CHAPITRE II

Sommes d'unitaires et analyse harmonique sur le groupe libre

Introduction

Let A and B be C^*-algebras. Denote $\| \|_{\min}$ the minimal C^*-cross norm on the tensor product $A \otimes B$, i.e. if A and B are represented on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the minimal norm is the norm induced by the canonical embedding of $A \otimes B$ into $B(\mathcal{H} \otimes_2 \mathcal{K})$. For a complex Hilbert space $\mathcal{H}$ denote by $\overline{\mathcal{H}}$ the complex conjugate Hilbert space, i.e. the same space $\mathcal{H}$ with complex conjugate scalar multiplication. Then $\overline{B(\mathcal{H})}$ can be identified with $B(\overline{\mathcal{H}})$. In **[45]** it is proved that for any sequence of unitary operators $u_1, u_2, \ldots, u_n \in B(\mathcal{H})$ the following inequality holds:

$$\left\| \sum_{i=1}^{n} u_i \otimes \overline{u_i} \right\|_{\min} \geq 2\sqrt{n-1}. \tag{II.0.12}$$

In this chapter we establish more general inequalities, exploiting the combinatorial principle behind (II.0.12). A large part is also devoted to an attempt to characterize unitaries, for which there is equality.
This chapter has six sections.
In section II.1 we give some examples and motivation.
In section II.2 the concept of a family nonnegative alternating mixed moments is introduced and several norm inequalities are derived, generalizing (II.0.12).
In section II.3 we characterize unitaries for which equality holds in (II.0.12) under the additional assumption that these unitaries come from the regular representation of a discrete group. It turns out that such sets of unitaries come from subsets of groups which have the Leinert property, i.e. they are up to one element translations of free subsets.
In section II.4 we disprove the following conjectures.

Conjecture II.0.23 ([**45**])**.** Let $u_1, u_2, \ldots, u_n$ be unitaries such that

$$\left\| \sum_{i=1}^{n} u_i \otimes \overline{u_i} \right\|_{\min} = 2\sqrt{n-1}.$$

Then the linear space spanned by $u_1, u_2, \ldots, u_n$ in $B(H)$ is (completely) isometric to the space spanned by $\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)$ in $B(\ell_2(\mathbf{F}_n))$.

Question II.0.24 ([**55**])**.** Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be unitary operators and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers. Then

$$\left\| \sum_{i=1}^{n} \alpha_i \, u_i \otimes v_i \right\|_{\min} \leq \left\| \sum_{i=1}^{n} \alpha_i \, u_i \otimes \overline{u_i} \right\|_{\min}. \tag{II.0.13}$$

In section II.5 we give estimates for the norm of subfamilies of unitaries for which equality holds in (II.0.12).
Finally, in section II.6 we give an estimate for the norm of free operators with operator coefficients, which involves a formula similar to the formula for the scalar case obtained by Akemann and Ostrand.
Our general reference for undefined terminology of C^*-algebras is [**58**].

II.1. Examples

Conjecture II.0.23 is motivated by theorem II.3.7 and conjecture II.0.24 by the following examples.

Example II.1.1. - Unitary matrices.

This is trivial because for $u_i \in M_n(\mathbf{C})$ (in fact, in any injective von Neumann algebra) and $\alpha_i \geq 0$ we have

$$\left\| \sum_{i=1}^{n} \alpha_i \, u_i \otimes \overline{u_i} \right\|_{\min} = \sum_{i=1}^{n} \alpha_i .$$

Example II.1.2. - Group representations. Let G and G' be discrete groups and denote their left regular representations by λ_G (resp. $\lambda_{G'}$). Then for any elements $t_i \in G$ and $t_i' \in G'$ and $\alpha_i \geq 0$ we have

$$\left\| \sum_{i=1}^{n} \alpha_i \, \lambda_G(t_i) \otimes \lambda_{G'}(t_i') \right\|_{\min} \leq \left\| \sum_{i=1}^{n} \alpha_i \, \lambda_G(t_i) \otimes \overline{\lambda_G(t_i)} \right\|_{\min} . \tag{II.1.1}$$

Note that the representation $\lambda_G \otimes \lambda_{G'}$ can be considered as the left regular representation $\lambda_{G\times G'}$ of the direct product of the two groups and that both G and G' are quotients of the direct product $G \times G'$. The following results prove the claim. The following proposition is an extension of [**30**, Lemma 3.1].

Proposition II.1.3. *Let G and H be discrete groups and $q : G \to H$ a group homomorphism. Then for every finite subset $\{t_1, t_2, \ldots , t_m\}$ of G and corresponding nonnegative real numbers $\alpha_1, \alpha_2, \ldots , \alpha_m$ the inequality*

$$\left\| \sum_{i=1}^{m} \alpha_i \, \lambda_H(q(t_i)) \right\|_{C^*_\lambda(H)} \geq \left\| \sum_{i=1}^{m} \alpha_i \, \lambda_G(t_i) \right\|_{C^*_\lambda(G)} \tag{II.1.2}$$

holds.

Proof. We will use the following well known fact about the noncommutative L^p-norms associated to τ_G:

$$\|X\|_{C^*_\lambda(G)} = \sup_{1\leq p<\infty} (\tau_G((X^*X)^p))^{1/2p} \tag{II.1.3}$$

It suffices to consider the integer values of p. Denote by $W_p^{alt}(A)$ the set of all alternating words of length $2p$ in the letters $A = \{t_1, t_2, \ldots , t_m\}$:

$$W_p^{alt}(A) = \left\{ t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_p}^{-1} t_{j_p} : i_1, j_1, i_2, j_2, \ldots , i_p, j_p \in \{1, 2, \ldots , m\} \right\} .$$

For $v = t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_p}^{-1} t_{j_p} \in W_p^{alt}(A)$ we use the following abbreviation:

$$\alpha_v = \alpha_{i_1}\alpha_{j_1}\alpha_{i_2}\alpha_{j_2} \cdots \alpha_{i_p}\alpha_{j_p} . \tag{II.1.4}$$

Then we can write (note that we identify words in $W_p^{alt}(A)$ with the corresponding elements in the group G):

$$\begin{aligned}
\left\| \sum_{i=1}^{m} \alpha_i \, \lambda_H(q(t_i)) \right\|_{C^*_\lambda(H)} &= \sup_p \left(\tau_H \left(\sum_{i,j=1}^{m} \alpha_i\alpha_j \, \lambda_H(q(t_i^{-1}t_j)) \right)^p \right)^{1/2p} = \\
&= \sup_p \left(\sum_{v\in W_p^{alt}(A)} \alpha_v \, \tau_H(\lambda_H(q(v))) \right)^{1/2p} = \sup_p \left(\sum_{\substack{v\in W_p^{alt}(A)\\ q(v)=e}} \alpha_v \right)^{1/2p} \\
&\geq \sup_p \left(\sum_{\substack{v\in W_p^{alt}(A)\\ v=e}} \alpha_v \right)^{1/2p} = \left\| \sum_{i=1}^{m} \alpha_i \, \lambda_G(t_i) \right\|_{C^*_\lambda(G)}
\end{aligned}$$

□

In [**45**] the following more general statement is proved:

Proposition II.1.4. *Let H be a Hilbert space and $U_1, U_2, \dots, U_n$ a finite sequence of unitary operators acting on H. Consider the representation π of $\mathbf{F}_n$ which is determined by the condition $\pi(g_i) = U_i$ for all $i \in \{1, 2, \dots, n\}$. Then for every set of words $\{w_1, w_2, \dots, w_m\} \in \mathbf{F}_n$ and every sequence $\alpha_1, \alpha_2, \dots, \alpha_m$ of positive real numbers we have*

$$\Big\|\sum_{i=1}^{m} \alpha_i\, \pi(w_i) \otimes \overline{\pi(w_i)}\Big\|_{\min} \geq \Big\|\sum_{i=1}^{m} \alpha_i\, \lambda(w_i)\Big\|_{C^*_\lambda(\mathbf{F}_n)}$$

For the next lemma we refer to [**15**, Lemma 2.1].

Lemma II.1.5 (Fell's principle, [**23**])**.** *Let $(\pi, \mathcal{H})$ be a unitary representation of the discrete group G. Then there is a unitary operator $W : \ell_2(G) \otimes_2 \mathcal{H} \to \ell_2(G) \otimes_2 \mathcal{H}$ such that for all $t \in G$*

$$W^{-1}\, \lambda_G(t) \otimes \pi(t)\, W = \lambda_G(t) \otimes I_{\mathcal{H}}.$$

In particular we have the identity

$$\Big\|\sum \alpha_i\, \lambda_G(t_i) \otimes \overline{\lambda_G(t_i)}\Big\| = \Big\|\sum \alpha_i\, \lambda_G(t_i)\Big\|$$

which finishes the poof of (II.1.1).

Example II.1.6. Unitary representations of the free group. The inequality (II.0.13) holds for $v_i = \lambda(g_i)$ and arbitrary u_i.

Indeed, denoting $g_1, g_2, \dots, g_n$ the generators of the free group $\mathbf{F}_n$ and λ its left regular representation, it follows from Fell's principle and Pisier's inequality [**45**, Theorem 1], that

$$\Big\|\sum \alpha_i\, \lambda(g_i) \otimes u_i\Big\|_{\min} = \Big\|\sum \alpha_i\, \lambda(g_i) \otimes \overline{\lambda(g_i)}\Big\| \leq \Big\|\sum \alpha_i\, u_i \otimes \overline{u_i}\Big\|.$$

The latter is a consequence of the following apparently more general inequality.

Theorem II.1.7 ([**55**])**.** *Let $u_1, u_2, \dots, u_n$ be unitaries in a C^*-algebra A and suppose that there is a state φ on A such that for any choice of indices $i_1, i_2, \dots, i_p$ and $j_1, j_2, \dots, j_p$ we have*

$$\varphi(u_{i_1}^{-1} u_{j_1} u_{i_2}^{-1} u_{j_2} \dots u_{i_p}^{-1} u_{j_p}) \geq 0 \tag{II.1.5}$$

Then for any sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ of real positive numbers we have

$$\Big\|\sum_{i=1}^{n} \alpha_i\, u_i\Big\| \geq \Big\|\sum_{i=1}^{n} \alpha_i\, \lambda(g_i)\Big\|.$$

PROOF. Using (II.1.3) it suffices to estimate all the moments of the square of $X = \sum \alpha_i \lambda(g_i)$ from above.

$$\begin{aligned}
\tau\big((X^*X)^p\big) &= \sum_{\substack{i_1,i_2,\dots,i_p\\ j_1,j_2,\dots,j_p\\ g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_p}^{-1}g_{j_p}=e}} \alpha_{i_1}\alpha_{j_1}\alpha_{i_2}\alpha_{j_2}\cdots\alpha_{i_p}\alpha_{j_p}\\
&\le \sum_{\substack{i_1,i_2,\dots i_p\\ j_1,j_2,\dots,j_p}} \alpha_{i_1}\alpha_{j_1}\alpha_{i_2}\alpha_{j_2}\cdots\alpha_{i_p}\alpha_{j_p}\,\varphi(u_{i_1}^*u_{j_1}u_{i_2}^*u_{j_2}\cdots u_{i_p}^*u_{j_p})\\
&= \varphi\left(\left(\Big(\sum \alpha_i\, u_i\Big)^*\Big(\sum \alpha_j\, u_j\Big)\right)^p\right)\\
&\le \left\|\sum_{i=1}^n \alpha_i\, u_i\right\|^{2p}.
\end{aligned}$$

□

Actually theorem II.2.3 together with Fell's principle implies the following more general result.

Example II.1.8. Let G be a group and $u_1, u_2, \dots, u_n$ be some unitaries on a Hilbert space $\mathcal{H}$. Then for all finite sequences $t_1, t_2, \dots, t_n$ of elements in G and corresponding nonnegative coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ we have

$$\left\|\sum_{i=1}^n \alpha_i\,\lambda_G(t_i)\otimes u_i\right\|_{\min} \le \left\|\sum_{i=1}^n \alpha_i\,\lambda_G(t_i)\otimes \overline{\lambda_G(t_i)}\right\|_{\min}.$$

II.2. Families with nonnegative alternating mixed moments

In this section we will exhibit the common principle behind all the examples in the introduction. It can be considered as a variant of Perron-Frobenius theory with some non-commutative aspects. See also [**53**].

Definition II.2.1. Let A be a C^*-algebra and φ a state on A. A family $(a_i)_{i\in I}$ of elements of A have *nonnegative alternating mixed moments* with respect to φ, if for any positive integer m and any choice of indices $i_1, j_1, i_2, j_2, \dots, i_m, j_m \in I$ we have

$$\varphi(a_{i_1}^*a_{j_1}a_{i_2}^*a_{j_2}\cdots a_{i_m}^*a_{j_m}) \ge 0.$$

If all mixed moments with respect to φ are nonnegative, we call φ a *jointly Perronian state* for the family $(a_i)_{i\in I}$, cf. [**37**].
A family of states S on A is said to be *faithful*, if for any nonzero $x \in A$ there is a state $\varphi \in S$ such that $\varphi(x^*x) > 0$. In particular, denoting by π_φ the GNS representation associated to φ, this implies that the representation $\pi_S = \bigoplus_{\varphi\in S}$ is faithful.

Remark II.2.2. Equivalently, we could consider families (a_i) in A together with a C^*-algebra B and a faithful completely positive map $\Phi : A \to B$ such that

$$\Phi(a_{i_1}^*a_{j_1}a_{i_2}^*a_{j_2}\cdots a_{i_m}^*a_{j_m}) \in B_+$$

for any choice of indices $i_1, j_1, i_2, j_2, \dots, i_m, j_m \in I$. Indeed, given such a map Φ, the family of states $\{\psi\circ\Phi :\ \psi \in S(B)\}$ is faithful for A; conversely, given a faithful set of states S, the map

$$\begin{aligned}\Phi : A &\to \ell_\infty(S)\\ a &\mapsto \big(\varphi(a)\big)_{\varphi\in S}\end{aligned}$$

is faithful and completely positive [**58**, Corollary 3.5].

For instance, the family $\big(\omega_{\xi\otimes\bar{\xi}}\big)_{\xi\in B_{\mathcal{H}}}$ is faithful for the algebra generated by the elements $a\otimes\bar{a}$ in $A\otimes\bar{A}$.

Theorem II.2.3. *Let A be a unital C^*-algebra with a faithful set of states S. Suppose that $a_1, a_2, \dots, a_n \in A$ have nonnegative alternating mixed moments with respect to all $\varphi\in S$. Then for any Hilbert space $\mathcal{H}$ and any sequence of operators $b_1, b_2, \dots, b_n$ in the unit ball of $B(\mathcal{H})$ we have*

$$\Big\|\sum_{i=1}^n a_i\otimes b_i\Big\|_{\min} \le \Big\|\sum_{i=1}^n a_i\Big\|.$$

Equivalently, denoting by $U_1, U_2, \dots, U_n$ the images under the universal representation of the free group generators $g_1, g_2, \dots, g_n$, we can restate the claim as

$$\Big\|\sum_{i=1}^n a_i\otimes U_i\Big\|_{\min} = \sup_{u_1,u_2,\dots,u_n \text{unitary}}\Big\|\sum_{i=1}^n a_i\otimes u_i\Big\|_{\min} = \Big\|\sum_{i=1}^n a_i\Big\|.$$

This implies in particular the following intermediate inequality between (II.1.5) and [**45**, Theorem 1]:

Corollary II.2.4. *Let A and S be as in Theorem II.2.3 and assume that the alternating mixed moments of the unitaries $u_1, u_2, \dots, u_n \in A$ with respect to S are nonnegative. Then we have for any sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ of nonnegative real numbers*

$$\Big\|\sum_{i=1}^n \alpha_i\,\lambda(g_i)\Big\| \le \Big\|\sum_{i=1}^n \alpha_i\, u_i\otimes\overline{u_i}\Big\|_{\min} \le \Big\|\sum_{i=1}^n \alpha_i\, u_i\Big\|.$$

In the same spirit we will prove the following extension of the remark after Theorem 1 in [**45**].

Theorem II.2.5. *Let A and B be unital C^*-algebras and S a faithful set of states for A. Let further $a_1, a_2, \dots, a_n$ be some elements of A with nonnegative alternating mixed moments with respect to S and $u_1, u_2, \dots, u_n$ be some unitaries in B whose alternating mixed moments with respect to some state ψ on B are nonnegative. Then*

$$\Big\|\sum_{i=1}^n a_i\otimes u_i\Big\| \ge \Big\|\sum_{i=1}^n a_i\otimes\lambda(g_i)\Big\|.$$

For the proofs of Theorem II.2.3 and II.2.5 we use the following method to calculate norms.

Proposition II.2.6. *Let A and B be unital C^*-algebras. Let $\Phi : A\to B$ be a faithful unital completely positive contraction. Then we can calculate norms in A as follows: For any $x\in A$ we have*

$$\|x\| = \sup_{1\le p<\infty}\big\|\Phi\big((x^*x)^p\big)\big\|^{\frac{1}{2p}} = \lim_{p\to\infty}\big\|\Phi\big((x^*x)^p\big)\big\|^{\frac{1}{2p}}.$$

As an immediate consequence we get

Corollary II.2.7. *Let A, A', B, B' be C^*-algebras and $\Phi : A \to A'$, $\Psi : B \to B'$ faithful completely positive contractions. Suppose that for $a \in A$ and $b \in B$ we have for all nonnegative integers p*

$$\left\|\Phi\big((a^*a)^p\big)\right\| \geq \left\|\Psi\big((b^*b)^p\big)\right\|$$

then $\|a\| \geq \|b\|$.

Proof of Proposition II.2.6. It suffices to show the claim for $x \in A$ positive self-adjoint. The spectrum $\sigma(x)$ of such an operator is contained in $\mathbf{R}^+$ and we define probability measures indexed by the unit ball of $\mathcal{H}$ and supported on $\sigma(x)$ as follows. The C^*-algebra generated by x and I in A is commutative and isomorphic to $C(\sigma(x))$. We can assume that B is acting on some Hilbert space $\mathcal{H}$. The states on A induced by the unit vectors in $\mathcal{H}$:

$$\omega_\xi(a) = \langle \Phi(a)\xi, \xi\rangle,$$

when restricted to $C^*(x, I)$, can be represented as probability measures on $C(\sigma(x))$, i.e. there is a measure μ_ξ whose support is contained in $\sigma(x)$ such that

$$\int_{\sigma(x)} f \, d\mu_\xi = \langle \Phi(f(x))\xi, \xi\rangle$$

for any function $f \in C(\sigma(x))$ and we claim that

$$\overline{\bigcup_{\xi\in\mathcal{H}} \operatorname{supp} \mu_\xi} = \sigma(x). \tag{II.2.1}$$

Suppose $t \in \sigma(x) \setminus \overline{\bigcup \operatorname{supp} \mu_\xi}$. Since the left hand side of (II.2.1) is closed, there is some $\varepsilon > 0$ such that $S_\varepsilon = [t-\varepsilon, t+\varepsilon] \cap \sigma(x)$ is still disjoint from it. Let f be a nonzero continuous positive function supported on S_ε. By functional calculus $f(x)$ is a nonzero positive element of A and by definition of μ_ξ we have

$$\langle \Phi(f(x))\xi, \xi\rangle = 0 \qquad \forall \xi \in \mathcal{H}$$

in contradiction to the faithfulness of Φ.
Fix $\varepsilon > 0$. We show that $\sup_p \|\Phi(x^p)\|^{1/p} > (1-\varepsilon)\,(\|x\| - \varepsilon)$. To this end choose some $\xi \in \mathcal{H}$ such that $\operatorname{supp}\mu_\xi \cap [\|x\| - \varepsilon, \|x\|]$ is nonempty. Such a ξ exists by (II.2.1). Next choose some integer p such that $\mu_\xi([\|x\| - \varepsilon, \|x\|])^{\frac{1}{p}} \geq 1 - \varepsilon$. It follows that

$$\begin{aligned} \|\Phi(x^p)\|^{\frac{1}{p}} \geq \langle \Phi(x^p)\xi, \xi\rangle^{\frac{1}{p}} = \left(\int_{\sigma(x)} t^p \, d\mu_\xi(t)\right)^{\frac{1}{p}} \geq \\ \geq \big(\|x\| - \varepsilon\big)\mu_\xi\big([\|x\| - \varepsilon, \|x\|]\big)^{\frac{1}{p}} \geq \big(\|x\| - \varepsilon\big)\big(1 - \varepsilon\big). \end{aligned}$$

□

Lemma II.2.8. *Let A, B, C be C^*-algebras and let $\Phi : A \to B$ be a faithful completely positive map. Then $\Phi \otimes I_C : A \otimes_{\min} C \to B \otimes_{\min} C$ is faithful and completely positive. More generally, if in addition $\Psi : C \to D$ is a faithful completely positive map into a C^*-algebra D, then the tensor product $\Phi \otimes \Psi : A \otimes_{\min} C \to B \otimes_{\min} D$ is still faithful.*

Proof. We only have to show faithfulness. Let $x \in A \otimes_{\min} C$ be nonzero. We have to show that $\Phi \otimes I_C(x^*x) \neq 0$. By [**24**, Proposition 10] there are states $\varphi \in S(A)$ and $\psi \in S(C)$ such that $\varphi \otimes \psi(x^*x) \neq 0$. In particular, $I_A \otimes \psi(x^*x) \neq 0$ and since the latter is still a positive operator, it follows that

$$(I_B \otimes \psi) \circ (\Phi \otimes I_C)(x^*x) = \Phi \circ (I_A \otimes \psi)(x^*x) \neq 0$$

and hence $\Phi \otimes I_C(x^*x) \neq 0$. The other part of the lemma is obvious by composing the faithful maps $\Phi \otimes I_C$ and $I_B \otimes \Psi$. □

PROOF OF THEOREM II.2.3: The map

$$\begin{aligned}\Phi_S : A &\to \ell_\infty(S)\\ x &\mapsto (\varphi(x))_{\varphi\in S}\end{aligned}$$

is completely positive and faithful (see remark II.2.2 and by lemma II.2.8 the same is true for $\Phi_S \otimes I_B$, where we write B for $B(\mathcal{H})$, so that we can apply Proposition II.2.6 to compute the norms of the operators $x = \sum_i a_i \otimes b_i$ and $y = \sum_i a_i$. First we observe that

$$\big\|(\Phi_S \otimes I_B)\big((x^*x)^p\big)\big\| = \sup_{\varphi\in S} \big\|(\varphi \otimes I_B)\big((x^*x)^p\big)\big\|$$

and the latter can be estimated as follows

$$\begin{aligned}\big\|\varphi \otimes I_B\big((x^*x)^p\big)\big\| &= \Bigg\| \sum_{\substack{i_1,i_2,\dots,i_p\\ j_1,j_2,\dots,j_p}} \varphi(a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p})\, b_{i_1}^* b_{j_1} b_{i_2}^* b_{j_2} \cdots b_{i_p}^* b_{j_p} \Bigg\|\\ &\le \sum_{\substack{i_1,i_2,\dots,i_p\\ j_1,j_2,\dots,j_p}} \varphi(a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p})\\ &= \varphi\left(\left(\Big(\sum_i a_i\Big)^* \Big(\sum_j a_j\Big)\right)^p\right)\\ &\le \Big\|\sum_i a_i\Big\|^{2p}\end{aligned} \tag{II.2.2}$$

where we used the assumption that the alternating mixed moments of the a_i's are non-negative and that the norm of the b_i's is bounded by 1. Since this holds for all positive integers p and for all $\varphi \in S$, we are done. □

PROOF OF THEOREM II.2.5. Let Φ_S be as in the proof of theorem II.2.3 $x = \sum a_i \otimes u_i$ and $y = \sum a_i \otimes \lambda(g_i)$. Then for any positive integer p we have

$$\begin{aligned}\big\|\Phi_S \otimes I_B\big((x^*x)^p\big)\big\| &\ge \big\|\Phi_S \otimes \psi\big((x^*x)^p\big)\big\| = \sup_{\varphi\in S} \big|\varphi \otimes \psi\big((x^*x)^p\big)\big| =\\ &= \sup_{\varphi\in S} \sum_{\substack{i_1,i_2,\dots,i_p\\ j_1,j_2,\dots,j_p}} \varphi(a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p})\, \psi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_p}^* u_{j_p})\\ &\ge \sup_{\varphi\in S} \sum_{\substack{i_1,i_2,\dots,i_p\\ j_1,j_2,\dots,j_p}} \varphi(a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p})\, \tau\big(\lambda(g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_p}^{-1} g_{j_p})\big)\\ &= \big\|\Phi_S \otimes \tau\big((y^*y)^p\big)\big\|\end{aligned}$$

and we can apply corollary II.2.7. □

Actually the second part of theorem II.2.3 can be derived from the following more general result.

Theorem II.2.9. *Let A be a unital C^*-algebra with a faithful set of states S. Suppose that the operators $a_1, a_2, \dots, a_n \in A$ have nonnegative alternating mixed moments with respect to all $\varphi \in S$. Suppose further that $u_1, u_2, \dots, u_n$ are unitaries such that $\|\sum_{i=1}^n u_i\| = n$. Then*

$$\Big\|\sum_{i=1}^n a_i \otimes u_i\Big\|_{\min} = \Big\|\sum_{i=1}^n a_i\Big\|.$$

We need the following folklore fact.

Lemma II.2.10. *If $u_1, u_2, \dots, u_n$ are unitaries in a C^*-algebra A for which there exist positive real numbers $\beta_1, \beta_2, \dots, \beta_n$ satisfying*

$$\Big\| \sum_{i=1}^{n} \beta_i \, u_i \Big\| = \sum_{i=1}^{n} \beta_i$$

then there exists a state φ of A such that for every $m \in \mathbf{N}$

$$\varphi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) = 1 \qquad \forall i_k, j_k = 1, 2, \dots, n.$$

Proof. By a compactness argument, there is a state on A such that $\varphi(\sum \beta_i \beta_j \, u_i^* u_j) = \| \sum \beta_i \beta_j \, u_i^* u_j \| = \sum \beta_i \beta_j$ It follows that $\varphi(u_i^* u_j) = 1$ for all $i, j = 1, 2, \dots, n$ and by induction on m we have $\varphi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) = 1$ for any choice of indices i_k, j_k. Indeed,

$$\big| \varphi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m} - u_i^* u_j u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) \big| \leq$$
$$\leq \varphi((I - u_i^* u_j)^* (I - u_i^* u_j))^{1/2} = 0.$$

Now in the case where $u_i = \lambda_G(t_i)$ this implies that the trivial representation of the group generated by $t_i^{-1} t_j$ extends to a representation of the reduced group C^*-algebra, which implies that it is amenable. □

Proof of theorem II.2.9. Replacing b_i by u_i, we can use the same notation as in the proof of theorem II.2.3. Then we apply the preceding lemma and instead of the inequality in (II.2.2) there is equality:

$$\begin{aligned} \big\| \varphi \otimes I_B\big((x^* x)^p\big) \big\| &= \Big\| \sum_{\substack{i_1, i_2, \dots, i_p \\ j_1, j_2, \dots, j_p}} \varphi(a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p}) \, u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_p}^* u_{j_p} \Big\| \\ &= \sum_{\substack{i_1, i_2, \dots, i_p \\ j_1, j_2, \dots, j_p}} \varphi(a_{i_1}^* a_{j_1} a_{i_2}^* a_{j_2} \cdots a_{i_p}^* a_{j_p}) \\ &= \varphi\left(\left(\Big(\sum_i a_i\Big)^* \Big(\sum_j a_j\Big) \right)^p \right) \end{aligned}$$

□

II.3. A Characterization of the Leinert Property

Notation II.3.1. Throughout in this section G will denote a discrete group with unit element e and $C^*_\lambda(G)$ the sub-C^*-algebra of $B(\ell_2(G))$ generated by its left regular representation λ_G. This algebra is equipped with the trace state $\tau_G(X) = \langle X \delta_e, \delta_e \rangle$. A subset $\{t_1, t_2, \dots, t_n\}$ of G is called *free* if it generates a copy of $\mathbf{F}_n$, the free group on n generators. We shall denote the canonical generators of the free group by $\{g_1, g_2, \dots, g_n\}$. When considering the free group we shall omit the subscript in $\lambda_{\mathbf{F}_n}$. We shall almost exclusively deal with finitely generated groups and repeatedly use the fact that given a generating set $\{h_1, h_2, \dots, h_n\}$ for the group G there is a (unique) quotient mapping $q : \mathbf{F}_n \to G$ with $q(g_i) = h_i \quad \forall i \in \{1, 2, \dots, n\}$.

The following proposition collects some results from [**30**, Lemma 3.1 and Theorem 3]:

Proposition II.3.2. *Let $\{h_1, h_2, \dots, h_n\}$ be a generating set of the group G and denote by $\alpha_1, \alpha_2, \dots, \alpha_n$ positive real numbers. Then*

$$\left\|\sum_{i=1}^{n} \alpha_i \left(\lambda_G(h_i) + \lambda_G(h_i)^*\right)\right\|_{C^*_\lambda(G)} \geq \left\|\sum_{i=1}^{n} \alpha_i \left(\lambda(g_i) + \lambda(g_i)^*\right)\right\|_{C^*_\lambda(\mathbf{F}_n)}$$

Moreover in the case of equal coefficients,

$$\left\|\sum_{i=1}^{n} (\lambda_G(h_i) + \lambda_G(h_i)^*)\right\|_{C^*_\lambda(G)} = \left\|\sum_{i=1}^{n} (\lambda(g_i) + \lambda(g_i)^*)\right\|_{C^*_\lambda(\mathbf{F}_n)} = 2\sqrt{2n-1}$$

if and only if $\{h_1, h_2, \dots, h_n\}$ is a free set.

This result has a graph-theoretical interpretation, where the operator in consideration corresponds to the combinatorial Laplacian on the Cayley graph of the group G, see [**39**] in particular for a generalization of the second statement. However it remained unclear whether the free group can be characterized by the norms of non-selfadjoint operators, and this question will be the subject of this section.
The set $\{g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}\}$ is only a special case of a *Leinert set*, a concept which appeared first in [**31**].

Definition II.3.3 ([**1**, Definition III B and Theorem III D]).
A subset $A = \{t_1, t_2, \dots, t_n\}$ of a discrete group G with unit element e is called *Leinert set* if it satisfies one of the following equivalent conditions:

- Every sequence $t_{i_1}, t_{j_1}, t_{i_2}, t_{j_2}, \dots, t_{i_m}, t_{j_m}$ with $i_k, j_k \in \{1, 2, \dots, n\}$ such that $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$ satisfies
$$t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_m}^{-1} t_{j_m} \neq e$$
- The set A can be written as $y(B \cup \{e\})$, where B is a free subset of G and $y \in G$.

The next proposition extends Kesten's formula for the norm (See also [**7**] for a weaker version). Simplified proofs can be found in [**44**] and [**60**]. For recent examples of Leinert sets in one-relator groups see [**10**].

Proposition II.3.4 ([**1**, Theorem IV J]). *Let $n \geq 2$ and $\{t_1, \dots, t_n\}$ be a Leinert set in a discrete group G. Then*

$$\left\|\sum_{i=1}^{n} \lambda_G(t_i)\right\|_{C^*_\lambda(G)} = 2\sqrt{n-1}$$

The following lemma and its corollary will play a key rôle in the generalization of Kesten's result.

Lemma II.3.5 ([**16**, Lemma 8]). *Let μ be a probability measure on $\mathbf{R}$ with compact support and suppose*

$$-m = \min \operatorname{supp} \mu < \max \operatorname{supp} \mu = M$$
$$\mu_n = \int_{\mathbf{R}} t^n \, d\mu(t) \geq 0 \ \text{for all } n \in \mathbf{N}$$

Then

$$\limsup_{n \to \infty} (\mu_n)^{1/n} = M = \max(m, M)$$

Corollary II.3.6 ([**30**, Lemma 3.2]). *For every subset $\{t_1, t_2, \dots, t_n\}$ of G and every sequence of positive real numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ we have*

$$\left\|\alpha_0 I + \sum_{i=1}^{n} \alpha_i \left(\lambda_G(t_i) + \lambda_G(t_i)^*\right)\right\| \quad = \quad \alpha_0 + \left\|\sum_{i=1}^{n} \alpha_i \left(\lambda_G(t_i) + \lambda_G(t_i)^*\right)\right\|$$

and a similar result holds for general unitaries:

$$\left\|\alpha_0 I + \sum_{i=1}^{n} \alpha_i \left(U_i \otimes \overline{U_i} + U_i^* \otimes \overline{U_i^*}\right)\right\| \quad = \quad \alpha_0 + \left\|\sum_{i=1}^{n} \alpha_i \left(U_i \otimes \overline{U_i} + U_i^* \otimes \overline{U_i^*}\right)\right\|$$

Proof. This follows from lemma II.3.5 applied to the probability measure μ on the spectrum of the self-adjoint operator

$$T = \sum_{i=1}^{n} \alpha_i \left(\lambda(t_i) + \lambda(t_i)^*\right)$$

which is determined by the moments

$$\mu_n = \tau_G(T^n).$$

For the case of general unitaries $\hat{U}_i = U_i \otimes \overline{U_i}$ we refer to [**50**, Example 5.6] where the following formula is proved: For any finite sequence of operators $x_1, x_2, \dots, x_n$ on some Hilbert space H we have

$$\left\|\sum x_i \otimes \overline{x_i}\right\|_{\min} = \sup_{y,z \in (S_2^+)_1} \sum \operatorname{tr}(x_i y x_i^* z) \tag{II.3.1}$$

where $(S_2^+)_1$ is the intersection of the unit ball of the Hilbert-Schmidt class operators on H with the cone of positive operators. In our case the operator is self-adjoint and we can restrict (II.3.1) to the symmetric part:

$$\begin{aligned}\left\|\alpha_0\, I + \sum_{i=1}^{n} \alpha_i \left(\hat{U}_i + \hat{U}_i^*\right)\right\|_{\min} &= \sup_{y \in (S_2^+)_1} \operatorname{tr}\left(\alpha_0\, y^2 + 2 \sum_{i=1}^{n} \hat{U}_i y \hat{U}_i^* y\right) \\ &= \alpha_0 + \left\|\sum_{i=1}^{n} \alpha_i \left(\hat{U}_i + \hat{U}_i^*\right)\right\|_{\min}\end{aligned}$$

□

Now we are ready to prove the main result.

Theorem II.3.7. *Let G be a discrete group, $n \geq 3$ an integer and let $t_1, t_2, \dots, t_n$ be some elements in G. Then for any sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ of strictly positive numbers we have*

$$\left\|\sum_{i=1}^{n} \alpha_i\, \lambda_G(t_i)\right\|_{C_\lambda^*(G)} \quad = \quad \left\|\sum_{i=1}^{n} \alpha_i\, \lambda(g_i)\right\|_{C_\lambda^*(\mathbf{F}_n)}$$

if and only if $\{t_1, t_2, \dots, t_n\}$ is a Leinert set. In particular, this is equivalent to the identity

$$\left\|\sum_{i=1}^{n} \lambda_G(t_i)\right\|_{C_\lambda^*(G)} \quad = \quad 2\sqrt{n-1}.$$

PROOF. We only have to show the "only if" part and we shall use the method introduced in [**30**, Theorem 3] (see also [**7**], where similar methods are used. We are grateful to A. Valette for bringing this article to our attention). Let us compare the norms of the operators

$$\tilde{T} = \sum_{i=1}^{n} \alpha_i \, \lambda_G(t_i) \qquad \text{and} \qquad T = \sum_{i=1}^{n} \alpha_i \, \lambda(g_i).$$

We can assume that G is generated by the subset $\{t_1, t_2, \ldots, t_n\}$. Suppose that the set $\{t_1, t_2, \ldots, t_n\}$ does not have the Leinert property. This means that there is an integer m and a sequence of indices $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \cdots \neq i_m \neq j_m$, with $i_k, j_k \in \{1, 2, \ldots, n\}$ and such that

$$t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_m}^{-1} t_{j_m} = e.$$

Let $g_1, g_2, \ldots, g_n$ be the generators of the free group $\mathbf{F}_n$ and let $q : \mathbf{F}_n \to G$ be the quotient mapping associated to the generating set $\{t_1, t_2, \ldots, t_n\}$. Taking the same indices as above, set

$$w := g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_m}^{-1} g_{j_m}.$$

Since $\{g_1, g_2, \ldots, g_n\}$ *is* a Leinert set, the word w does not reduce to the identity in $\mathbf{F}_n$ but it is in the kernel of q. We can assume without loss of generality that w is cyclically reduced, i.e. $j_m \neq i_1$. For if this is not the case, we can consider the word

$$g_{i_m}^{-1} g_{i_1} w \, g_{i_1}^{-1} g_{i_m} = g_{i_m}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_{m-1}}^{-1} g_{j_{m-1}}$$

which is cyclically reduced and still in the kernel of q. Note that it cannot be reduced to the empty word in this way, since the conjugacy class of the latter in $\mathbf{F}_n$ is trivial.
So we can assume for the sake of clarity that $i_1 = 1$ and $j_m = 2$. Now consider the elements

$$w' = g_1^{-1} g_3 \, w \, g_3^{-1} g_1$$
$$w'' = g_2^{-1} g_3 \, w \, g_3^{-1} g_2$$

which are both in the kernel of q and have infinite order, because w has, being cyclically reduced. Moreover, they form a free set, since there are no possible cancellations in non-trivial reduced words built out of them. Thus we can apply lemma II.2.10 to the operator

$$T_1 = \alpha_{w'} \left(\lambda(w') + \lambda(w')^* \right) + \alpha_{w''} \left(\lambda(w'') + \lambda(w'') \right)^*$$

and we get

$$\|T_1\| < 2(\alpha_{w'} + \alpha_{w''}) \tag{II.3.2}$$

because the group generated by $\{w'^{-1} w'', w'^2, w''^2\}$ is not amenable.
To compare the norms of T and $\tilde{T}$ we use the identity

$$\|X\|^2 = \|X^* X\| = \|(X^* X)^p\|^{1/p},$$

which is a consequence of Gel'fand's theorem. Thus it suffices to show that for some p

$$\|(T^* T)^p\| < \|(\tilde{T}^* \tilde{T})^p\|$$

We consider $p = m + 2$ so that $2p = |w'| = |w''|$. Denote by $W_p^{alt}(n)$ the set of all alternating words of length $2p$ in n generators:

$$W_p^{alt}(n) = \{ g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_p}^{-1} g_{j_p} : i_1, j_1, i_2, j_2, \ldots, i_p, j_p \in \{1, 2, \ldots, n\} \}$$

Using the notation of (II.1.4) we can write

$$\begin{aligned}(T^*T)^{m+2} &= \sum_{v\in W^{alt}_{m+2}(n)} \alpha_v \lambda(v)\\ &= T_0 + T_1\end{aligned}$$

where

$$T_0 = \sum_{v\in W^{alt}_{m+2}(n)\setminus\{w', w'^{-1}, w'', w''^{-1}\}} \alpha_v \lambda(v)$$

and T_1 is the same as above. Obviously, setting

$$\tilde{T}_0 = \sum_{v\in W^{alt}_{m+2}(n)\setminus\{w', w'^{-1}, w'', w''^{-1}\}} \alpha_v \lambda_G(q(v))$$

we have

$$(\tilde{T}^*\tilde{T})^{m+2} = \tilde{T}_0 + 2(\alpha_{w'} + \alpha_{w''})I$$

Observe that T_0 (and hence $\tilde{T}_0$) is selfadjoint with positive coefficients so that we can apply (II.1.2) and Corollary II.3.6 and we get

$$\begin{aligned}\|(T^*T)^{m+2}\| \le \|T_0\| + \|T_1\| < \|\tilde{T}_0\| + 2(\alpha_{w'} + \alpha_{w''}) = \\ = \|\tilde{T}_0 + 2(\alpha_{w'} + \alpha_{w''})I\| = \|(\tilde{T}^*\tilde{T})^{m+2}\|.\end{aligned}$$

□

Remark II.3.8. Actually in the case of equal coefficients the above proof gives the following quantitative estimate (cf. [**30**, (4.15)] and [**39**, Theorem 3.2]): With the notation as in the proof, let m denote the minimal length of a nontrivial word in the kernel of the quotient mapping q. Then

$$\|\tilde{T}\| \ge \|T\| + \frac{2(n-1) - 2\sqrt{2n-3}}{(2m+4)\, n^{2m+3}}.$$

Indeed, when there are n free generators, we can build a free set $\{w_1, w_2, \dots, w_{n-1}\}$ in $\mathbf{F}_n$ by setting $w_i = g_i^{-1} g_n\, w\, g_n^{-1} g_i$ for $i \in \{1, 2, \dots, n-1\}$. Then

$$\left\|\sum_{i=1}^{n-1} (\lambda(w_i) + \lambda(w_i)^*)\right\| = 2\sqrt{2n-3}$$

and by modifying T_0 and T_1 (resp. $\tilde{T}_0$ and $\tilde{T}_1$ appropriately, we have the inequality

$$\|\tilde{T}\|^{2(m+2)} \ge \|T\|^{2(m+2)} + 2(n-1) - 2\sqrt{2n-3}.$$

Now a simple convexity argument and the fact that $\|\tilde{T}\| \le n$ yields the claim.

The following corollary gives a characterization of the free group in terms of the norms operators in the reduced C^*-algebra. Note that the free group cannot be characterized by the spectrum of the sum of the generators, as shown in [**17**, sect.4].

Corollary II.3.9. *Let G be a discrete group and $S = \{t_0 = e, t_1, \dots, t_n\}$ be a generating subset with $n \ge 2$. Then*

$$2\sqrt{n} \le \left\|\sum_{i=0}^{n} \lambda_G(t_i)\right\|$$

with equality if and only if G is the free group on n generators.

A similar proof yields the following result about unitaries which do not necessarily arise from a regular representation of a discrete group, as for example in [**33**].

Theorem II.3.10. *Let $U_1, \dots, U_n$ be unitary operators acting on some Hilbert space H and suppose*

$$\left\| \sum_{i=1}^{n} U_i \otimes \overline{U_i} \right\|_{\min} = 2\sqrt{n-1}$$

Then for all products of the form $V = U_{i_1}^{-1} U_{j_1} U_{i_2}^{-1} U_{j_2} \cdots U_{i_m}^{-1} U_{j_m}$ with $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$ we have

$$\|V \otimes \overline{V} - I\| \geq 1$$

Proof. To simplify notation we use the unitary representation

$$\hat{\pi} = \pi \otimes \overline{\pi} : \mathbf{F}_n \to B(H \otimes_2 \overline{H})$$

which we already introduced in proposition II.1.4 and which maps the generators g_i to the corresponding unitaries $\hat{U}_i = U_i \otimes \overline{U_i}$. Now let

$$v_0 = g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_m}^{-1} g_{j_m}$$

be a word as in the claim and set $\hat{V} = \hat{\pi}(v_0) = V \otimes \overline{V}$. We can assume v_0 to be cyclically reduced, because $\|\hat{V} - I\| = \|W\hat{V}W^{-1} - I\|$ for any unitary W and in the case v_0 is not cyclically reduced, we can take a cyclically reduced one in its conjugacy class. We can of course assume that $i_1 = 1$ and $j_m = 2$. Now for the cyclically reduced word v_0 the set

$$F_k = \{(g_1^{-1} g_3)^r \, v_0 \, (g_1^{-1} g_3)^{-r} : r = 1, 2, \dots, k\}$$

is a free set and we can do the same estimate as in the above proof. Setting $T = \sum_{i=1}^{n} \lambda(g_i)$ as above and $\hat{T} = \sum_{i=1}^{n} \hat{U}_i$ we can identify F_k with some subset of the set of unreduced words $W_{m+2k}^{alt}(n)$, e.g. the set $\{(g_1^{-1}g_3)^r v_0 (g_1^{-1}g_3)^{-r} (g_1^{-1}g_1)^{k-r} : r = 1, 2, \dots, k\}$, and we get

$$\begin{aligned}
\|(T^*T)^{m+2k}\| &= \left\| \sum_{v \in W_{m+2k}^{alt}(n)} \lambda(v) \right\| \\
&\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n) \setminus (F_k \cup F_k^{-1})} \lambda(v) \right\| + \left\| \sum_{v \in F_k} \lambda(v) + \lambda(v)^* \right\| \\
&\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n) \setminus (F_k \cup F_k^{-1})} \hat{\pi}(v) \right\| + 2\sqrt{2k-1}.
\end{aligned}$$

The last inequality follows from (II.1.2) and proposition II.3.4. We can now apply Corollary II.3.6 to the first term on the right hand side and we obtain

$$\begin{aligned}
\|(T^*T)^{m+2k}\| &\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n) \setminus (F_k \cup F_k^{-1})} \hat{\pi}(v) + \sum_{v \in F_k} 2I \right\| + 2\sqrt{2k-1} - 2k \\
&\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n)} \hat{\pi}(v) \right\| + \left\| \sum_{v \in F_k} 2I - \hat{\pi}(v) - \hat{\pi}(v)^* \right\| + 2\sqrt{2k-1} - 2k \\
&\leq \|(\hat{T}^*\hat{T})^{m+2k}\| + 2k\, \|\hat{\pi}(v_0) - I\| + 2\sqrt{2k-1} - 2k
\end{aligned}$$

This together with the assumption $\|T\| = \|\hat{T}\|$ yields

$$\|\hat{V} - I\| \geq 1 - \frac{\sqrt{2k-1}}{k}$$

and since this holds for all k, the proof is complete. $\square$

Remark II.3.11. There is a connection to the paper of V. Flory, "Estimating norms in C^*-algebras of discrete groups", Math. Ann. **224** (1976) 41–52. Using our results, Theorem 8 in the latter paper can be sharpened for finite sets as follows. Let G be a discrete group and $E \subset G$ a finite subset. Theorem 8 can be sharpened for finite subsets as follows. Define for $K \subset G$ the *Leptin constant* $\omega(K) = \inf\{\frac{\#(KU)}{\#(U)} : \emptyset \neq U \subset G \quad \text{finite}\}$. Then the following are equivalent.

1. E has the Leinert property.
2. For every $y \in E$ the set $\{y^{-1}x : x \in E \text{ and } x \neq y\}$ is a free subset of G.
3. $\omega(E) = \#(E) - 1$.

II.4. A Counterexample.

In this section we exhibit an example showing that there actually can occur strict inequality in Corollary II.2.4 and that Conjectures II.0.23 and II.0.24 are false.

Theorem II.4.1. *For every positive integer r there are unitaries $u_1, u_2, \dots, u_r$ such that*

$$\left\| \sum_{i=1}^{r} \left(u_i \otimes \overline{u_i} + u_i^* \otimes \overline{u_i^*} \right) \right\| = 2\sqrt{2r-1}$$

and

$$\left\| \sum_{i=1}^{r} (u_i + u_i^*) \right\| > 2\sqrt{2r-1}$$

Remark II.4.2. Note that this is a nontrivial example of equality in Haagerup's Cauchy-Schwarz inequality for operators (cf. [**25**, Lemma 2.4]), which reads

$$\left\| \sum_i a_i \otimes \overline{b_i} \right\| \leq \left\| \sum_i a_i \otimes \overline{a_i} \right\|^{1/2} \left\| \sum_i b_i \otimes \overline{b_i} \right\|^{1/2}.$$

Indeed, taking $a_i = u_i$ from theorem II.4.1 and $b_i = \lambda(g_i)$ there is equality.

Proof. We shall consider the unitary representations of the free group on r generators associated to the Haagerup functions $x \mapsto t^{|x|}$ as realized in [**52**]. We need the series of unitary representations π_t presented there which is indexed by $t \in]0,1[$ and which is defined on the canonical basis of $\ell_2(\mathbf{F}_r)$ as follows. Every $x \in \mathbf{F}_r$ different from the neutral element e can be represented as a reduced word in the generators $g_1, g_2, \dots, g_r$ and their inverses. Denote by $\bar{x}$ the word obtained by removing the last letter, so that $|\bar{x}| = |x| - 1$ and define the operator P on $\ell_2(\mathbf{F}_r)$ by $P\delta_x = \delta_{\bar{x}}$ and $P\delta_e = 0$. Then the Pytlik-Szwarc representations act as follows on $\ell_2(\mathbf{F}_r)$. First we consider the unit vector δ_e, where we define

$$\pi_t(x)\delta_e = t^{|x|}\,\delta_e + \sum_{k=0}^{|x|-1} t^k \sqrt{1-t^2}\, P^k \delta_x \tag{II.4.1}$$

This implicitly defines $\pi_t(x)$ on the other basis vectors $\delta_y = \frac{1}{\sqrt{1-t^2}}\big(\pi_t(y)\delta_e - t\,\pi_t(\bar{y})\delta_e\big)$ for $y \neq e$, namely

$$\pi_t(x)\delta_y = \frac{1}{\sqrt{1-t^2}}\big(\pi_t(xy)\delta_e - t\,\pi_t(x\bar{y})\delta_e\big).$$

In particular, δ_e is a cyclic vector and the matrix coefficients

$$\begin{aligned}\langle\pi_t(x)\delta_y,\delta_z\rangle &= \frac{1}{1-t^2}\left\langle\big(\pi(z^{-1})-t\pi(\bar z^{-1})\big)\big(\pi(xy)-t\pi(x\bar y)\big)\delta_e,\delta_e\right\rangle\\ &= \frac{t^{|z^{-1}xy|}-t^{1+|\bar z^{-1}xy|}-t^{1+|z^{-1}x\bar y|}+t^{2+|\bar z^{-1}x\bar y|}}{1-t^2}\end{aligned}$$

are all of the order $\psi_t(x)=t^{|x|}$. We can calculate the ℓ_p-norm of the latter as follows. It is easy to see that the number of words of length n in $\mathbf{F}_r$ is $2r\,(2r-1)^{n-1}$ for $n\ge 1$, so that

$$\|\psi_t\|_p^p=\sum_{x\in\mathbf{F}_r}t^{p|x|}=1+\sum_{n=1}^{\infty}2r\,(2r-1)^{n-1}t^{np}=1+\frac{2r}{2r-1}\sum_{n=1}^{\infty}\left(t^p\,(2r-1)\right)^n, \tag{II.4.2}$$

which is finite if and only if $t<(2r-1)^{-1/p}$. Now the proof of [**13**, Theorem 1] shows that any unitary representation π of a discrete group G whose matrix coefficients with respect to a cyclic subset of vectors are in $\ell_{2+\varepsilon}(G)$ for arbitrary positive ε is weakly contained in the left regular representation λ. This means that for any function f of finite support the inequality

$$\|\pi(f)\|\le\|\lambda(f)\| \tag{II.4.3}$$

holds. Formula (II.4.2) shows that for $t\le\frac{1}{\sqrt{2r-1}}$ the representation π_t is weakly contained in the left regular representation of the $\mathbf{F}_r$. However, this is not true for $t>\frac{1}{\sqrt{2r-1}}$. In [**57**, Theorem 5] it is shown that the spectrum of the operator $\pi_t(\chi_1)=\sum\big(\pi_t(g_i)+\pi_t(g_i^{-1})\big)$ is the set $\big[-2\sqrt{2r-1},2\sqrt{2r-1}\big]\cup\big\{(2r-1)t+\frac{1}{t}\big\}$, which for $t>\frac{1}{\sqrt{2r-1}}$ is different from $\operatorname{spec}\big(\lambda(\chi_1)\big)=\big[-2\sqrt{2r-1},2\sqrt{2r-1}\big]$. For completeness we shall give an elementary proof of the inequality

$$\|\pi_t(\chi_1)\|\ge(2r-1)t+\frac{1}{t}$$

using the method of [**44**]. We shall exhibit a family of vectors $(\xi_{t,\alpha})_{\alpha<\alpha_0}\subseteq\ell_2(\mathbf{F}_r)$ such that

$$\lim_{\alpha\to\alpha_0}\frac{\|\pi_t(\chi_1)\xi_{t,\alpha}\|}{\|\xi_{t,\alpha}\|}=(2r-1)t+\frac{1}{t}$$

for $t>\frac{1}{\sqrt{2r-1}}$. Denote by χ_n the characteristic function of the words of length n on $\mathbf{F}_r$. We define

$$\xi_{t,\alpha}=\sum_{n=0}^{\infty}\alpha^n\pi_t(\chi_n)\delta_e$$

or equivalently, using formula (II.4.1),

$$\begin{aligned}\xi_{t,\alpha}(e) &= \frac{2r}{(2r-1)(1-(2r-1)\alpha t)}\\ \xi_{t,\alpha}(x) &= \frac{\sqrt{1-z^2}}{1-(2r-1)\alpha t}\,\alpha^{|x|}\qquad\text{for}\quad x\ne e.\end{aligned}$$

In particular, setting $\alpha_0=\min\left\{\frac{1}{\sqrt{2r-1}},\frac{1}{(2r-1)t}\right\}$, we have

$$\begin{aligned}\|\xi_{t,\alpha}\|_2 &< \infty\qquad\text{for}\quad\alpha<\alpha_0\\ \lim_{\alpha\to\alpha_0}\|\xi_{t,\alpha}\|_2 &= \infty,\end{aligned} \tag{II.4.4}$$

and the relations

$$\begin{aligned}\pi_t(\chi_1)\,\pi_t(\chi_1) &= \pi_t(\chi_2) + 2r\,I\\ \pi_t(\chi_1)\,\pi_t(\chi_n) &= \pi_t(\chi_{n+1}) + (2r-1)\,\pi_t(\chi_{n-1})\end{aligned}$$

imply

$$\pi_t(\chi_1)\,\xi_{t,\alpha} = \left((2r-1)\alpha + \frac{1}{\alpha}\right)\xi_{t,\alpha} + \left(\alpha - \frac{1}{\alpha}\right)\delta_e,$$

i.e. $\xi_{t,\alpha}$ is almost an eigenvector. Hence by (II.4.4)

$$\lim_{\alpha\to\alpha_0}\frac{\|\pi_t(\chi_1)\,\xi_{t,\alpha}\|_2}{\|\xi_{t,\alpha}\|_2} = (2r-1)\alpha_0 + \frac{1}{\alpha_0}.$$

Now observe that the function $\alpha \mapsto (2r-1)\alpha + \frac{1}{\alpha}$ defined on the positive real axis attains its unique minimum at $\frac{1}{\sqrt{2r-1}}$, taking the value $2\sqrt{2r-1}$ there, and that for $t > \frac{1}{\sqrt{2r-1}}$ the radius of convergence of $\xi_{t,\alpha}$ becomes $\alpha_0 = \frac{1}{(2r-1)t}$, and thus

$$\|\pi_t(\chi_1)\| \geq \frac{1}{t} + (2r-1)\,t > 2\sqrt{2r-1}.$$

This will actually yield our counterexample. Consider the representation $\pi_t \otimes \overline{\pi_t}$. The elementary matrix coefficients

$$\left\langle \left(\pi_t(x) \otimes \overline{\pi_t(x)}\right)\delta_y \otimes \overline{\delta_{y'}}, \delta_z \otimes \overline{\delta_{z'}}\right\rangle$$

are of the order $t^{2|x|}$ and hence $\pi_t \otimes \overline{\pi_t}$ is weakly contained in λ for $t \leq \frac{1}{\sqrt[4]{2r-1}}$. Thus taking $t = \frac{1}{\sqrt[4]{2r-1}}$ we get

$$\|\pi_t(\chi_1)\| = \sqrt{2r-1}\left(\frac{1}{\sqrt[4]{2r-1}} + \sqrt[4]{2r-1}\right) > 2\sqrt{2r-1} = \|(\pi_t \otimes \overline{\pi_t})\,(\chi_1)\|\,.$$

□

II.5. Unitaries with nonnegative alternating mixed moments

In this section we derive some estimates for norms of sums of operators with nonnegative mixed moments, in particular unitaries. The methods are combinatorial. There is still hope to prove the following conjecture.

Conjecture II.5.1. Let $u_1, u_2, \dots, u_n$ be some unitaries with nonnegative alternating mixed moments with respect to a faithful set of states and such that

$$\left\|\sum_{i=1}^{n} u_i\right\| = 2\sqrt{n-1}$$

then the linear space spanned by $u_1, u_2, \dots, u_n$ is (completely) isometric to the space spanned by $\lambda(g_1), \lambda(g_2), \dots, \lambda(g_n)$.

We begin with an elementary Perron-Frobenius type proposition.

Proposition II.5.2. *Let A be a C^*-algebra with a faithful set of states S and let a and b be elements of A.*

1. *If a and b have nonnegative alternating mixed moments with respect to S, then $\|a+b\| \geq \|a\|$.*
2. *If a is self-adjoint and has nonnegative moments with respect to S, then $\|a\| \in \sigma(a)$.*

PROOF. The first assertion follows from Corollary II.2.7 applied to the faithful completely positive map $\Phi_S : A \to l_\infty(S)$ which sends x to $(\varphi(x))_{x\in S}$. For the second assertion note that a slight modification of the argument of the proof of Proposition II.2.6 yields the following result. Denote for each $\varphi \in S$ the unique measure on $\sigma(a)$ satisfying

$$\varphi(f(a)) = \int_{\sigma(a)} f \, d\mu_\varphi$$

for every $f \in C(\sigma(a))$. Then by the faithfulness of the family S we have

$$\overline{\bigcup_{\varphi\in S} \operatorname{supp} \mu_\varphi} = \sigma(a)$$

and applying lemma II.3.5 to each μ_φ yields

$$\|a\| = \sup_{\varphi\in S} \max\{|\min \operatorname{supp} \mu_\varphi|, |\max \operatorname{supp} \mu_\varphi|\} = \sup_{\varphi\in S} \max \operatorname{supp} \mu_\varphi \in \sigma(a).$$

□

Taking a and b orthogonal projections shows that we cannot have strict inequality in general; however, if b in the last proposition is unitary, we can get a quantitative estimate.

Theorem II.5.3. *Let A be a C^*-algebra. Suppose that a, $u \in A$ have nonnegative alternating mixed moments with respect to a faithful set of states and that u is unitary. Then*

$$\|a + u\| \geq \frac{1}{2}\left(\|a\| + \sqrt{\|a\|^2 + 4}\right). \tag{II.5.1}$$

Remark II.5.4. Observe that the inequality $\|a + u\| \geq \sqrt{\|a\|^2 + 4}$ would imply with the assumptions of Conjecture II.5.1 (by induction) that $\|\sum_1^m u_i\| = 2\sqrt{m-1}$ for all $2 \leq m \leq n$; however, inequality (II.5.1) is asymptotically best possible as $a \to 0$.

We will need a combinatorial lemma.

Lemma II.5.5. *Let $\beta(p, k)$ = the number of possibilities to pick k disjoint pairs of consecutive numbers out of $\{1, 2, \ldots, 2p\}$. Then $\beta(p, k) = \binom{2p-k}{k}$ for $1 \leq k \leq p$ and the generating function is given by the formula*

$$F(x, y) = \sum_{p=1}^{\infty} \sum_{k=1}^{p} \beta(p, k)\, x^p\, y^k = \frac{xy(xy - x - 1)}{(x-1)(x^2y^2 - 2xy - x + 1)}.$$

PROOF. There is a one to one correspondence between choices of k pairs out of a line $2p$ points and choices of k points out of a line of $2p - k$ points. Hence $\beta(p, k) = \binom{2p-k}{k}$. To get a recursion for these numbers we use the following notation.

$\beta^0(p, k) = \#$ choices such that the first point is part of a pair.
$\beta^1(p, k) = \#$ choices such that the first point is free.

Then after prepending two more points we obtain the following recursion:

$$\beta^0(p + 1, k) = \beta(p, k - 1)$$
$$\beta^1(p + 1, k) = \beta(p, k) + \beta^1(p, k - 1).$$

Setting

$$F^i(x, y) = \sum_{p=1}^{\infty} \sum_{k=1}^{p} \beta^i(p, k)\, x^p\, y^k$$

for $i = 1, 2$ we have $F(x, y) = F^0(x, y) + F^1(x, y)$ and the claim follows after solving the equations

$$F^0(x,y) = xy\left(\frac{1}{1-x} + F(x,y)\right)$$
$$F^1(x,y) = x\left(\frac{xy}{1-x} + F(x,y) + y\,F^1(x,y)\right).$$

$\square$

Proof of Theorem II.5.3. We use Proposition II.2.6 to calculate the norm. To this end for each state φ in the faithful family we estimate

$$m_p = \varphi\left(\left((a+u)^*(a+u)\right)^p\right)$$

in terms of the moments

$$\alpha_p = \varphi\big((a^*a)^p\big)$$

as follows. Set $a_1 = a$ and $a_2 = u$, then

$$m_p = \sum_{i_1,j_1,\dots,i_p,j_p\in\{1,2\}} \varphi(a_{i_1}^* a_{j_1}\cdots a_{i_p}^* a_{j_p})$$

Now we collect all the terms where no u remains after cancelling out uu^* and u^*u wherever possible. There are exactly $\beta(p, p-k)$ terms such that $(a^*a)^k$ remains. By the positiveness of the remaining moments we get

$$m_p \geq \alpha_p + \sum_{k=1}^{p} \beta(p,k)\,\alpha_{p-k}.$$

We define the moment generating functions

$$G_a(z) = \sum_{p=0}^{\infty} \alpha_p\, z^{2p} \qquad G_{a+u}(z) = \sum_{p=0}^{\infty} m_p\, z^{2p}$$

and

$$\tilde{G}_{a+u}(z) = \sum_{p=0}^{\infty}\left(\alpha_p + \sum_{k=1}^{p}\beta(p,k)\,\alpha_{p-k}\right) z^{2p}.$$

The radius of convergence of G_{a+u} is $\frac{1}{\|a+u\|}$, and by Pringsheim's theorem [**34**, Theorem 17.13] it is smaller than $\tilde{r}$, the radius of convergence of $\tilde{G}_{a+u}$, which we shall estimate from above in terms of $\|a\|$. This will yield a lower bound for $\|a+u\|$. Now we have

$$\tilde{G}_{a+u}(z) = G_a(z) + \sum_{k=0}^{\infty}\sum_{p=k+1}^{\infty} \beta(p, p-k)\,\alpha_k\, z^{2p}.$$

The sum $\sum_{k=1}^{\infty} \beta(p, p-k)\, z^{2p}$ can actually be calculated. Putting

$$F_q(\zeta, z) = \frac{1}{\zeta^q}\,F(\zeta\, z^2, \tfrac{1}{\zeta}) = \sum_{p=1}^{\infty}\sum_{k=1}^{p} \beta(p,k)\,\zeta^{p-k-q}\, z^{2p}$$

and applying Cauchy's integral formula we have

$$\frac{1}{2\pi i}\oint F_q(\zeta,z)\frac{d\zeta}{\zeta} = \sum_{p=1}^{\infty}\sum_{\substack{k\\ p-k-q=0}} \beta(p,k)\,z^{2p} = \sum_{p=1}^{\infty}\beta(p,p-q)\,z^{2p};$$

on the other hand,

$$F_k(\zeta,z) = \frac{1}{\zeta^k}\frac{z^2(z^2-z^2\zeta-1)}{(z^2\zeta-1)(z^4-2z^2-z^2\zeta+1)}$$
$$= \frac{1}{\zeta^k}\left(\frac{1-z^2}{(1-z^2)^2-z^2\zeta}-\frac{1}{1-z^2\zeta}\right)$$
$$= \frac{1}{\zeta^k}\left(\frac{1}{1-z^2}\sum_{m=0}^{\infty}\left(\frac{z^2}{(1-z^2)^2}\,\zeta\right)^m-\sum_{m=0}^{\infty}(z^2\zeta)^m\right)$$

and hence

$$\frac{1}{2\pi i}\oint F_k(\zeta,z)\frac{d\zeta}{\zeta} = \frac{1}{1-z^2}\left(\frac{z^2}{(1-z^2)^2}\,\zeta\right)^k - z^{2k} = z^{2k}\left(\frac{1}{(1-z^2)^{2k+1}-1}-1\right).$$

Now the radius of convergence of second summand in

$$\tilde{G}_{a+u}(z) = G_a(z) + \sum_{k=0}^{\infty}\alpha_k\, z^{2k}\left(\frac{1}{(1-z^2)^{2k+1}}-1\right)$$

is determined by

$$\limsup_{k\to\infty}\left(\alpha_k\,|z|^{2k}\left(\frac{1}{(1-z^2)^{2k+1}}-1\right)\right)^{1/2k}\le 1$$

i.e.

$$\|a\|\,|z|\le 1-|z|^2$$

and hence

$$\tilde{r}\le\min\left\{\frac{1}{\|a\|},\frac{\sqrt{\|a\|^2+4}-\|a\|}{2}\right\};$$

this implies

$$\|a+u\|\ge\max\left\{\|a\|,\frac{1}{2}\left(\|a\|+\sqrt{\|a\|^2+4}\right)\right\}.$$

□

Remark II.5.6. Assuming that the states in consideration are tracial does not improve the estimate; indeed, the combinatorics in this case are given by the numbers $\tilde{\beta}(p,k)$ of possibilities to choose k pairs out of a circle of $2p$ points. Similar to Lemma II.5.5 one can see that $\tilde{\beta}(p,k)=\binom{2p-k}{k}+\binom{2p-k-1}{k-1}\le 2\binom{2p-k}{k}$ and asymptotically this is of the same order.

If a is the sum of unitaries there are far more cancellations and we can get a better estimate. In order to do this we will do an analysis of radial functions associated to the operator $\sum\lambda(g_i)$ which is quite similar to the theory of radial functions on free groups (cf. [**11**], [**12**] and [**51**]). Instead of calculating the norm of $T=\sum u_i$ directly, we shall consider the operator $T^*T-nI=\sum_{i\ne j}u_i^*u_j$. This leads to the following definition.

Definition II.5.7. On the free group $\mathbf{F}_n$ we set for $k>0$

$$\tilde{E}_k^{(n)}=\left\{g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_k}^{-1}g_{j_k} : i_1\ne j_1\ne i_2\ne\ldots\ne i_k\ne j_k\right\}$$

and $\tilde{E}_0^{(n)}=\{e\}$; we denote by

$$\tilde{\chi}_k^{(n)}=\chi_{\tilde{E}_k^{(n)}}$$

its characteristic function.

The next proposition is the analogue of [**11**, Theorem 1].

Proposition II.5.8. *The functions $\tilde{\chi}_k^{(n)}$ satisfy the following convolution identities.*

(II.5.2a) $$\tilde{\chi}_0^{(n)} * \tilde{\chi}_k^{(n)} = \tilde{\chi}_k^{(n)}$$

(II.5.2b) $$\tilde{\chi}_1^{(n)} * \tilde{\chi}_1^{(n)} = \tilde{\chi}_2^{(n)} + (n-2)\tilde{\chi}_1^{(n)} + n(n-1)\tilde{\chi}_0^{(n)}$$

(II.5.2c) $$\tilde{\chi}_1^{(n)} * \tilde{\chi}_k^{(n)} = \tilde{\chi}_{k+1}^{(n)} + (n-2)\tilde{\chi}_k^{(n)} + (n-1)^2\tilde{\chi}_{k-1}^{(n)} \quad \textit{for } k \geq 2$$

Proof. Since (II.5.2a) is trivial we begin by showing (II.5.2b). We expand the product and split it as follows:

$$\begin{aligned}\tilde{\chi}_1^{(n)} * \tilde{\chi}_1^{(n)} &= \sum_{i_1\neq j_1}\sum_{i_2\neq j_2} g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2} \\ &= \sum_{i_1\neq j_1\neq i_2\neq j_2} g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2} + \sum_{i_1\neq j_1\neq j_2} g_{i_1}^{-1}g_{j_2}\end{aligned}$$

(the sums run over all indices appearing under the $\sum$ sign which satisfy the indicated relations; sums of group elements are performed in the group algebra $\mathbf{C}G$). In the second sum, for fixed i_1, j_2, $g_{i_1}^{-1}g_{j_2}$ appears once for every $j_1 \notin \{i_1, j_2\}$, that is $(n-2)$ times if $i_1 \neq j_2$ and $(n-1)$ times if $i_1 = j_2$. In the latter case there is one more cancellation and we get $n(n-1)\tilde{\chi}_0^{(n)}$ and hence (II.5.2b).
We can treat (II.5.2c) similarly; namely,

$$\begin{aligned}\tilde{\chi}_1^{(n)} * \tilde{\chi}_k^{(n)} &= \sum_{i\neq j}\sum_{\substack{i_1\neq j_1\neq\dots\\ \dots\neq i_k\neq j_k}} g_i^{-1}g_jg_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_k}^{-1}g_{j_k} \\ &= \sum_{\substack{i\neq j\neq i_1\neq j_1\neq\dots\\ \dots\neq i_k\neq j_k}} g_i^{-1}g_jg_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_k}^{-1}g_{j_k} + \sum_{\substack{i\neq j\neq j_1\neq\dots\\ \dots\neq i_k\neq j_k}} g_i^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_k}^{-1}g_{j_k}.\end{aligned}$$

In the last summand for $g_i^{-1}g_{j_1}$ appears $(n-2)$ times if $i \neq j_1$ and $(n-1)$ times if $i = j_j$. Contrary to the case $k = 2$, there is the additional condition $j_1 \neq i_2$, thus (II.5.2c) holds. □

Similar to [**11**] the following corollary is an immediate consequence of the relations (II.5.2a) – (II.5.2c).

Corollary II.5.9. *1. There are polynomials $P_k^{(n)}$ of degree k such that*

(II.5.3a) $$P_k^{(n)}\big(\tilde{\chi}_1^{(n)}\big) = \tilde{\chi}_k^{(n)};$$

they satisfy the relations

(II.5.3b) $$P_0^{(n)}(x) = 1 \qquad P_1^{(n)}(x) = x \qquad P_2^{(n)}(x) = x^2 - (n-2)x - n(n-1)$$

(II.5.3c) $$P_{k+1}^{(n)}(x) = (x-n+2)P_k^{(n)}(x) - (n-1)^2P_{k-1}^{(n)}(x) \quad \textit{for } k \geq 2.$$

2. Conversely there are coefficients $b_{k,p}^{(n)}$ such that

(II.5.4a) $$\left(\tilde{\chi}_1^{(n)}\right)^p = \sum_{k=0}^{p} b_{k,p}^{(n)}\tilde{\chi}_k^{(n)};$$

they satisfy the relations

(II.5.4b) $$b_{0,p+1}^{(n)} = n(n-1)b_{1,p}^{(n)}$$

(II.5.4c) $$b_{k,p+1}^{(n)} = b_{k-1,p}^{(n)} + (n-2)b_{k,p}^{(n)} + (n-1)^2b_{k+1,p}^{(n)}$$

and their generating function is given by

$$\tilde{F}^{(n)}(x,z) = \sum_{p=0}^{\infty}\sum_{k=0}^{p} b_{k,p}^{(n)} x^k z^p = \frac{\left(\frac{(n-1)^2 z}{x} - (n-2)z + \frac{1}{n}\right)\tilde{F}_0^{(n)}(z) + 1 - \frac{1}{n}}{1 - \left(x + n - 2 + \frac{(n-1)^2}{x}\right) z} \tag{II.5.5}$$

where

$$\tilde{F}_0^{(n)}(z) = \sum_{p=0}^{\infty} b_{0,p}^{(n)} z^p = \frac{2 - n + n\sqrt{\frac{1-(3n-4)z}{1+nz}}}{2(1 - n(n-1)z)}$$

Proof. The recursions (II.5.3b-c) and (II.5.4b-c) follow readily from (II.5.2). As for the generating function, note that $b_{0,p}^{(n)} = \tau\big(\lambda(\tilde{\chi}_1^{(n)})^p\big)$. The generating function of this sequence can be easily deduced from the computations of [**60**]. Denoting the moment generating function of an operator T in a finite von Neumann algebra with respect to a trace τ by $F_T(z) = \sum_{k=0}^{\infty} \tau(T^k) z^k = \tau\big((I - zT)^{-1}\big)$, it is easy to deduce from [**60**, Theorem 2] that the moment generating function of $\sum_{i,j=1}^{n} \lambda(g_i^{-1} g_j) = nI + \tilde{\chi}_1^{(n)}$ is

$$F_{nI+\lambda(\tilde{\chi}_1^{(n)})} = \frac{2 - n + n\sqrt{1 + 4z - 4nz}}{2(1 - n^2 z)}. \tag{II.5.6}$$

Now considering the moment generating function of a translation of T by a multiple of the identity:

$$F_{T+\alpha I}(z) = \tau\Big(\big((1 - z\alpha)I - zT\big)^{-1}\Big) = \frac{1}{1 - z\alpha} F_T\left(\tfrac{z}{1-z\alpha}\right)$$

formula (II.5.5) is an immediate consequence of (II.5.6). □

Actually it will be convenient to work with translations of the polynomials $P_k^{(n)}$ by a fixed difference.

Proposition II.5.10. *The polynomials $Q_k^{(n)}(x) = P_k^{(n)}(x + n - 2)$ fulfill the following properties.*

1. *The recursion*

$$Q_0^{(n)}(x) = 1 \qquad Q_1^{(n)}(x) = x + n - 2 \qquad Q_2^{(n)}(x) = x^2 + (n-2)x - n(n-1) \tag{II.5.7a}$$

$$Q_{k+1}^{(n)}(x) = x Q_k^{(n)}(x) - (n-1)^2 Q_{k-1}^{(n)}(x) \quad \textit{for } k \geq 2; \tag{II.5.7b}$$

2. *the inequalities*

$$Q_{k+1}^{(n)}(x) \geq (n-1) Q_k^{(n)}(x) \geq 0 \quad \textit{for} \quad x \geq 2(n-1) \tag{II.5.8a}$$

$$Q_{k+1}^{(n)\prime}(x) \geq (n-1) Q_k^{(n)\prime}(x) \quad \textit{for} \quad x \geq 2(n-1); \tag{II.5.8b}$$

3. *there is a constant C depending only on n such that for $2(n-1) \le x \le n^2 - 2n + 2$ we have*

(II.5.9)

$$\begin{aligned} Q_k^{(n)}(x) = {} & \frac{n+2-\frac{x}{n-1}}{\sqrt{x^2-4(n-1)^2}} \left(\left(\frac{x+\sqrt{x^2-4(n-1)^2}}{2} \right)^k - \left(\frac{x-\sqrt{x^2-4(n-1)^2}}{2} \right)^k \right) \\ & + \frac{n}{n-1} \left(\left(\frac{x+\sqrt{x^2-4(n-1)^2}}{2} \right)^{k+1} - \left(\frac{x-\sqrt{x^2-4(n-1)^2}}{2} \right)^{k+1} \right) \\ & \ge C \left(\frac{x+\sqrt{x^2-4(n-1)^2}}{2} \right)^k . \end{aligned}$$

Proof. The recursions (II.5.7a) and (II.5.7b) are immediate consequences of (II.5.3b) and (II.5.3c). The inequalities (II.5.8a) and (II.5.8b) follow by induction. We only prove (II.5.8a), which for $k = 0$ is obvious; we even have $Q_1^{(n)}(x) \ge n$ for x in the indicated range. The case $k = 2$ now follows easily:

$$\begin{aligned} Q_2^{(n)}(x) &= xQ_1^{(n)}(x) - n(n-1) \\ &= (n-1)Q_1^{(n)}(x) + (x-(n-1))Q_1^{(n)}(x) - n(n-1) \\ &\ge (n-1)Q_1^{(n)}(x) + (n-1)(Q_1^{(n)}(x) - n) \\ &\ge (n-1)Q_1^{(n)}(x); \end{aligned}$$

similarly, for $k \ge 2$, using (II.5.7b), we get

$$\begin{aligned} Q_{k+1}^{(n)}(x) &= xQ_k^{(n)}(x) - (n-1)^2 Q_{k-1}^{(n)}(x) \\ &= 2(n-1)Q_k^{(n)}(x) + (n-1)^2 Q_{k-1}^{(n)}(x) \\ &\ge (n-1)Q_k^{(n)}(x) + (n-1)(Q_k^{(n)}(x) - (n-1)Q_{k-1}^{(n)}(x) \\ &\ge (n-1)Q_k^{(n)}(x) \end{aligned}$$

Similarly one can prove (II.5.8b). The general formula for the solution of the recurrence relation

(II.5.10) $$a_{n+1} = \alpha a_n - \beta a_{n-1},$$

has the form (cf. [**11**, Proposition 2])

$$a_n = \lambda A_n + \mu A_{n+1}$$

where $\lambda, \mu \in \mathbf{C}$ and $A_n = a_+^n - a_-^n$ with

$$a_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

is the basic solution of (II.5.10) for $\alpha^2 - 4\beta \neq 0$. The formula (II.5.9) follows from this by determining λ and μ from $Q_1^{(n)}$ and $Q_2^{(n)}$. □

Now we are ready to prove a strengthened version of Theorem II.5.3 for sums of unitaries.

Theorem II.5.11. *Let $u_1, u_2, \dots, u_n$ be unitaries in a C^*-algebra A with nonnegative alternating mixed moments with respect to a faithful set of states. For $2 \le m \le n$ we denote $T_m = \sum_1^m u_i$, $y_m = \|T_m\|^2 - 2(m-1)$, and*

$$w_m = y_m + \sqrt{y_m^2 - 4(m-1)^2} = \|T_m\|^2 - 2(m-1) + \|T_m\|\sqrt{\|T_m\|^2 - 4(m-1)}$$

Then

$$w_m \leq w_n$$

Remark II.5.12. In particular, if $u_1, u_2, \ldots, u_n$ are as above and $\left\|\sum_1^n u_i\right\| = 2\sqrt{n-1}$ then $\left\|\sum_1^m u_i\right\| \leq 2\sqrt{m - 1 + \frac{(n-m)^2}{4(n-1)}}$.

Proof. We introduce some additional notation. Denote π the representation of the free group $\mathbf{F}_n$ which is uniquely determined by the requirement $\pi(g_i) = u_i$. Further for $1 \leq m \leq n$ denote $\alpha_m = \|T_m\|$, $x_m = \|\pi(\tilde{\chi}_1^{(m)})\| = \|T_m^* T_m - mI\| = \alpha_m^2 - m$ and $y_m = x_m - m + 2 = \alpha_m^2 - 2(m-1)$. Then the radius of convergence of the power series

$$\sum_{p=0}^{\infty} \pi(\tilde{\chi}_1^{(n)})^p \, z^p$$

with values in A is $\frac{1}{\|\pi(\tilde{\chi}_1^{(n)})\|}$. We will obtain a lower bound for this in terms of $\|\pi(\tilde{\chi}_1^{(n)})\|$. Indeed, for $t > 0$ we get

$$\begin{aligned}
\left\|\sum_{p=0}^{\infty} \pi(\tilde{\chi}_1^{(n)})^p \, t^p\right\| &= \left\|\sum_{p=0}^{\infty} \sum_{k=0}^{p} b_{kp}^{(n)} \, \pi(\tilde{\chi}_k^{(n)}) \, t^p\right\| && \text{by (II.5.4a)} \\
&\geq \left\|\sum_{p=0}^{\infty} \sum_{k=0}^{p} b_{kp}^{(n)} \, \pi(\tilde{\chi}_k^{(m)}) \, t^p\right\| && \text{by Proposition II.5.2 (1.)} \\
&= \left\|\sum_{p=0}^{\infty} \sum_{k=0}^{p} b_{kp}^{(n)} \, P_k^{(m)}(\pi(\tilde{\chi}_1^{(m)})) \, t^p\right\| && \text{by (II.5.3a)} \\
&\geq \sum_{p=0}^{\infty} \sum_{k=0}^{p} b_{kp}^{(n)} \, P_k^{(m)}(x_m) \, t^p && \text{by Proposition II.5.2 (2.)} \\
&= \sum_{p=0}^{\infty} \sum_{k=0}^{p} b_{kp}^{(n)} \, Q_k^{(m)}(y_m) \, t^p && \\
&\geq C \sum_{p=0}^{\infty} \sum_{k=0}^{p} b_{kp}^{(n)} \, \left(\frac{w_m}{2}\right)^k \, t^p && \text{by (II.5.9)} \\
&= C\tilde{F}^{(n)}\left(\tfrac{w_m}{2}, t\right) && \text{by (II.5.5)}
\end{aligned}$$

Since the power series expansion of $\tilde{F}$ has only nonnegative coefficients, the rational function $\tilde{F}\left(\frac{w_m}{2}, \,.\,\right)$ cannot explode for $t < \frac{1}{\|\pi(\tilde{\chi}_1^{(n)})\|}$, i.e. the denominator in formula (II.5.5) must be nonnegative at $x = \frac{w_m}{2}$ and $z = \frac{1}{\|\pi(\tilde{\chi}_1^{(n)})\|}$. It follows that

$$\frac{w_m}{2} + \frac{2(n-1)^2}{w_m} \leq y_n$$

and consequently

$$\frac{4(n-1)^2}{w_n} = y_n - \sqrt{y_n^2 - 4(n-1)^2} \leq w_m \leq y_n + \sqrt{y_n^2 - 4(n-1)^2} = w_n$$

$\square$

II.6. Free operators with operator coefficients[1]

Let $g_1, g_2, \dots, g_n$ be the generators of the free group $\mathbf{F}_n$. C. Akemann and P. Ostrand proved in [**1**] a formula for the norm of free operators, i.e. operators of the form $\sum \alpha_i \lambda(g_i)$. This formula improved estimates of M. Leinert [**31**] and M. Bożejko [**7**]. It was previously known in the case of equal coefficients [**30**] and simpler proofs were found in [**60**] and [**44**]. In this section we show how a part of the proof of [**44**] can be generalized to obtain estimates for the operator valued case, which improve the bounds in [**26**, Proposition 1.1] (see also [**9**] for related recent results).

Theorem II.6.1. *Let $n \geq 2$ and $g_1, \dots, g_n$ be the generators of the free group $\mathbf{F}_n$ and let further $a_1, a_2, \dots, a_n$ be some operators on a Hilbert space H which can be approximated by invertible operators, then*

$$\text{(II.6.1)}\qquad \begin{aligned} \left\| \sum_{i=1}^n \lambda(g_i) \otimes a_i \right\|_{\min} \leq \inf_{s>0} &\left\| \sum_{i=1}^n \left(s^2 I + a_i a_i^*\right)^{1/2} - (n-2)sI \right\|^{1/2} \times \\ &\times \left\| \sum_{i=1}^n \left(s^2 I + a_i^* a_i\right)^{1/2} - (n-2)sI \right\|^{1/2} \end{aligned}$$

Remark: If the Hilbert space H is finite dimensional, any operator can be approximated with invertible ones. In the infinite dimensional case this is not generally true, see [**27**, Problem 140]. However, we have

Corollary II.6.2. *For an arbitrary family of operators $a_1, a_2, \dots, a_n$ on a Hilbert space H we have*

$$\text{(II.6.2)}\qquad \begin{aligned} \left\| \sum_{i=1}^n \lambda(g_i) \otimes a_i \right\|_{\min} &\leq 2\sqrt{1 - \frac{1}{n}} \left(\frac{\|\sum_{i=1}^n a_i a_i^*\| + \|\sum_{i=1}^n a_i^* a_i\|}{2} \right)^{1/2} \\ &\leq 2\sqrt{1 - \frac{1}{n}} \max \left\{ \left\| \sum_{i=1}^n a_i a_i^* \right\|^{1/2}, \left\| \sum_{i=1}^n a_i^* a_i \right\|^{1/2} \right\} \end{aligned}$$

The proof of (II.6.1) is an adaptation to the non-commutative situation of the first part of [**44**]. Denote by $HS(H)$ the space of Hilbert-Schmidt operators on the Hilbert space H and by tr the usual (unbounded) trace. We need the following version of the Cauchy-Schwarz inequality.

Lemma II.6.3. *For any $x_1, x_2, \dots, x_n \in B(H)$ and $y_1, y_2, \dots, y_n \in HS(H)$ we have the inequality*

$$\text{(II.6.3)}\qquad \left\| \sum_{i=1}^n x_i y_i \right\|_{HS} \leq \left\| \sum_{i=1}^n x_i x_i^* \right\|^{1/2} \operatorname{tr} \left(\sum_{i=1}^n y_i^* y_i \right)^{1/2}$$

Proof. After writing the sum as

$$\begin{bmatrix} \sum_{i=1}^n x_i y_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ y_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \cdots & 0 \end{bmatrix}$$

the claim follows from the operator ideal property of $HS(H^n)$. $\square$

[1] voir la note p. xii.

Proof of Theorem. We consider first the case where all the operators a_i are invertible. The general case follows then by a topological argument.
Let $T = \sum_{i=1}^n \lambda(g_i) \otimes a_i$ act on the Hilbert space $\ell_2(\mathbf{F}_n; HS(H))$. For every word $y \in \mathbf{F}_n$, $i \in \{1, 2, \dots n\}$ and any positive real number s we define the following operator:

$$p_i(y,s) = a_i^{-1}\big((s^2 + a_i a_i^*)^{1/2} \mp s\big) = \big((s^2 + a_i^* a_i)^{1/2} \mp s\big)\, a_i^{-1}$$

with

"$-$" if there is no cancellation in the word $g_i^{-1}y$, i.e. the first letter of y is different from g_i.
"$+$" if the first letter of y *is* g_i and there is cancellation.

Here and in the following scalars appearing in operator expressions mean the corresponding multiple of the identity operator and the square root of a positive operator is always the unique positive square root. Then one can easily check that $p_i(y,s)$ is invertible and that its inverse is

$$\begin{aligned} p_i(y,s)^{-1} &= \big((s^2 + a_i a_i^*)^{1/2} \pm s\big)\, a_i^{*\,-1} \\ &= a_i^{*\,-1}\big((s^2 + a_i^* a_i)^{1/2} \pm s\big) \end{aligned}$$

(note the change of sign).
Now pick $h \in \ell_2(\mathbf{F}_n; HS(H))$ with finite support. In order to get the upper bound for $\|Th\|_2^2 = \sum_y \|Th(y)\|_{HS}^2$ we first give an estimate of

$$\|Th(y)\|_{HS}^2 = \left\| \sum_{i=1}^n a_i\, p_i(y,s)\, p_i(y,s)^{-1}\, h(g_i^{-1}y) \right\|_{HS}^2$$

for fixed y. Now the operator $a_i\, p_i(y,s)$ is positive and we can apply the above lemma by setting $x_i = (a_i\, p_i(y,s))^{1/2}$ its (positive) square root and denoting the rest by $y_i = (a_i\, p_i(y,s))^{1/2}\, p_i(y,s)^{-1}\, h(g_i^{-1}y)$:

$$\|Th(y)\|_{HS}^2 \le \left\| \sum_{i=1}^n a_i\, p_i(y,s) \right\| \operatorname{tr}\left(\sum_{i=1}^n h(g_i^{-1}y)^*\, p_i(y,s)^{*\,-1}\, a_i\, h(g_i^{-1}y) \right)$$

Note that among the words $\{ g_1^{-1}y, g_2^{-1}y, \dots g_n^{-1}y \}$ there is at most one cancellation so that there is always the "$-$" sign in $p_i(y,s)$ except maybe one case and since all the summands $a_i\, p_i(y,s)$ are positive operators, we have the operator inequality

$$\sum_{i=1}^n a_i\, p_i(y,s) \le 2s + \sum_{i=1}^n \big((s^2 + a_i a_i^*)^{1/2} - s\big) =: c_0(s)$$

This upper bound does not depend on the word y and we can estimate the norm of Th as follows.

$$\begin{aligned} \|Th\|_2^2 &\le \|c_0(s)\| \sum_y \operatorname{tr}\left(\sum_{i=1}^n p_i(g_i y, s)^{*\,-1}\, a_i\, h(y)\, h(y)^* \right) \\ &\le \|c_0(s)\| \sum_y \left\| \sum_{i=1}^n p_i(g_i y, s)^{*\,-1}\, a_i \right\| \|h(y)\|_{HS}^2 \end{aligned}$$

Now $p_i(g_i y, s)$ has always the "$+$" sign with at most one exception possible in the case when there is already cancellation in $g_i y$. Thus $p_i(g_i y, s)^{-1}$ has the "$-$" signs and with the formula for the inverse we get

$$\sum_{i=1}^{n} p_i(g_i y, s)^{*-1} a_i \leq 2s + \sum_{i=1}^{n} \left((s^2 + a_i^* a_i)^{1/2} - s\right) =: c_1(s)$$

The bound $c_1(s)$ is again independent of y and finally we get the inequality we wanted: For all positive real numbers s

$$\|Th\|_2^2 \leq \|c_0(s)\| \, \|c_1(s)\|.$$

Let us now consider the case where the a_i's are not invertible but approximable by invertible operators. Consider the topological space $B(H)^n$ equipped with the product topology and define the functions

$$\begin{array}{rrcl} & B(H)^n & \to & \mathbf{R} \\ g: & x = (a_1, a_2, \ldots, a_n) & \mapsto & \inf\limits_{s \geq 0} \left(\|c_0(s)\| \, \|c_1(s)\|\right)^{1/2} \\ f: & x = (a_1, a_2, \ldots, a_n) & \mapsto & \left\| \sum\limits_{i=1}^{n} \lambda(g_i) \otimes a_i \right\|. \end{array}$$

g is upper semicontinuous, i.e. the set

$$\{\, x \in B(H)^n \mid g(x) < t \,\}$$

is open for any $t \in \mathbf{R}$. Since f is continuous, the set

$$\{\, x \in B(H)^n \mid g(x) - f(x) \geq 0 \,\}$$

is closed and hence contains the closure of all the n-tuples of invertible operators. □

Note that the infimum over all positive scalars s could be replaced by an infimum over all positive operators S which commute with the a_i's. However in view of the examples below this does not seem to improve the inequality very much.

Proof of Corollary. We will prove first the case where the a_i's are approximable by invertibles. In fact we will show that the bound (II.6.1) is sharper than (II.6.2) just as in the commutative case, cf. [**1**]. We recall the following facts from non-commutative analysis. A function f is called *operator-monotone* if for any positive selfadjoint operators a, b the implication

$$a \geq b \;\Rightarrow\; f(a) \geq f(b)$$

holds. It is *operator-concave*, if the operator inequality

$$f(\lambda a + (1-\lambda) b) \geq \lambda f(a) + (1-\lambda) f(b)$$

holds for all positive operators a, b and any $0 < \lambda < 1$. By Löwner's theorem [**35**, p.464], the function $t \mapsto t^\alpha$ is operator-monotone for $0 \leq \alpha \leq 1$ (see also [**43**]). Ando showed in [**2**] (see also [**36**]) that any operator-monotone function is necessarily operator-concave. We can now apply this to the function $t \mapsto \sqrt{t}$ and any sequence of positive operators $x_i = a_i^* a_i$

$$\frac{1}{n} \sum_{i=1}^{n} x_i^{1/2} \leq \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^{1/2}$$

and our bound becomes

$$
\begin{aligned}
(\|c_0(s)\|\,\|c_1(s)\|)^{1/2} &\le \left((2-n)s + n\sqrt{s^2+\frac{1}{n}\Big\|\sum_{i=1}^n a_i a_i^*\Big\|}\right)^{1/2} \times \\
&\qquad \times \left((2-n)s + n\sqrt{s^2+\frac{1}{n}\Big\|\sum_{i=1}^n a_i^* a_i\Big\|}\right)^{1/2} \\
&\le (2-n)s + \frac{n}{2}\left(\sqrt{s^2+\frac{1}{n}\Big\|\sum_{i=1}^n a_i a_i^*\Big\|} + \right. \\
&\qquad \left. + \sqrt{s^2+\frac{1}{n}\Big\|\sum_{i=1}^n a_i^* a_i\Big\|}\right) \\
&\le (2-n)s + n\sqrt{s^2+\frac{1}{2n}\left(\Big\|\sum_{i=1}^n a_i a_i^*\Big\| + \Big\|\sum_{i=1}^n a_i^* a_i\Big\|\right)}.
\end{aligned}
$$

The infimum of the last expression over all $s \ge 0$ is the expression in the claim.
Let us now consider general operators $a_1, a_2, \ldots, a_n$ which are not necessarily approximable by invertible ones. Denote by $P_f(H)$ the set of all finite dimensional projections on H. It is easy to see that the embedding

$$
\begin{aligned}
\Phi : B(H) \to & \bigoplus_{p\in P_f(H)} pB(H)p \\
a \mapsto & (pap)_{p\in P_f(H)}
\end{aligned}
$$

is a complete isometry, i.e. for any operators $b_1, b_2, \ldots, b_n$ acting on some Hilbert space K we have

$$
\Big\|\sum_{i=1}^n a_i \otimes b_i\Big\|_{\min} = \Big\|\sum_{i=1}^n \Phi(a_i) \otimes b_i\Big\|_{\min}
$$

Now $\Phi(a_i)$ lying in a direct sum of matrix algebras can be approximated by invertibles, so that inequality (II.6.2) holds when we replace a_i by $\Phi(a_i)$. Now we use the fact that the right hand side of (II.6.2) comes from an operator space structure. Indeed, denoting e_{ij} the canonical basis of $\boldsymbol{M}_n$, it is easy to see that

$$
\Big\|\sum_{i=1}^n a_i \otimes e_{1i}\Big\| = \Big\|\sum_{i=1}^n a_i a_i^*\Big\|^{\frac{1}{2}} \qquad \text{and} \qquad \Big\|\sum_{i=1}^n a_i \otimes e_{i1}\Big\| = \Big\|\sum_{i=1}^n a_i^* a_i\Big\|^{\frac{1}{2}};
$$

hence we have

$$
\begin{aligned}
\Big\|\sum_{i=1}^n \lambda(g_i)\otimes a_i\Big\|_{\min} &= \Big\|\sum_{i=1}^n \lambda(g_i)\otimes \Phi(a_i)\Big\|_{\min} \\
&\le 2\sqrt{1-\frac{1}{n}}\left(\frac{\|\sum \Phi(a_i)\otimes e_{1i}\| + \|\sum \Phi(a_i)\otimes e_{i1}\|}{2}\right)^{\frac{1}{2}} \\
&= 2\sqrt{1-\frac{1}{n}}\left(\frac{\|\sum a_i\otimes e_{1i}\| + \|\sum a_i\otimes e_{i1}\|}{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

and this proves the claim. We do not see how to generalize (II.6.1) to non-approximable operators with a similar trick, since the assignment $(a_i) \mapsto \left\|\sum |a_i|\right\|$ fails to be a norm and hence is not a complete invariant. $\square$

Examples. There is actually equality not only for scalar coefficients and it would be interesting to characterize such families of operators.

Example II.6.4. Commuting normal operators.

If the operators $a_1, a_2, \dots, a_n$ generate a commutative C^*-algebra, the scalar formula of Akemann-Ostrand is applicable as a simple consequence of Gel'fand's theorem.

Example II.6.5. For unitaries $u_1, u_2, \dots, u_n$ there is equality:

$$\begin{aligned}\left\|\sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i u_i\right\| &= \left\|\sum_{i=1}^{n} \alpha_i \, \lambda(g_i)\right\| \\ &= \min_{s \geq 0} 2s + \sum_{i=1}^{n} \left(\sqrt{s^2 + |\alpha_i|^2} - s\right) \\ &= \min_{s \geq 0} (\|c_0(s)\| \, \|c_1(s)\|)^{1/2}\end{aligned}$$

The first identity is Fell's lemma **[23]** applied to the left regular representation and the unitary representation which is uniquely determined by $\pi(g_i) = u_i$.
The following example uses the following simple identity. For any projection p and positive real numbers σ, α

$$(\sigma I + (\sqrt{\sigma^2 + \alpha^2} - \sigma)p)^2 = \sigma^2 I + \alpha^2 p \tag{II.6.4}$$

Example II.6.6. For the basis of the row-space R_n and equal coefficients there is equality:

$$\left\|\sum_{i=1}^{n} \lambda(g_i) \otimes e_{1i}\right\| = \sqrt{n} = \min_{s \geq 0} (\|c_0(s)\| \, \|c_1(s)\|)^{1/2} \tag{II.6.5}$$

However if the coefficients are different there may be strict inequality, e.g.

$$\begin{aligned}\|\lambda(g_1) \otimes e_{11} + \lambda(g_2) \otimes e_{12} + 2\lambda(g_3) \otimes e_{13}\| &= \sqrt{6} \\ &< \sqrt{8} = \min_{s \geq 0} (\|c_0(s)\| \, \|c_1(s)\|)^{1/2}\end{aligned} \tag{II.6.6}$$

The bound for the general operator $\sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i e_{1i}$ is determined by the norms

$$\begin{aligned}\|c_0(s)\| &= \left\|2s + \sum_{i=1}^{n} \left((s^2 + |\alpha_i|^2 e_{11})^{1/2} - s\right)\right\| \\ &= 2s + \sum_{i=1}^{n} (\sqrt{s^2 + |\alpha_i|^2} - s) \\ \|c_1(s)\| &= \left\|2s + \sum_{i=1}^{n} \left((s^2 + |\alpha_i|^2 e_{ii})^{1/2} - s\right)\right\| \\ &= \left\|2s + \sum_{i=1}^{n} \left((s^2 + |\alpha_i|^2)^{1/2} - s\right) e_{ii}\right\| \\ &= s + \sqrt{s^2 + \max_i |\alpha_i|^2}\end{aligned}$$

We must find the minimum of the function

$$g : s \mapsto \|c_0(s)\| \, \|c_1(s)\|$$

If $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$ this is

$$g(s) = \left(s + \sqrt{s^2+1}\right)\left(2\,s + n\left(\sqrt{s^2+1} - s\right)\right)$$

with strictly positive derivative

$$g'(s) = \frac{2\,\left(s + \sqrt{s^2+1}\right)^2}{\sqrt{s^2+1}}$$

and thus the minimum is attained at $s = 0$, which yields (II.6.5). For (II.6.6) where $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 2$ we consider

$$g(s) = \left(2\sqrt{s^2+1} + \sqrt{s^2+4} - s\right)\left(s + \sqrt{s^2+4}\right)$$

which has the derivative

$$g'(s) = \frac{\left(2\,s + 2\sqrt{s^2+4}\right)\left(s\sqrt{s^2+4} + s^2 + 1\right)}{\sqrt{s^2+1}\sqrt{s^2+4}}$$

This is again strictly positive and the infimum of g is $g(0) = 8$.

Example II.6.7. The Cuntz algebra.

In [**27**, Problem 140] it is shown that the unilateral shift S on ℓ_2 cannot be approximated by invertible operators. Consider the Cuntz algebra, which is generated by n "free" copies of the shift. A priori we cannot apply theorem II.6.1 to the sum

$$\sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i S_i.$$

However since this norm is trivially equal to $\left(\sum |\alpha_i|^2\right)^{1/2}$, the inequality holds even in this case.