Tight Euclidean t-designs and tight relative t-designs in certain association schemes

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This talk is based on the following three papers.

(1) Eiichi Bannai, Etsuko Bannai, Sho Suda and Hajime Tanaka:

On relative t-designs in polynomial association schemes, arXiv:1303.7163

(2) Eiichi Bannai, Etsuko Bannai and Hideo Bannai:

On the existence of tight relative 2-designs on binary Hamming association schemes, Discrete Mathematics, 314 (2014), 17–37.

(3) Yan Zhu, Eiichi Bannai, Etsuko Bannai:

On tight relative 2-designs on two shells on the Johnson association schemes (preprint).

The purpose of Design Theory is to find "good" finite subsets which approximate the whole space M.

Spherical t-designs
($M =$ the sphere S^{n-1})

Combinatorial t-designs
$(M = \binom{V}{k}\ (= S_v/S_k \times S_{v-k}))$

Euclidean t-designs
$(M = S^{n-1}(r_1) \cup \cdots \cup S^{n-1}(r_p))$

Relative t-designs
$(M = \binom{V}{k_1} \cup \binom{V}{k_2} \cup \cdots \cup \binom{V}{k_p})$

Spherical *t*-designs (Delsarte-Goethals-Seidel,1977)

Let $t \in \mathbb{N}$ and $S^{n-1} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$.
Then, $Y \subset S^{n-1}, |Y| < \infty$, is called a spherical t-design on S^{n-1}, if and only if

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)\, d\sigma(x) = \frac{1}{|Y|}\sum_{x\in Y} f(x)$$

for any polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree $\leq t$.
Here, $|S^{n-1}|$ denotes the area of the sphere S^{n-1}, and the integral is the surface integral on S^{n-1}.

This condition is equivalent to the condition:

$$\sum_{x\in Y} f(x) = 0, \quad \text{for all } f(x) \in \mathrm{Harm}_i(\mathbb{R}^n), (1 \leq i \leq t),$$

where $\mathrm{Harm}_i(\mathbb{R}^n)$ is the space of homogeneous harmonic polynomials of degree i.

It is known that spherical t-designs do exist for any t and for any n. (Seymour-Zaslavsky, 1984). However, the explicit constructions of spherical t-designs for large t are not easy in general.

Combinatorial t-designs (t-(v, k, λ) designs)

Let $1 \le t \le k \le v$ be all natural numbers. Let V be a set with $|V| = v$, and let $V^{(k)} = \binom{V}{k}$ be the set of all k-element subsets of V.
A subset $Y (\subset V^{(k)})$ is called a t-design (or t-(v, k, λ) design), if and only if

$$|\{y \in Y \mid z \subset y\}| = \lambda \text{ (constant)}, \quad \text{for all } z \in V^{(t)}.$$

(In what follows, we consider only nontrivial t-designs Y, i.e. we assume $Y \neq V^{(k)}$).

It is known that non-trivial t-designs exist *for any* t, and for some v, k, and λ (Teirlinck, 1987). Recently, further existence theorems are known for more general t, v, k, λ, by the probabilistic method.

Delsarte (1973) gave an equivalent algebraic definition of combinatorial t-designs.

Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le k})$ be the Johnson association scheme $J(v, k)$, with $X = V^{(k)}$. Then, a subset $Y \subset X$ is a t-design in $J(v, k)$ if and only if,

$$E_i \phi_Y = 0, \text{for all } i, (1 \le i \le t),$$

where E_i is the i-th primitive idempotent, and ϕ_Y is the characteristic vector of Y, defined by

$$\phi_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

Weighted spherical t-designs and weighted combinatorial t-designs

We can consider weighted designs, for spherical designs and for combinatorial t-designs.

Let $Y \subset S^{n-1}$, and let

$$w : Y \longrightarrow \mathbb{R}_{>0}$$

be a weight function. Then the pair (Y, w) is called a weighted spherical t-design, if and only if:

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x) = \sum_{x \in Y} w(x) f(x)$$

for any polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree $\leq t$.

Similarly, let Y be a subset of $X = V^{(k)}$ of the Johnson association schemes $J(v,k)$. Let

$$w : Y \longrightarrow \mathbb{R}_{>0}$$

be a weight function. Then the pair (Y, w) is called a weighted combinatorial t-design, if

$$E_i \phi_{(Y,w)} = 0, \text{for all } i (1 \leq i \leq t),$$

where $\phi_{(Y,w)}$ is defined by:

$$\phi_{(Y,w)}(x) = \begin{cases} w(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

Fisher type lower bounds for spherical t-designs and combinatorial t-designs

(For simplicity, here we consider only the case $t = 2e$.)

(Delsarte-Goethals-Seidel, 1977) If Y is a $2e$-design in S^{n-1}, then

$$|Y| \geq \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1}$$
$$= m_e + m_{e-1} + \cdots + m_1 + m_0,$$

where $m_i = \dim(\mathrm{Harm}_i(\mathbb{R}^n)) = \binom{n-1+i}{i} - \binom{n-1+i-2}{i-2}$.

Y is called a **tight spherical t-design** if the equality " $=$" holds in the above inequality.

Remarks. (i) Exactly the same lower bound holds for weighted spherical t-designs (Y, w) on S^{n-1}.

(ii) If (Y, w) is a tight weighted t-design on S^{n-1}, then X is a tight spherical t-design on S^{n-1}, i.e., w is a constant function.

Similarly, a Fisher type lower bound is given for combinatorial t-designs. Here, we assume $t = 2e$.

(Ray-Chaudhuri and Wilson, 1975) If $Y (\subset X = V^{(k)})$ is a non-trivial combinatorial $2e$-design in $J(v, k)$, then

$$|Y| \geq \binom{v}{e} = m_e + m_{e-1} + \cdots + m_1 + m_0,$$

where $m_i =$ rank of $E_i = \binom{v}{i} - \binom{v}{i-1}$, is the degree of the irreducible representation of S_v corresponding to the Young diagram of type $(n-i, i)$.

A $2e$-design Y is called a tight $2e$-design if

$$|Y| = \binom{v}{e}.$$

Similarly as in the weighted spherical design case, we have:

Remarks. (i) Exactly the same lower bound (as combinatorial $2e$-designs) holds for weighted combinatorial $2e$-designs (Y, w) on $J(v, k)$.

(ii) If (Y, w) is a tight weighted $2e$-design on $J(v, d)$, then Y is a tight combinatorial $2e$-design in $J(v, k)$, i.e., w is a constant function.

Classification of tight spherical t-designs.

Tight spherical t-designs are classified except for $t = 4, 5, 7$.
Delsarte-Goethals-Seidel (1977),
Bannai-Damerell (1979/80),
Bannai-Sloane (1981).

For $t = 4, 5, 7$, the classification is still **open**.
Cf. Bannai-Munemasa-Venkov (Algebra i Analiz, 2004), and
Nebe-Venkov (Algebra i Analiz, 2012).

Classification of tight combinatorial $2e$-designs.

$e = 1 \Longrightarrow$ There are many examples (symmetric 2-designs) and the classification is hopeless.

$e = 2 \Longrightarrow$ They are 4-$(23, 7, 1)$ design and 4-$(23, 16, 52)$ design (Enomoto-Ito-Noda, and Bremner, 1979)

No non-trivial tight $2e$-designs ($e \geq 3$) are known.

$e = 3 \Longrightarrow \nexists$ (Peterson, 1977)

$4 \leq e \Longrightarrow$ There are only finitely many tight $2e$-designs for each fixed e. (Bannai, 1977)

$5 \leq e \leq 9 \Longrightarrow \nexists$ (Dukes and Short-Gershman, J. Algebraic Comb., 2013)

$e = 4 \Longrightarrow \nexists$ (Z. Xiang, unpublished recent work).

The complete classification for $e \geq 10$ is still open.

Euclidean t-design is a two step generalization of spherical t-designs (Neumaier-Seidel, 1988)

Step 1. Allow weight function $w : Y \longrightarrow \mathbb{R}_{>0}$. Namely,

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x) = \sum_{x\in Y} w(x)f(x)$$

for any polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree up to t.

Step 2. Allow several shells of spheres (p shells).
Let $Y \subset S^{n-1}(r_1) \cup S^{n-1}(r_2) \cup \cdots \cup S^{n-1}(r_p)$, and let $w : Y \longrightarrow \mathbb{R}_{>0}$. Then (Y, w) in a Euclidean t-design (on p shells), if and only if

$$\sum_{i=1}^{p} \frac{W(Y \cap S^{n-1}(r_i))}{|S^{n-1}(r_i)|}\int_{S^{n-1}(r_i)} f(x)d\sigma_i(x) = \sum_{x\in Y} w(x)f(x)$$

for any polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree up to t.

Fisher type lower bound for Euclidean t-designs.

Here we assume $t = 2e$ for simplicity.
If (Y, w) is a Euclidean $2e$-design on p shells, then we have

$$|Y| \geq \overline{m}_e + \overline{m}_{e-1} + \cdots + \overline{m}_{e-p+1},$$

where $\overline{m}_i = \binom{n-1+i}{i} + \binom{n-1+i-1}{i-1} (= \sum_{j=0}^{i} m_j)$.
(Note: formula for odd t is more complicated.)

There are many interesting examples of tight Euclidean t-designs.
For example, $t = 6,\ n = 22,\ p = 2,\ Y = Y_1 \cup Y_2$,
with $Y_1 \cong \mathrm{McL}/U_4(3)$ $(|Y_1| = 275)$, $Y_2 \cong \mathrm{McL}/M_{22}$ $(|Y_2| = 2025)$.

We tried to work on the classification problems of tight Euclidean t-designs, in particular on small number of shells (e.g. for $p = 2$.)
(See, Bannai-Bannai, many papers, 2005-present).

Relative t-design is a two step generalization of combinatorial t-designs.

There are two kinds of relative t-designs, one using the P-structure, and another one using the Q-structure. Cf. Bannai-Bannai-Suda-Tanaka: arXiv:1303.7163

But here we only consider the one using the Q-structure.

Relative t-design (for Q-polynomial association schemes.)

(Delsarte, Pairs of vectors in association schemes, 1977)

Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme.
Let $u_0 \in X$ (fixed), Y be a subset of X, and $w : Y \longrightarrow \mathbb{R}_{>0}$.
We defined

$$\phi_{(Y,w)}(x) = \begin{cases} w(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

(Then $\phi_{(Y,w)}$ is identified as a column vector.)

• (Delsarte, 1973) Recall that (Y, w) is called a weighted t-design in Q-polynomial association scheme $\mathfrak{X}$, if and only if

$$E_i \phi_{(Y,w)} = 0, \text{ for } i = 1, 2, \ldots, t.$$

• (Delsarte, 1977) (Y, w) is called a relative t-design (for Q-structure) in $\mathfrak{X}$ w.r.t. a fixed point $u_0 (\in X)$, if and only if $E_i \phi_{(Y,w)}$ and $E_i \phi_{u_0}$ are linearly dependent, for $i = 1, 2, \ldots, t$.

This definition of relative t-design (for Q-structue) is equivalent to the following condition:
$Y \subset X_{\nu_1} \cup X_{\nu_2} \cup \cdots \cup X_{\nu_p}$, and

$$\sum_{i=1}^{p} \frac{W(Y \cap X_{\nu_i})}{|X_{\nu_i}|} \sum_{x \in X_{\nu_i}} f(x) = \sum_{x \in Y} w(x) f(x)$$

for any

$$f \in L_0(X) + L_1(X) + \cdots + L_t(X),$$

where L_j is the space spanned by the column vectors of E_j.

It is shown (B-B, 2012) that if (Y, w) is a relative $2e$-design (for Q-structure), then

$$|Y| \geq \dim(L_0(S) + L_1(S) + \cdots + L_e(S)),$$

where $S = X_{\nu_1} \cup X_{\nu_2} \cup \cdots \cup X_{\nu_p}$.

We remark that it is not easy to calculate the RHS of the above inequality explicitly. We conjecture that the RHS of the above inequality is equal to

$$m_e + m_{e-1} + \cdots + m_{e-p+1} \quad \text{(in most cases)},$$

where m_i = rank of E_i.

(True for $H(n, 2)$ (Z. Xiang, 2012). True for some other cases (cf. B-B-Suda-Tanaka), but the general case is still open.)

Remark. It is known that a relative t-design (Y, w) in $H(n, 2)$ is equivalent to a weighted regular t-wise balanced design (Y, w), namely,

$$Y \subset X_{\nu_1} \cup X_{\nu_2} \cup \cdots \cup X_{\nu_p},$$

and

$$\sum_{y \in Y, z \subset y} w(y) = \lambda_i, \text{ for all } z \in V^{(i)}, \text{for all } i \ (1 \leq i \leq t).$$

(namely, blocks are allowed to have different sizes.)

Association schemes and coherent configurations

The following facts are well known.

For any tight spherical t-design and for any tight combinatorial t-design, the structure of an association scheme is always attached. (Delsarte-Goethals-Seidel (1977), and Delsarte (1973).)

Then, how about for tight Euclidean t-designs and for tight relative t-designs? We obtained the following results so far.

(i) (B-B) For any tight Euclidean t-design (Y, w), the weight function w is constant on each shell.
(ii) (B-B) For any tight Euclidean t-design on 2 shells, a coherent configuration is always attached, although this is not true in general if $p \geq 3$.

Question: How about for tight relative t-designs?

Answers:

(i) The weight function is constant on each shell for any tight relative t-design in a wide class of Q-polynomial association schemes. In particular, this is true if G_{u_0} is transitive on each shell X_i of the Q-polynomial association scheme.

(ii) True for the following very special case:
tight relative 2-designs (Y, w) on 2 shells in $H(n, 2)$.

This implies that a coherent configuration (of very special type) is attached to such a relative 2-design on 2 shells in $H(n,2)$.

Review of B-B-B (2014) on tight relative 2-designs on two shells on $H(n,2)$

(a) We could find many such examples, starting from known (combinatorial) symmetric $2-(n+1,k,\lambda)$ designs. (Woodall, 1970). Such examples have $|Y| = n+1$ and the weight function w must be constant.

(b) We found the following new family of examples with $w_2 \neq w_1$.

(So far, these are the only known examples of such tight relative 2-designs on two shells with $w_2 \neq w_1$.)

Let $m \equiv -1 \pmod 4$ and suppose that there exists an Hadamard matrix of size $m+1$ by $m+1$. Then we can construct a tight relative 2-design Y (on 2 shells in $H(n,2)$) with the following parameters.

$$n = 2m, \nu_1 = 2, \nu_2 = m, N_1 = |Y \cap X_2| = m, N_2 = |Y \cap X_m| = m+1, |Y| = n+1$$

with

$$\frac{w_2}{w_1} = \frac{8}{n+2}.$$

So, we get examples (with non-constant weight function) for $n = 14, 22, 30, 38, \ldots$

(c) We classified tight relative 2-designs on two shells on $H(n, 2)$ with $n \leq 30$. (This result was recently generalized by Hong Yue, a student at Hebei Normal University, for $n \leq 50$.)

There are many examples of tight relative 2-designs in $H(n, 2)$, similarly as there are many symmetric combinatorial 2-designs. However, it seems that tight relative $2e$-designs $e \geq 2$ in $H(n, 2)$ are very rare.

I believe there is a good hope that we can approach the classification of tight relative t-designs, as we have studied the classification problem (although it is not yet complete) of tight combinatorial t-designs.

Methods!

In Euclidean t-design case of (Y, w) on p shells in $\mathbb{R}^n$, we can find a good explicit orthonormal basis of the space of polynomials of degree at most e (if $t = 2e$) restricted to the union of shells

$$S = S^{n-1}(r_1) \cup S^{n-1}(r_2) \cup \cdots \cup S^{n-1}(r_p),$$

with respect to the inner product

$$\langle f, g \rangle = \sum_{i=1}^{p} \frac{W(Y \cap S^{n-1}(r_i))}{|S^{n-1}(r_i)|} \int_{S^{n-1}(r_i)} f(x)g(x)d\sigma_i(x)$$

In relative t-design case of (Y, w) on p shells in a Q-polynomial association scheme, we need to find an explicit orthonormal basis of the space

$$L_0(S) + L_1(S) + \cdots + L_e(S)$$

with

$$S = X_{\nu_1} \cup X_{\nu_2} \cup \cdots \cup X_{\nu_p},$$

with respect to the inner product

$$\langle f, g \rangle = \sum_{i=1}^{p} \frac{W(Y \cap X_{\nu_i})}{|X_{\nu_i}|} \sum_{x \in X_{\nu_i}} f(x)g(x).$$

But finding the explicit formula for this orthonormal basis is very difficult for large e.

Our ongoing work (Yan Zhu, and B-B.)

We are trying to obtain similar results (as in $H(n,2)$) for tight relative 2-designs on two shells in $J(v,k)$.

(a) We could solve the representation part of the problem (although the obtained formulas are very complicated, compared with $H(n,2)$ case).

(b) So, we can list possible parameters for the existence of tight relative 2-designs on two shells on $J(v,k)$, for small v, say $v \leq 100$.
(In all these small cases, we have shown that coherent configuration is attached, but we are not yet successful in proving this in general.)

More on (tight) relative t-designs

Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be $H(n, 2)$, $J(v, k)$,
or more generally some other Q-polynomial scheme.

Facts

(i) $Y \subset X$ is a t-design $\Longrightarrow$ Y is a relative t-design
w.r.t. any $u_0 \in X$.

(ii) Y is a relative t-design (w.r.t. u_0) and $Y \subset X_l$
(where $X_l = \{x \in X \mid (u_0, x) \in R_l\}$)

$\Longrightarrow$ Y is a t-design of X_l.

$\mathfrak{X} = H(n,2) \Longrightarrow X_l \approx J(n,l)$
$\mathfrak{X} = J(v,k) \Longrightarrow X_l \approx J(k,l) \times J(v-k,l)$

$Y =$ tight relative 2-design on one shell X_l in $H(n,2)$
$\Longrightarrow Y =$ symmetric 2-design of $J(n,l)$ (many examples)

$Y =$ tight relative 2-design on one shell X_l in $J(v,k)$
$\Longrightarrow Y =$ tight 2-design of $J(k,l) \times J(v-k,l)$
(they do not seem to exist in general, cf. W. Martin (1998))

So, we are interested in tight relative 2-designs on two shells (in $H(n,2)$, in $J(v,k)$, and in other cases).

More special tight relative t-design on two shells in $J(v, k)$.

Let Y be a tight 2-design of $J(v, k)$.
(i.e. a symmetric 2-(v, k, λ) design.)
Then Y is a tight relative 2-design of $J(v, k)$ w.r.t. ${}^{\forall}u_0 \in X$.

If there is a $u_0 \in X$ such that $Y \subset X_{r_1} \cup X_{r_2}$
(with respect to u_0),
then Y is a tight relative 2 -design on 2 shells in $J(v, k)$.

So the problem is reduced to the following problem.

Problem:

Find a symmetric 2-(v, k, λ) design (V, Y)
such that there exists a k element subset $K \notin Y$
with the following property $(*)$:

$(*)$ $\{|K \cap y| \mid y \in Y\}$ consists of exactly 2 values (say $k - r_1$ and $k - r_2$).

We say such a symmetric 2-(v, k, λ) design, admissible or has the property $(*)$.

This problem has been studied extensively without the assumption that $|K| = k$ in finite geometries.
(cf. Calderbank-Kantor(1986), Martin (2001), etc.)

Our observation

If a symmetric 2-(v, k, λ) design has the property $(*)$ and $v \leq 100$, then the only possible cases are as follows

$$\begin{aligned}&2\text{-}(16, 6, 2)\\&2\text{-}(36, 15, 6)\\&2\text{-}(45, 12, 3)\\&2\text{-}(64, 28, 12)\\&2\text{-}(96, 20, 4)\\&2\text{-}(100, 45, 20)\end{aligned}$$

This is obtained from the possible list of tight relative 2-design on 2 shells in $J(v, k)$.

2-$(16, 6, 2)$ designs: 2 of them (out of 3) have the property $(*)$.
2-$(36, 15, 6)$ designs: E. Spence's homepage gives many such designs.

- 31 (out of 32548 coming from SRGs of type $(36, 15, 6, 6)$)

- 178 (out of 180 coming from SRGs of type $(36, 21, 12, 12)$) have the property $(*)$.

- 339 (out of 617 coming from (known) designs with polarities with absolute points 6, 12, 18, 24, 30)) have the property $(*)$. (It seems that those designs with polarity as well as those without polarity are not classified.)

Similar results for 2-$(45, 12, 3)$ designs. Namely, 29 (out of 78 coming from SRGs of type (45,12,3,3) have the property $(*)$.

Z. Xiang gave a very fast program to check this. For each given symmetric design of v at most around 100, about one hour is enough to check this. We also learned that Ziv-Av also did similar calculations before, (through W. Martin and M. Klin.)

Fact

Menon symmetric designs $2\text{-}(4u^2, 2u^2-u, u^2-u)$

(Such symmetric designs are conjectured to exist for all $u \geq 2$.)

We conjecture that there should exist those Menon symmetric designs with the property $(*)$ for all $u \geq 2$..

True for $u = 2^m$ (from linked symmetric designs, cf. Kerdock codes and/or Cameron-Seidel construction), but not yet shown for general u.

Other candidates (Cf. Wallis, 1971)

$2\text{-}(q^2(q+2), q(q+1), q)$ designs ($q =$ prime power)

$$\begin{aligned}
&2\text{-}(16,6,2), && q=2,\\
&2\text{-}(45,12,3), && q=3,\\
&2\text{-}(96,20,4), && q=4,\\
&2\text{-}(175,30,5), && q=5, \quad \text{etc.}
\end{aligned}$$

Finally, we remark that some examples of tight relative 2-designs on two shells in $H(n,2)$ are constructed from tight 2-design in $H(n,2)$ (i.e., Hadamard matrices of size $n+1$.) Namely, such examples in $H(n,2)$ (with $n = 4u^2 - 1$) are constructed, if there exists a Menon symmetric design 2-$(4u^2, 2u^2 - u, u^2 - u)$.)

Conclusion:

We believe the time is ripe to study tight relative t-designs in Q-polynomial association schemes more systematically, in particular of those on two shells.

Thank You