i,j_1,j_2,\dots,j_p} \Big) \\
&= \|\tilde{K}(z)\| \operatorname{tr}\Big(\sum_{p=0}^{\infty} \sum_{j_1,j_2,\dots,j_p} h_{j_1,j_2,\dots,j_p}^* \Big(z + a_0 + \sum a_i^* z^{-1} a_i \Big) h_{j_1,j_2,\dots,j_p} \Big) \\
&\leq \|\tilde{K}(z)\|^2 \|h\|^2 .
\end{aligned}$$

$\square$

Proof of the lower bound in Theorem 1.3. To show the reverse inequality we cannot make use of lemma 2.3, since the vacuum state is neither tracial nor faithful. However, the alternative argument at the end of section 3 translates exactly to the Fock space framework, using now the proposition 4.2 to find z_0. The vectors ξ_z and η_z can be taken to be $\omega_{z^{-1},z^{-1}}$ and ω_{z^{-1},I_m} respectively from proposition 4.1. □

Proof of Theorem 1.8. In order to obtain the upper bound we only have to slightly modify the above proof. Since the vectors $e_{j_1,j_2,\dots,j_p}$ $(p \geq 1)$ do not feel the projection p_Ω, only the term $\langle T_0 h, \Omega\rangle$ is different from its counterpart $\langle Th, \Omega\rangle$. Using the positivity of $a+b$ and choosing z such that both z and $b+z$ are positive and invertible, we can estimate similarly as before

$$\begin{aligned}\|\langle T_0 h, \Omega\rangle\|_2^2 &= \left\|(a_0+b)h_\Omega + \sum a_i^* h_i\right\|_2^2 \\ &= \left\|(a_0+b)^{\frac12}(a_0+b)^{\frac12}h_\Omega + \sum a_i^*(b+z)^{-1}a_i a_i^{-1}(b+z)h_i\right\|_2^2 \\ &\leq \left\|a_0+b+\sum a_i^*(b+z)^{-1}a_i\right\| \operatorname{tr}\left(h_\Omega^*(a+b)h_\Omega + \sum h_i^*(b+z)h_i\right)\end{aligned}$$

We also replace z by $z+b$ in the other terms and obtain for any index set $j_1,\dots,j_p$, $p \geq 1$,

$$\begin{aligned}&\left\|\langle T_0 h, e_{j_1,j_2,\dots,j_p}\rangle\right\|_{HS}^2 \leq \\ &\qquad \left\|a_0+b+z+\sum a_i^*(b+z)^{-1}a_i\right\| \operatorname{tr}\Bigg(h_{j_1,j_2,\dots,j_p}^* a_0 h_{j_1,j_2,\dots,j_p} + \\ &\qquad\qquad h_{j_2,j_3,\dots,j_p}^* a_{j_1}^*(b+z)^{-1}a_{j_1}h_{j_2,j_3,\dots,j_p} + \sum_i h_{i,j_1,j_2,\dots,j_p}^*(b+z)h_{i,j_1,j_2,\dots,j_p}\Bigg).\end{aligned}$$

Now note that $0 \leq a_0+b+\sum a_i^*(b+z)^{-1}a_i \leq a_0+b+z+\sum a_i^*(b+z)^{-1}a_i$ and we get in exactly the same manner as before that $\|T_0 h\| \leq \|\tilde K_0(z)\|\,\|h\|$.
In order to get the lower bound we can assume that one of the coefficients a_i $(i \geq 1)$, say a_1, is invertible. For if this is not the case, we can consider for any $\varepsilon > 0$ the perturbated operator $T_0^{(\varepsilon)} = T_0 + \varepsilon I_m \otimes (l_{n+1}^* + l_{n+1})$. Now εI_m is invertible and assuming that we have proved the claim when at least one coefficient is invertible, we can choose some z_ε with the required positivity properties such that

$$\|T_0^{(\varepsilon)}\| \geq \|K_0^{(\varepsilon)}(z_\varepsilon)\| - \varepsilon.$$

Then the operator inequality $K_0^{(\varepsilon)}(z_\varepsilon) = K_0(z_\varepsilon) + \varepsilon^2 z_\varepsilon^{-1} \geq K_0(z_\varepsilon) \geq 0$ implies that

$$\|T_0\| \geq \|T_0^{(\varepsilon)}\| - 2\varepsilon \geq \|K_0^{(\varepsilon)}(z_\varepsilon)\| - 3\varepsilon \geq \|K_0(z_\varepsilon)\| - 3\varepsilon$$

Since ε can be chosen arbitrary small, the claim follows.
In order to show the claim when a_1 is invertible, consider again a curve $z(t)$ starting at some remote point $z(0) = z_0$ where $\tilde K(z_0) = \kappa_0 I_m$ and with derivative $\frac{dz}{dt}(t) = -D\tilde K_0(z(t))^{-1}[I_m]$. As $\|\tilde K_0(z(t))\| = \kappa_0 - t$ decreases, it cannot happen that $b+z$ comes close to a singular point, since in this case $\kappa_0 - t = \tau_m(\tilde K_0(z(t))) \geq \tau_m(a_1 a_1^*(b+z)^{-1})$ would explode. The two other possibilities which make $z(t)$ stop at t_1 are that $D\tilde K_0(z(t))^{-1}[I_m]$ explodes or that z becomes singular. The former leads to the asymptotic eigenvector $\tilde\omega_{b+z}$ as before; in the latter case the family $\omega_{z^{-1},z^{-1}}$ satisfies $T_0\omega_{z^{-1},z^{-1}} = \tilde K_0(z)\omega_{z^{-1},z^{-1}} - I_m \otimes \Omega$ and $\|\omega_{z^{-1},z^{-1}}\|_2^2 \geq \operatorname{tr}(z^{-2})$ explodes, so that again we get a lower bound for the norm, namely $\lim_{t\to t_1}\|\tilde K_0(z(t))\|$. □

Remark 4.3. Although stated only for matrix coefficients, both formulae (1.1) and (1.4) hold with matrices replaced by operators. Indeed one can easily verify that the proof for the inequality "$\leq$" carries over without modification, whereas lemma 2.3 does not seem to have an analogue in this setting. This can be overcome as follows. Let y_i denote either the free unitaries $\lambda(g_i)$ or the free semicircular operators x_i and let $T = a_0 \otimes I + \sum a_i \otimes y_i$ have coefficients in some matrix algebra $\boldsymbol{M}_m$. First observe that embedding $\boldsymbol{M}_m$ into $B(H)$ for some infinite dimensional Hilbert space and identifying the matrices a_i with the operators $\tilde{a}_i = \left[\begin{smallmatrix} a_i & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we have that the norm of the operator $\tilde{T} = \sum \tilde{a}_i \otimes x_i$ is equal to the norm of $\|T\|$. On the other hand, we have seen that

$$\|\tilde{T}\| \leq \inf_{\tilde{z} \in B(H)_+^{-1}} \|K_{\tilde{T}}(\tilde{z})\| \tag{4.9}$$

and the latter is smaller than

$$\inf_{z \in \boldsymbol{M}_m{}_+^{-1}} \|K_T(z)\| = \|T\|, \tag{4.10}$$

since we may take $\tilde{z}$ of the form

$$\tilde{z} = \begin{bmatrix} z & 0 \\ 0 & z' \end{bmatrix}$$

with suitable positive invertible z'. This shows that equality holds in (4.9) if the coefficients are finite rank operators. This implies that the operator space structure defined the right hand side of (4.9) is completely isometric to the operator space structure defined by (4.10). Now Lemma 1.5 in [dCH85] implies that equality holds in (4.9) for arbitrary operator coefficients.

5. Applications to harmonic analysis on the free group and on Fock space

In this section we give applications of theorems 1.1 and 1.8 to compute the norms of arbitrary finitely supported operators in the C^*-algebra generated by the free operators in consideration.

5.1. **Free Group.** It is known that the norm structure of a C^*-algebra generated by unitaries $u_1, u_2, \dots, u_n$ is determined by the operator space structure of the subspace spanned by $I, u_1, u_2, \dots, u_n$, see [Pis96]. The following gives a constructive proof of this fact. The claim is that one can compute the norm of arbitrary finite linear combinations of products (also called words) of the generating unitaries, provided that one knows how to compute matrix-valued linear combinations of the generating unitaries and the identity operator, i.e. the operator space structure of the spanned subspace. It is enough to give a procedure to compute matrix-valued linear combinations of products of length at most two of the generating unitaries. Then one can iterate this procedure and reduce the length of the involved words by a factor two. This is the object of the current section.
We shall first consider the special case of scalar linear combinations of words of length smaller than or equal to two. To simplify notation we will assume that g_0 is the identity element and hence $\lambda(g_0)$ is the identity operator. Suppose that we are given a self-adjoint convolution operator T supported on the words of length 2, i.e. T has the form

$$T = \sum_{i,j=0}^{n} c_{ij}\, \lambda(g_i g_j)$$

with $c_{ij} \in \mathbf{C}$ and $c_{ij} = \overline{c_{ji}}$. Assume moreover that the matrix $C = [c_{ij}] \in \boldsymbol{M}_{n+1}$ is positive definite, so that it has a self-adjoint square root $A = [\alpha_{ij}]$ (in what follows one can choose any matrix A such that $A^*A = C$). Setting

$$G = \begin{bmatrix} \lambda(g_0) & 0 & \dots & 0 \\ \lambda(g_1) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda(g_n) & 0 & \dots & 0 \end{bmatrix}$$

it follows that

$$\begin{aligned} \|T\| &= \left\| \sum \lambda(g_i) c_{ij} \lambda(g_j) \right\| \\ &= \|G^* C G\| \\ &= \|AG\|^2 \\ &= \left\| \sum A_i \otimes \lambda(g_i) \right\| \end{aligned}$$

where the coefficients are given by

$$A_i = \begin{bmatrix} \alpha_{i0} & 0 & \dots & 0 \\ \alpha_{i1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{in} & 0 & \dots & 0 \end{bmatrix}.$$

The norm can then be calculated by formula (1.1). Next assume that the coefficients c_{ij} are positive real numbers, C still being symmetric. It follows from [dlHRV93, Lemma 3.2] that the spectrum of T has nonvoid intersection with the positive real numbers and moreover $\max \sigma(T) \geq |\min \sigma(T)|$. Hence for any positive real number λ we have $\|\lambda I + T\| = \lambda + \|T\|$ and we can choose λ such that $\lambda I_{n+1} + C$ is positive definite. Then

$$\|T\| = \|G^*(\lambda I_{n+1} + C) G\| - (n+1)\lambda$$

and we are back in the case of a positive definite coefficient matrix.
A trick similar to the one used in section 1 helps with a non-selfadjoint coefficient matrix C. In this case we have that

$$\|T\| = \left\| \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \right\|$$

The matrix $\tilde{T} = \left[\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right]$ satisfies $U\tilde{T}U^* = -\tilde{T}$, where $U = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$. Thus the spectrum of $\tilde{T}$ is symmetric around 0 and for any real number λ we have

$$\left\| \begin{bmatrix} \lambda & T^* \\ T & \lambda \end{bmatrix} \right\| = |\lambda| + \left\| \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \right\|; \tag{5.1}$$

choosing λ positive big enough so that the matrix

$$\tilde{C}_\lambda = \begin{bmatrix} \frac{\lambda}{n+1} I_{n+1} & C^* \\ C & \frac{\lambda}{n+1} I_{n+1} \end{bmatrix}$$

is positive definite leads again to the case of a positive coefficient matrix, except that now we have to work with an extended matrix $\tilde{G} = \left[\begin{smallmatrix} G & 0 \\ 0 & G \end{smallmatrix}\right]$ instead of G. Here instead of taking the positive square root of the extended matrix $\tilde{C}_\lambda$, it seems to be more efficient to take a Cholesky-type factor

$$R = \begin{bmatrix} \frac{\sqrt{\lambda}}{\sqrt{n+1}} I_{n+1} & \frac{\sqrt{n+1}}{\sqrt{\lambda}} C^* \\ 0 & D \end{bmatrix}$$

where D is such that $D^*D + \frac{n+1}{\lambda} CC^* = \frac{\lambda}{n+1} I_{n+1}$ and hence $R^*R = \tilde{C}_\lambda$.
This completes the special case of words of length two. In order to iterate this scheme, we have to extend it to matrix coefficients. Consider an operator of the form

$$T = \sum C_{ij} \otimes \lambda(g_i g_j)$$

where $C_{ij} \in \boldsymbol{M}_m$, $i, j \in \{0, \dots, n\}$. Again we have that

$$\begin{aligned}
\|T\| &= \left\| \begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} (n+1)\lambda I_m \otimes I & T \\ T^* & (n+1)\lambda I_m \otimes I \end{bmatrix} \right\| - (n+1)\lambda \\
&= \left\| \begin{bmatrix} G^{(n+1)^*} & 0 \\ 0 & G^{(n+1)^*} \end{bmatrix} \begin{bmatrix} \lambda I_{n+1} \otimes I_m \otimes I & C^{(n+1)} \otimes I \\ C^{(n+1)^*} \otimes I & \lambda I_{n+1} \otimes I_m \otimes I \end{bmatrix} \begin{bmatrix} G^{(n+1)} & 0 \\ 0 & G^{(n+1)} \end{bmatrix} \right\| - (n+1)\lambda
\end{aligned}$$

where $G^{(n+1)} = \sum e_{i1} \otimes I_m \otimes \lambda(g_i)$ and $C^{(n+1)} = \sum e_{ij} \otimes C_{ij}$. By choosing λ large enough the $(n+1)m \times (n+1)m$ matrix

$$\begin{bmatrix} \lambda I_{n+1} \otimes I_m & C^{(n+1)} \\ C^{(n+1)^*} & \lambda I_{n+1} \otimes I_m \end{bmatrix}$$

can be made positive definite and we can reduce the norm computation to words of length one. Obviously this technique can be applied iteratively to convolution operators supported on arbitrary (finitely many) elements in the group Γ_n.
Finally observe that the elements $g_0g_1, g_0g_2, \dots, g_0g_n$ generate a copy of the free group $\mathbf{F}_n = \mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z}$ inside Γ_{n+1}, thus the above scheme can also be used to calculate the norm of finitely supported convolution operators on $\mathbf{F}_n$.

5.2. **Fock Space.** The situation in Fock space is similar to the previous one. Any operator T supported on the words of length ≤ 2 can be represented as $T = LCL^*$ with some coefficient matrix $C \in \boldsymbol{M}_{2n+1}$ and where

$$L = \left[I, l_1^*, l_1, \dots, l_n^*, l_n\right].$$

If the coefficient matrix C is positive definite, we can proceed as before. If it is not, we repeat the matrix trick with the following slight modification. Again the norm of T is the same as the norm of the amplified operator

$$\tilde{T} = \begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \begin{bmatrix} L^* & 0 \\ 0 & L^* \end{bmatrix}.$$

Now we cannot immediately pass to the positive definite matrix $\left[\begin{smallmatrix} \lambda & C \\ C^* & \lambda \end{smallmatrix}\right]$ since $LL^* = I + \sum l_i^* l_i + l_i l_i^* = (n+2)I - p_\Omega$ is not quite a multiple of the identity. Thus we have to include p_Ω into L to get the identity matrix. Setting

$$\tilde{L} = \left[p_\Omega, I, l_1^*, l_1, \dots, l_n^*, l_n\right]$$

and

$$\tilde{C} = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \in \boldsymbol{M}_{2n+2}$$

we get that

$$\begin{bmatrix} \tilde{L} & 0 \\ 0 & \tilde{L} \end{bmatrix} \begin{bmatrix} \lambda I_{2n+2} & \tilde{C} \\ \tilde{C}^* & \lambda I_{2n+2} \end{bmatrix} \begin{bmatrix} \tilde{L}^* & 0 \\ 0 & \tilde{L}^* \end{bmatrix} = \begin{bmatrix} (n+2)\lambda & T \\ T^* & (n+2)\lambda \end{bmatrix}$$

and from here one can proceed as before, using now theorem 1.8. Again this construction can be adapted to the matrix valued case which makes computation with words of arbitrary length possible.

6. **Numerical computation**

In view of the matrix manipulations in section 5 there might be some interest in evaluating the infima (1.1) and (1.4) for concrete matrices a_i – at least numerically – on a computer. This amounts to solving a constrained nonlinear optimization problem:

$$\|\tilde{K}(z)\| \to \text{min!}$$
$$z \in \boldsymbol{M}_m, z > 0$$

Moreover, since the operator norm of a positive definite matrix is equal to its largest eigenvalue and since the eigenvalue map $a \mapsto \lambda_i(a)$ is differentiable at a if and only if the i-th eigenvalue of a is simple, the last assertions of theorem 1.1 and theorem 1.3 indicate that this optimization problem is essentially non-differentiable, so that standard Newton methods do not apply. However, the latter remarks also indicate where to look for the optimum, namely among the matrices z for which $\tilde{K}(z)$ is a scalar multiple of the identity. Such a matrix is a zero of the matrix-valued function

$$z \mapsto \Phi(z) = \tilde{K}(z) - \tau_m(\tilde{K}(z)) I_m,$$

which can be found using propositions 3.5 and 4.2 respectively. The function Φ is differentiable and thus can be put into a quasi-Newton method, which has the advantage of local quadratic convergence, while fix point iteration is only linearly convergent. Both methods together yield an efficient method to find a zero of Φ. Once such a zero is found, we can compute the tangent of the curve

$$\gamma = \{z \in \boldsymbol{M}_m : z > 0 \text{ and } \tilde{K}(z) = \lambda I_m, \lambda \in \mathbf{C}\} \tag{6.1}$$

and minimize $||\tilde{K}(z)||$ along the tangent. Iteration of these two steps then leads to an optimal z^*. This is essentially equivalent to the solution of the differential equation considered at the end of the proof of theorem 1.1 in section 3.

Recall that the Newton-Raphson method can be formulated as follows. Given a differentiable function $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ and a point $x_k \in \mathbf{R}^n$, one makes a linear approximation of Ψ around x_k

$$\Psi(x_k + h) = \Psi(x_k) + D_h\Psi(x_k) + o(||h||)$$

and takes the zero of this linearized version as the next candidate. It is given by

$$x_{k+1} = x_k - D\Psi(x_k)^{-1}[\Psi(x_k)]$$

where $D\Psi(x_k) = [D_i\Psi_j(x_k)]$ is the Jacobian of Ψ at the point x_k. Then it can be shown that provided x_0 is close enough to a zero x^* this sequence converges quadratically to x^*, i.e. there is a constant C such that

$$||x_{k+1} - x^*|| \leq C||x_k - x^*||^2.$$

This method requires the Jacobian to be invertible. In our case we know from lemma 2.3 that the derivative of $\tilde{K}(z)$ is invertible on the curve (6.1) and hence in some neighbourhood of it. This implies that the derivative of Φ, which is given by $D_h\Phi(z) = D_h\tilde{K}(z) - \tau_m(D_h\tilde{K}(z))I_m$, has nontrivial null space (containing $h_0 = D\tilde{K}(z)^{-1}[I_m]$)

and Newton-Raphson cannot be applied. Hence we have to find another descent direction for the *master function* $f(z) = ||\Phi(z)||_2^2 = \operatorname{tr}(\Phi(z)^2)$. A good candidate is $h = -D\tilde{K}(z)^{-1}[\Phi(z)]$, since

$$D_h\Phi(z) = D_h\tilde{K}(z) - \tau_m\big(D_h\tilde{K}(z)\big)I_m = -\Phi(z)$$

and hence

$$D_h f(z) = 2\operatorname{tr}\left(\Phi(z)D_h\Phi(z)\right) = -2f(z).$$

Indeed, this gives rise to the following quadratic convergent iteration.

Proposition 6.1. *Let z_0^* be a point on the curve γ given by* (6.1) *such that the Jacobian $D\tilde{K}(z_0^*)$ is regular. Then there exists a neighbourhood V of z_0^* such that for any $z_0 \in V$ the iteration sequence z_k given by*

$$z_{k+1} = z_k - D\tilde{K}(z_k)^{-1}[\Phi(z_k)]$$

is well defined and the sequence $\delta_k = \operatorname{dist}(z_k, \gamma)$ satisfies $\delta_{k+1} \leq C(\delta_k)^2$, i.e. z_k converges quadratically to a point on γ.

Proof. The proof is along the lines of the standard proof of quadratic convergence of the Newton method. We replace the euclidean metric around γ by the following one. Set

$$\langle h, h'\rangle_z = \tau_m\left(D_h\tilde{K}(z)D_{h'}\tilde{K}(z)\right) \quad \text{and} \quad ||h||_z^2 = \langle h, h\rangle_z$$

Since $D\tilde{K}(z)$ is invertible, this metric is locally uniformly equivalent to the standard metric. Hence it suffices to show that the iteration is quadratically convergent for this metric. We will abbreviate $\langle\, .\,,\,.\,\rangle_k = \langle\, .\,,\,.\,\rangle_{z_k}$ and $||\,.\,||_k = ||\,.\,||_{z_k}$ respectively. It will be convenient to work with $\delta_k = \operatorname{dist}(z_k, \gamma)$ according to this metric rather than the standard euclidean metric. Let z_k be close to γ and let $z^*(t)$ be a parametrization of γ. Since K has full rank, γ is differentiable and so is the function $t \mapsto ||z^*(t) - z_k||_k^2$. Thus if we choose t_k such that $\|z_k - z^*(t_k)\|_k = \delta_k$, the following necessary first order condition must be satisfied.

$$\begin{aligned}
0 &= \frac{1}{2}\,\frac{d}{dt}\,\|z^*(t) - z_k\|_k^2\bigg|_{t=t_k} \\
&= \tau_m\Big(D\tilde{K}(z_k)\left[\frac{dz^*}{dt}(t_k)\right]\,D\tilde{K}(z_k)\left[z^*(t_k) - z_k\right]\Big)
\end{aligned}$$

Now $\tilde{K}(z^*(t)) = \kappa(t)I_m$ for some scalar valued function κ and $D\tilde{K}(z^*(t))\left[\frac{dz^*}{dt}\right] = \kappa'(t)I_m$, whence it follows that

$$\begin{aligned}
\tau_m\left(D\tilde{K}(z_k)[z^*(t_k) - z_k]\right) &= \tau_m\Big(\Big(D\tilde{K}(z^*(t_k)) - D\tilde{K}(z_k)\Big)\left[\tfrac{dz^*}{dt}(t_k)\right]\,D\tilde{K}(z_k)[z^*(t_k) - z_k]\Big) \\
&= O\left(\|z^*(t_k) - z_k\|_k^2\right)
\end{aligned}$$

since $D\tilde{K}$ Lipschitz continuous and $||D\tilde{K}(z_k) - D\tilde{K}(z^*(t_k))|| = O\left(\|z^*(t_k) - z_k\|_k\right)$. Next we use a first order Taylor approximation of Φ around z_k and we get

$$\begin{aligned}
0 &= \Phi(z^*(t_k)) \\
&= \Phi(z_k) - D\Phi(z_k)[z_k - z^*(t_k)] + O(||z_k - z^*(t_k)||_k^2) \\
&= -D\tilde{K}(z_k)[z_{k+1} - z_k] - D\tilde{K}(z_k)[z_k - z^*(t_k)] + \tau_m\big(D\tilde{K}(z_k)[z_k - z^*(t_k)]\big) + O(||z_k - z^*(t_k)||_k^2) \\
&= -D\tilde{K}(z_k)[z_{k+1} - z^*(t_k)] + O(||z_k - z^*(t_k)||_k^2)
\end{aligned}$$

and hence

$$||z_{k+1} - z^*(t_k)||_k \leq C||z_k - z^*(t_k)||_k^2.$$

□

This iteration can be combined with the globally convergent fix-point iteration from propositions 3.5 and 4.2 gives an efficient method to find a point z_0 on γ. Now given a point z_k on γ, the next step involves a linesearch on the positive part of the tangent, i.e. $\{z_k + \lambda h_k : \lambda > 0\}$ to decrease $\tilde{K}(z)$. Here $h_k = -D\tilde{K}(z_k)^{-1}[I_m]$ is a direction of descent for $\tilde{K}(z)$, since $\tilde{K}(z_k)$ is a multiple of the identity matrix I_m and h_k is chosen such that all the eigenvalues of $\tilde{K}(z)$ decrease. Finding a parameter $\tilde{\lambda}$ which decreases $\|\tilde{K}(z)\|$ on the tangent will yield a point $\tilde{z}_{k+1} = z_k + \tilde{\lambda} h_k$ satisfying $\|\tilde{K}(\tilde{z}_{k+1})\| < \|\tilde{K}(z_k)\|$. In the hope that this is not too far off the curve γ we will then try to find a point $z_{k+1} \in \gamma$ using the iteration from proposition 6.1 with starting point $\tilde{z}_{k+1}$. It is however not a good idea to take $\tilde{\lambda}$ to be the minimizer λ^* of the function $\lambda \mapsto \|\tilde{K}(z_k + \lambda h_k)\|$ for the following reason. The critical value $b = \frac{1}{\|T\|} I_m$ is not the only singularity of the Cauchy transform G, and as soon as $\tilde{K}(z)$ is close to another singularity of G one might expect that the Jacobian $D\tilde{K}(z)$ becomes ill-conditioned if not singular, thus it will be difficult to get back to γ from such a point. This is what actually happens in numerical experiments, especially when z_k is close to the optimal z^* and $z_k + \lambda^*$ gets very close to a singular point not on γ. It is therefore advisable to add a penalty function, for example a multiple of $\|\tilde{K}(z) - \tau_m(\tilde{K}(z))\|_2^2$ in order to stay close enough to γ and away from other singular points. Numerically this works out fine and we have the following iteration scheme. After finding a point $z_0 \in \gamma$ repeat the following three steps

(i) Using the quasi-Newton iteration starting at $\tilde{z}_k$, find a point $z_k \in \gamma$.
(ii) Compute the tangent vector $h_k = -D\tilde{K}(z_k)^{-1}[I_m]$.
(iii) Minimize $\|\tilde{K}(z)\| + \mu \operatorname{tr}(\Phi(z)^2)$ on the tangent $\{z_k + \lambda h_k : \lambda \geq 0\}$ to get a point $\tilde{z}_{k+1}$ which decreases $\|\tilde{K}(z)\|$.

until some stopping criterion is fulfilled. A natural stopping criterion is given by the tangent vector $h = D\tilde{K}(z)^{-1}[I_m]$. Since its trace is equal to the norm of $\tilde{\omega}_z$ (see (4.3) and (3.15) above), one gets a good estimate from the identities (3.14) and (4.6), namely

$$\|T\| \geq \min \operatorname{spec}(\tilde{K}(z)) - 2\frac{\|z\|_{HS}}{\operatorname{tr}(h)}$$

in the case of the free group and

$$\|T\| \geq \min \operatorname{spec}(\tilde{K}(z)) - \frac{\|z\|_{HS}}{\operatorname{tr}(h)}$$

in the case of Fock space.

The author has implemented this method in the *Rlab* environment [Sea] and can be contacted for further information.

References

[AO76] C. A. Akemann and P. A. Ostrand, *Computing norms in group C^*-algebras*, Amer. J. Math. **98** (1976), 1015–1047.

[Boż75] M. Bożejko, *On $\Lambda(p)$ sets with minimal constant in discrete noncommutative groups*, Proc. Amer. Math. Soc. **51** (1975), 407–412.

[dCH85] J. de Canniere and U. Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. **107** (1985), 455–500.

[dlHRV93] P. de la Harpe, A. G. Robertson, and A. Valette, *On the spectrum of the sum of generators for a finitely generated group I*, Isr. J. Math. **81** (1993), 65–96.

[Flo76] M. Flory, *Estimating norms in C^*-algebras of discrete groups*, Math. Ann. **224** (1976), 41–52.

[FTS94] A. Figa-Talamanca and T. Steger, *Harmonic analysis on for anisotropic random walks on homogeneous trees*, Mem. Amer. Math. Soc. **531** (1994), xii+68 pp.

[GdlH97] R. Grigorchuk and P. de la Harpe, *On problems related to growth, entropy and spectrum in group theory*, J. of Dynamical and Control Systems **3** (1997), 51–89.

[Haa97] U. Haagerup, *On Voiculescu's R- and S-transforms for free non-commuting random variables*, Free probability theory, Fields Institute proceedings, vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 127–148.

[HP93] U. Haagerup and G. Pisier, *Bounded linear operators between C^*-algebras*, Duke Math. J. **71** (1993), 889–925.

[Kes59] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.

[Leh97] F. Lehner, *Free operators with operator coefficients*, Coll. Math. **74** (1997), 321–328.

[Lei74] M. Leinert, *Faltungsoperatoren auf gewissen diskreten Gruppen*, Studia Math. **52** (1974), 149–158.

[Pis96] G. Pisier, *A simple proof of a theorem of Kirchberg and related results on C^*-norms*, J. Oper. Theory **35** (1996), 317–335.

[PP88] M. A. Picardello and T. Pytlik, *Norms of free operators*, Proc. Amer. Math. Soc. **104** (1988), 257–261.

[Sea] I. Searle, *RLab 2*, `http://www.eskimo.com/~ians/rlab.html`.

[VDN92] D. Voiculescu, K. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series, vol. 1, Amer. Math. Soc., Providence, 1992.

[Voi95] D. Voiculescu, *Operations on certain non-commutative operator-valued random variables*, Recent advances in operator algebras, Orléans 1992, Astérisque, vol. 232, 1995, pp. 243–275.

[Woe86a] W. Woess, *A short computation of the norms of free convolution operators*, Proc. Amer. Math. Soc. **96** (1986), 167–170.

[Woe86b] W. Woess, *Nearest neighbour random walks on free products of discrete groups*, Boll. Un. Mat. It. **(6) 5-B** (1986), 961–982.

IMADA, Odense Universitet, Campusvej 55, DK-5230 Odense M, Denmark

E-mail address: `lehner@imada.ou.dk`