40. Bestimmen Sie den maximalen Definitionsbereich D und die partiellen Ableitungen erster Ordnung nach allen auftretenden Variablen im Innern B von D.

    (a) $f(x,y) = x^3 e^{xy} + y^x$

    (b) $f(x,y,z) = z\sin(xy) - y\cos(yz) + ze^{x+y}$

    (c) $f(x,y) = \dfrac{x-y}{\sqrt{x+2y}}$;

41. Die Abbildung $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ sei gegeben durch

$$\mathbf{f}(x,y,z) = \begin{pmatrix} xz + xy + yz \\ xe^y + ze^x \end{pmatrix}.$$

   Bestimmen Sie die Ableitungsmatrix $\frac{\partial(f_1,f_2)}{\partial(x,y,z)}(1,0,-1)$.

42. Die reellen Funktionen f seien in ihrem Definitionsbereich zweimal stetig differenzierbar. Berechnen Sie $\frac{\partial^2 u}{\partial x \partial y}$ für

    (a) $u = f(x^2 + y^2 + z^2)$

    (b) $u = f(x, xy, xyz)$.

43. Stellen Sie fest, dass $(1, \frac{1}{2})$ und $(2,1)$ kritische Punkte der Funktion

$$f(x,y) = x^3 - 4x^2 - 2xy + 2y^2 + 6x$$

   sind und bestimmen Sie und ihren Typ.

401

a) $f(x,y) = x^3 e^{xy} + y^x$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} x^3 e^{xy} + y^x = \frac{\partial}{\partial x} x^3 e^{xy} + \frac{\partial}{\partial x} \underbrace{y^x}_{= e^{\ln(y)x}} = 3x^2 e^{xy} + x^3 y e^{xy} + \ln(y) y^x = e^{xy}(3x^2 + x^3 y) + \ln(y) y^x$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} x^3 e^{xy} + \frac{\partial}{\partial y} y^x = x^4 e^{xy} + x y^{x-1}$$

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{pmatrix} = \begin{pmatrix} e^{xy}(3x^2 + x^3) + \ln(y) y^x \\ x^4 y e^{xy} + x y^{x-1} \end{pmatrix}$$

$D = \mathbb{R} \times \mathbb{R}_+$

b) $f(x,y,z) = z\sin(xy) - y\cos(yz) + ze^{x+y}$

$$\nabla f(x,y,z) = \begin{pmatrix} zy\cos(xy) + ze^{x+y} \\ zx\cos(xy) - \cos(yz) + yz\sin(yz) + ze^{x+y} \\ \sin(xy) + y^2 \sin(yz) + e^{x+y} \end{pmatrix}$$

$D = \mathbb{R}^3$

c) $f(x,y) = \dfrac{x-y}{\sqrt{x+2y}}$

$\frac{\partial}{\partial x} \frac{x-y}{\sqrt{x+2y}}$? Quotientenregel: $\frac{\partial}{\partial x}(x-y) = 1$ $\quad \frac{\partial}{\partial x} \sqrt{x+2y}$? Kettenregel $\frac{\partial}{\partial x} (x+2y)^{1/2} = 1 \cdot \frac{1}{2}(x+2y)^{-1/2}$

$$\Rightarrow \frac{\partial}{\partial x} \frac{x-y}{\sqrt{x+2y}} = \frac{\sqrt{x+2y} - (x-y)\frac{1}{2}(x+2y)^{-1/2}}{x+2y} = \frac{\sqrt{x+2y} - \frac{x-y}{2\sqrt{x+2y}}}{x+2y} = \frac{\frac{2(x+2y)-(x-y)}{2\sqrt{x+2y}}}{x+2y} = \frac{2(x+2y)-x+y}{(x+2y)2\sqrt{x+2y}} = \frac{x+5y}{2(x+2y)^{3/2}}$$

$\frac{\partial}{\partial y} \frac{x-y}{\sqrt{x+2y}}$: $\frac{\partial}{\partial y}(x-y) = -1$, $\frac{\partial}{\partial y} (x+2y)^{1/2} = (x+2y)^{-1/2}$

$$\Rightarrow \frac{\partial}{\partial y} \frac{x-y}{\sqrt{x+2y}} = \frac{-(x+2y)^{1/2} - (x-y)(x+2y)^{-1/2}}{x+2y} = \frac{-\frac{x+2y}{(x+2y)^{1/2}} - \frac{x-y}{(x+2y)^{1/2}}}{x+2y} = \frac{\frac{-x-2y-x+y}{(x+2y)^{1/2}}}{x+2y} = \frac{-2x-y}{(x+2y)^{3/2}} = -\frac{2x+y}{(x+2y)^{3/2}}$$

$$\nabla f(x,y) = \begin{pmatrix} \frac{x+5y}{2(x+2y)^{3/2}} \\ -\frac{2x+y}{(x+2y)^{3/2}} \end{pmatrix}$$

$D = \{(x,y) \in \mathbb{R}^2 : x+2y > 0\}$

4.1)

$$f(x,y,z)=\begin{pmatrix} xz+xy+yz \\ xe^y+ze^x \end{pmatrix}$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y,z)}(1,0,-1)$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y,z)}=\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}$$

$$\frac{\partial f_1}{\partial x}=\frac{\partial}{\partial x}\, xz+xy+yz=y+z,\quad \frac{\partial f_1}{\partial y}=\frac{\partial}{\partial y}\, xz+xy+yz=x+z,\quad \frac{\partial f_1}{\partial z}=x+y$$

$$\frac{\partial f_2}{\partial x}=\frac{\partial}{\partial x}\, xe^y+ze^x=e^y+ze^x,\quad \frac{\partial f_2}{\partial y}=\frac{\partial}{\partial y}\, xe^y+ze^x=xe^y,\quad \frac{\partial f_2}{\partial z}=e^x$$

$$\Rightarrow \frac{\partial(f_1,f_2)}{\partial(x,y,z)}=\begin{pmatrix} y+z & x+z & x+y \\ e^y+ze^x & xe^y & e^x \end{pmatrix}$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y,z)}(\overset{x}{1},\overset{y}{0},\overset{z}{-1})=\begin{pmatrix} 0-1 & 1-1 & 1+0 \\ 1-e & 1\cdot 1 & e \end{pmatrix}=\begin{pmatrix} -1 & 0 & 1 \\ 1-e & 1 & e \end{pmatrix}$$

4.2) Berechne $\frac{\partial^2 u}{\partial x \partial y}$ für

a) $u=f(x^2+y^2+z^2)$

$$\frac{\partial}{\partial x} f(x^2+y^2+z^2)=2x\, f'(x^2+y^2+z^2)$$ (Kettenregel, $u=f(v)$, $v=x^2+y^2+z^2$)

$$\frac{\partial}{\partial y}\left(2x f'(x^2+y^2+z^2)\right)=4xy\, f''(x^2+y^2+z^2)$$ (Kettenregel, $u=f'(v)$, $v=x^2+y^2+z^2$)

b) $u = f(x, xy, xyz)$

Sei $u(x,y,z) = f(x, xy, xyz) = f(a(x,y,z), b(x,y,z), c(x,y,z)) = f(g(x,y,z))$ mit $g(x,y,z) = \begin{pmatrix} a(x,y,z) \\ b(x,y,z) \\ c(x,y,z) \end{pmatrix}$

$$\Rightarrow \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = Du(x,y,z) \overset{\text{Kettenregel}}{=} Df(g(x,y,z)) \cdot Dg(x,y,z) =$$

$$= \left(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c}\right) \cdot \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial z} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial z} \end{pmatrix}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial f}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial f}{\partial b} \cdot \frac{\partial b}{\partial x} + \frac{\partial f}{\partial c} \cdot \frac{\partial c}{\partial x} = \frac{\partial f}{\partial a} \cdot 1 + \frac{\partial f}{\partial b} \cdot y + \frac{\partial f}{\partial c} \cdot yz$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b} \cdot y + \frac{\partial f}{\partial c} \cdot yz\right) =$$

$$= \left(\frac{\partial^2 f}{\partial a^2} \cdot \frac{\partial a}{\partial y} + \frac{\partial^2 f}{\partial b \partial a} \cdot \frac{\partial b}{\partial y} + \frac{\partial^2 f}{\partial c \partial a} \cdot \frac{\partial c}{\partial y}\right) \cdot 1 + 0 \cdot \frac{\partial f}{\partial a} +$$

$$+ \left(\frac{\partial^2 f}{\partial a \partial b} \cdot \frac{\partial a}{\partial y} + \frac{\partial^2 f}{\partial b^2} \cdot \frac{\partial b}{\partial y} + \frac{\partial^2 f}{\partial c \partial b} \cdot \frac{\partial c}{\partial y}\right) \cdot y + 1 \cdot \frac{\partial f}{\partial b} +$$

$$+ \left(\frac{\partial^2 f}{\partial a \partial c} \cdot \frac{\partial a}{\partial y} + \frac{\partial^2 f}{\partial b \partial c} \cdot \frac{\partial b}{\partial y} + \frac{\partial^2 f}{\partial c^2} \cdot \frac{\partial c}{\partial y}\right) \cdot yz + z \frac{\partial f}{\partial c} =$$

$$= \frac{\partial^2 f}{\partial a^2} \cdot 0 + \frac{\partial^2 f}{\partial b \partial a} \cdot x + \frac{\partial^2 f}{\partial c \partial a} \cdot xz + 0 \cdot \frac{\partial f}{\partial a} +$$

$$+ \left(\frac{\partial^2 f}{\partial a \partial b} \cdot 0 + \frac{\partial^2 f}{\partial b^2} \cdot x + \frac{\partial^2 f}{\partial c \partial b} \cdot xz\right) \cdot y + \frac{\partial f}{\partial b} +$$

$$+ \left(\frac{\partial^2 f}{\partial a \partial c} \cdot 0 + \frac{\partial^2 f}{\partial b \partial c} \cdot x + \frac{\partial^2 f}{\partial c^2} xz\right) \cdot yz + z \cdot \frac{\partial f}{\partial c} =$$

$$= \frac{\partial f}{\partial b} + z \cdot \frac{\partial f}{\partial c} + \frac{\partial^2 f}{\partial b \partial a} x + \frac{\partial^2 f}{\partial c \partial a} xz + \frac{\partial^2 f}{\partial b^2} xy + \frac{\partial^2 f}{\partial c \partial b} \cdot 2xyz + \frac{\partial^2 f}{\partial c^2} xyz^2$$

43) $f(x,y) = x^3 - 4x^2 - 2xy + 2y^2 + 6x$. z.z.: $(1, \frac{1}{2})$ und $(2,1)$ sind kritische Punkte von f. Bestimme Typ

Kritischer Punkt $P = (p_x, p_y) \rightsquigarrow \nabla f(p_x, p_y) = 0$

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 - 8x - 2y + 6 \\ -2x + 4y \end{pmatrix} \quad (= \text{Gradient von } f(x,y))$$

Um Typ zu bestimmen brauchen wir die Hessematrix:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x - 8 & -2 \\ -2 & 4 \end{pmatrix}$$

$P_1 = (1, \frac{1}{2})$

$$\nabla f(1, \tfrac{1}{2}) = \begin{pmatrix} 3 - 8 \cdot 1 + 6 \\ -2 + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark \quad (1, \tfrac{1}{2}) \text{ ist ein kritischer Punkt von } f(x,y)$$

Klassifizierung:

$$\text{Sei } \Delta_2 = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \quad (\text{Skript S. 157, Satz 5.7.4})$$

1) $\Delta_2 > 0 \Rightarrow$ Extremum
   a) $\frac{\partial^2 f}{\partial x^2} > 0 \Rightarrow$ Minimum
   b) $\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow$ Maximum
2) $\Delta_2 < 0 \Rightarrow$ Sattelpunkt
3) $\Delta_2 = 0 \Rightarrow$ keine Aussage

$$\Delta_2 = (6x - 8) \cdot 4 - (-2)^2 = 24x - 36$$

$$\Delta_2\big|_{(1, \frac{1}{2})} = 24 - 36 < 0 \Rightarrow \text{Sattelpunkt in } P_1 = (1, \tfrac{1}{2})$$

$$\nabla f(2,1) = \begin{pmatrix} 3 \cdot 2^2 - 8 \cdot 2 - 2 \cdot 1 + 6 \\ -2 \cdot 2 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow P_2 = (2,1) \text{ ist ein kritischer Punkt von } f(x,y)$$

$$\Delta_2 = 24x - 36$$

$$\Delta_2\big|_{(2,1)} = 12 > 0 \Rightarrow \text{Extremum}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x - 8 = 12 - 8 = 4 > 0 \Rightarrow \text{Minimum in } P_2$$