Tight Euclidean t-designs on two concentric spheres.

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1. **Definition of Euclidean t-designs.**

2. **Known facts on Euclidean t-designs and definition of tightness.**

3. **The classification problem of tight $(2e+1)$-design on two concentric spheres.**

Let $X \subset \mathbb{R}^n$ be a finite point set and $w : X \longrightarrow \mathbb{R}_{>0}$. be a weight function. Let X be on p concentric spheres $S_1 \cup \cdots \cup S_p$ centered at the origin.

Definition(Neumaier-Seidel, 1988)
A pair (X, w) is a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{S_i} f(x) d\sigma_i(x) = \sum_{x \in X} w(x) f(x)$$

for any polynomial $f(x)$ of degree at most t, where $w(X_i) = \sum_{x \in X_i} w(x)$.

One of the spheres could be $S_i = \{0\}$, with $\frac{1}{|S_i|} \int_{S_i} f(x) d\sigma_i(x) = f(0)$.

Remarks:

- $p = 1$, $X \neq \{0\}$, $w(x) \equiv 1 \quad \Longrightarrow$ Spherical t-designs.
- Assume $0 \notin X$.
  Then (X, w) is a Euclidean t-design if and only if $(X \cup \{0\}, w)$ is a Euclidean t-design ($w(0)$ can be any positive real number).

Equivalent definition

The following are equivalent.

(1) (X, w) is a Euclidearn t-design.

(2) $\sum_{x \in X} w(x) f(x) = 0$ for any $f \in \|x\|^{2j} \mathrm{Harm}_l(\mathbb{R}^n)$ with $1 \leq l \leq t$ and $2j + l \leq t$.

Natural lower bounds

Theorem (Möller 1976)

(Original theorem was given interms of general cubature formula)

$\mathbb{R}^n \supset X$: a finite set

1. X: Euclidean $2e$-design $\Longrightarrow |X| \geq \dim(\mathcal{P}_e(\mathbb{R}^n)|_S)$.
2. X: Euclidean $(2e+1)$-design.
   (a) e odd, or e even and $0 \notin X \Longrightarrow |X| \geq 2\dim(\mathcal{P}_e^*(S))$.
   (b) e even and $0 \in X \Longrightarrow |X| \geq 2\dim(\mathcal{P}_e^*(S)) - 1$,

where $\mathcal{P}_e(\mathbb{R}^n) = \oplus_{i=0}^{e} \mathrm{Hom}_i(\mathbb{R}^n)$,
$\mathcal{P}_e^*(\mathbb{R}^n) = \oplus_{i=0}^{[\frac{e}{2}]} \mathrm{Hom}_{e-2i}(\mathbb{R}^n)$, and $S = S_1 \cup \cdots \cup S_p$ (support of X).

Definition of Tight designs

If " $=$ " holds then (X, w) is a tight (Euclidean) t-design on p concentric spheres in $\mathbb{R}^n$

In particular tight Euclidean $(2e+1)$-designs are always antipodal.

Moreover if

(1) $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$ (for $t = 2e$),

or

(2) $\dim(\mathcal{P}^*_e(S)) = \dim(\mathcal{P}^*_e(\mathbb{R}^n))$ (for $t = 2e+1$)

holds, then (X, w) is a (Euclidean) tight t-design of $\mathbb{R}^n$

(If $p \geq [\frac{t}{4}] + \epsilon$ (where $\epsilon = 0$ or 1),
then (1) and (2) are always satisfied.)

Explicit formula of the dimensions of the polynomial spaces. $(S = S_1 \cup \cdots \cup S_p)$

$$\dim(\mathcal{P}_e(\mathbb{R}^n)) = \sum_{i=0}^{e} \dim(\mathrm{Hom}_i(\mathbb{R}^n)) = \binom{n+e}{e}$$

If $p \leq [\frac{e+\varepsilon_S}{2}]$,

$$\dim(\mathcal{P}_e(S)) = \varepsilon_S + \sum_{i=0}^{2(p-\varepsilon_S)-1} \binom{n+e-i-1}{e-i}$$

If $p \geq [\frac{e+\varepsilon_S}{2}] + 1$,

$$\dim(\mathcal{P}_e(S)) = \binom{n+e}{e}$$

$$\dim(\mathcal{P}_e^*(\mathbb{R}^n)) = \sum_{i=0}^{[\frac{e}{2}]} \binom{n+e-2i-1}{e-2i}$$

If $p \geq [\frac{e}{2}+1]$:

$$\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$$

If $p \leq [\frac{e}{2}]$ and $0 \notin S$:

$$\dim(\mathcal{P}_e^*(S)) = \sum_{i=0}^{p-1} \binom{n+e-2i-1}{e-2i}$$

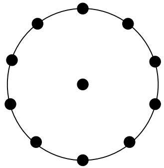
If $p \leq [\frac{e}{2}]$, e is even and $0 \in S$:

$$\dim(\mathcal{P}_e^*(S)) = 1 + \sum_{i=0}^{p-2} \binom{n+e-2i-1}{e-2i}.$$

Example: tight 9-design on p concentric spheres in $\mathbb{R}^2$

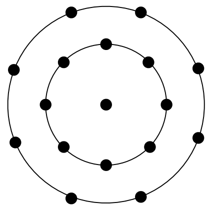
Case $0 \in X$

regular 10-gon and 0



$t = 9$, $p = 2$

$|X| = 11$



regular 8-gons and 0

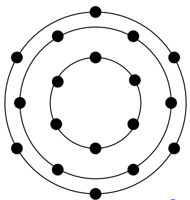
$t = 9$, $p = 3$

$|X| = 17$

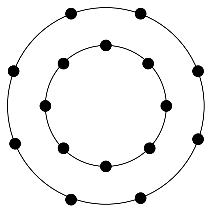
tight 9-design of $\mathbb{R}^2$

Case $0 \notin X$

hexagons
$t = 9$,
$p = 3$
$|X| = 18$



tight 9-design of $\mathbb{R}^2$



regular 8-gons
$t = 9$, $p = 2$
$|X| = 16$

Ratio of the radii can be any real number $\neq 1$, and the weight is constant on each circle and the weights are $w_i = w/(nr_i^2)$ for any $w > 0$.

Known results for the classification of tight Euclidean t-designs

- $n = 2$:

Verlinden-Cools (1992)
Bajnok (2006)
B-B-Hirao-Sawa (2010)
} ← those with $p \le [\frac{t}{2}] + 1$ are completely described

- $t = 2$:

B-B-Suprijanto (2007, Europ. J. Comb)

- $t = 3$:

B (2005, J. Alg. Comb.)
} ← completely described

- We cannot expect the complete classification for $t \ge 4$ in general, if p is not too small, since many deformations (non-rigidity) are usually possible (B-B-Suprijanto, 2007)

So, here we mainly study the cases where $t \ge 4$ and $p = 2$ (or $p =$ small)

Known results (continued)

- $t = 5,\ p = 2:$ B (2006, Europ. J. Comb)
  completely described modulo classification
  of tight spherical 4-designs
- $t = 7,\ p = 2:$ B-B (2009, ALM vol 8), completely described
- $t = 9, p = 2:$ B-B (2011, J. Math. Soc. Japan) completely described,
  i.e., non-existence for $n \geq 3$.
- $t = 4,\ p = 2:$ B (2009, Europ J. Comb),
  several examples and partial classification
  B-B (2010, Contemporary Mathematics)
  further partial classification
- $t = 6,\ p = 2:$ B-B-Shigezumi (2012 Ann. Comb.),
  one new example with $n = 22$ and
  $|X| = 275(\mathrm{McL}/U_4(3)) + 2025(\mathrm{McL}/M_{22})$

For $p \geq 3$ (and $t \geq 4$), some sporadic examples are known.

- $p = 3,\ t = 7, n = 3,\ |X| = 26:$ **Bajnok (2007, J. Alg. Comb.)**
- $p = 3,\ t = 5, n = 4,\ |X| = 22:$ **Hirao-Sawa-Zhu (2011, JCT(A))**

Non rigidity of Euclidean designs

A Euclidean t-design $X = (\{x_i\}_{i=1}^{N}, w)$ is non-rigid

$\Longleftrightarrow$

For any $\varepsilon > 0$ there exists another Euclidean t-design $X' = (\{x'_i\}_{i=1}^{N}, w') \subset \mathbb{R}^n$ satisfying the following 2 conditions:

1. $\|x_i - x'_i\| < \varepsilon$ **and** $|w(x_i) - w(x'_i)| < \varepsilon$ **for** $1 \leq i \leq N$.

2. **There is no transformation** $g \in O^*(\mathbb{R}^n)$ **satisfying** $g(x_i) = x'_i$ **for** $1 \leq i \leq N$.

Here $O^*(\mathbb{R}^n) = \langle O(\mathbb{R}^n), g_\lambda, g^\mu \rangle$

$g_\lambda : (X, w) \longrightarrow (X', w')$ **(a scaling)**

$x \longmapsto x' = \lambda x$, $w'(x') = w(x)$.

$g^\mu : (X, w) \longrightarrow (X', w')$ **(an adjustment of** w**)**

$x \longmapsto x' = x$, $w'(x') = \mu w(x)$.

Strong non rigidity of Euclidean designs

A Euclidean t-design $X = (\{x_i\}_{i=1}^N, w)$ is strongly non-rigid

$$\iff$$

For any $\varepsilon > 0$ there exists another Euclidean t-design $X' = (\{x_i'\}_{i=1}^N, w') \subset \mathbb{R}^n$ satisfying the following 2 conditions:

1. $\|x_i - x_i'\| < \varepsilon$ **and** $|w(x_i) - w(x_i')| < \varepsilon$ **for** $1 \leq i \leq N$.
2. **There exist distinct** i, j **satisfying** $\|x_i\| = \|x_j\|$ **and** $\|x_i'\| \neq \|x_j'\|$

Note: strong non-rigid $\Longrightarrow$ non-rigid.

The transformation given above is not in $O^*(\mathbb{R}^n)$.

Any tight spherical $2e$-design is rigid as a Euclidean design for $e \geq 2$.

The following tight Euclidean t-designs are non-rigid.

1. **Tight spherical 2-design on S^{n-1} considered as tight Euclidean 2-design. ($\Rightarrow$ tight 2-design on $p = 2, 3, \ldots, n+1$ spheres)**
2. **Tight spherical 3-design on S^1 considered as tight Euclidean 3-design. ($\Rightarrow$ tight 3-design on 2 spheres)**
3. **Tight Euclidean 4-designs in $\mathbb{R}^2$ supported by 2 concentric spheres. ($\Rightarrow$ tight 4-design on 3 or 4 spheres)**
4. **Tight Euclidean 5-designs in $\mathbb{R}^2$ supported by 2 concentric spheres.($\Rightarrow$ tight 5-design on 3 or 4 spheres)**

Basic properties of tight Euclidean t-designs on p-concentric spheres.

1. **Weight w is constant on each shell $X_i = X \cap S_i$, $1 \leq i \leq p$.**
2. **If $t = 2e$, then each shell is at most an e-distanse set.**
3. **If $t = 2e+1$, then X is antipodal (i.e. $x \in X \Rightarrow -x \in X$), then X_i is at most an $(e+1)$-distanse set**

   and X_i^* is at most an e-distanse set where X_i^* is the set of representative of the antipodal pairs in X_i.

The "antipodal" property in 3 was proved by Möller for cubature formula with centrally symmetric measure.

Tight t-design on 2 concentric spheres in $\mathbb{R}^n$.

$X = X_1 \cup X_2$. We assume $X_i \neq \{0\}$.

For X_1, X_2, we define

$$A(X_i) := \left\{ \frac{x \cdot y}{\|x\|\|y\|} \;\middle|\; x, y \in X_i,\ x \neq y \right\} \text{ for } i = 1,\ 2$$

$$A(X_1, X_2) := \left\{ \frac{x \cdot y}{\|x\|\|y\|} \;\middle|\; x \in X_1, y \in X_2 \right\} \ (= A(X_2, X_1)),$$

Remark:

If $p = 2$, each X_i is a spherical $(t-2)$-design.

Each X_i is associated with a Q-polynomial association scheme. Moreover X has the structure of a coherent configuration.

(Relations are defined by distances between the points $x, y \in X_i$ or $x \in X_1, y \in X_2$.)

Classification problem of tight Euclidean t-designs on 2-concentric spheres.

The followings hold.

I. **For $t = 2e + 1 \le 7$, complete classification is given.**

II. **If $n \ge 3$, there is no tight 9-design on 2-concentric spheres in $\mathbb{R}^n$ with positive radii.**

III, **For each $n \ge 3$ and $2e + 1 \ge 13$, there are only finitely many tight $(2e + 1)$-designs on 2-concentric spheres in $\mathbb{R}^n$ with positive radii.**

IV. **The case for $t = 11$ is still open.**

- **We want to prove that there are no tight $(2e + 1)$-designs on 2-concentric spheres if $n \ge 3$ and $2e + 1 \ge 11$.**

The case, $t = 2e$, the problem seems much more difficult.

Key Lemma we use for the classification problem

Let (X, w) be a tight $(2e+1)$-design on 2 concentric spheres in $\mathbb{R}^n$. Assume $0 \notin X$.

Then the followings hold.

(1) $\left(\frac{|X_i|}{2} - \binom{n+e-3}{e-2}\right) Q_{n,e}(\alpha) + Q_{n,e}(1) \sum_{l=1}^{[\frac{e}{2}]} Q_{n,e-2l}(\alpha) = 0$ **for any** $\alpha \in A(X_i)$ **satisfying** $\alpha \neq -1$ **for** $i = 1,\ 2$.

(2) $Q_{n,e}(\gamma) = 0$ **for any** $\gamma \in A(X_1, X_2)$.

If $e \geq 5$, **then** $A(X_1, X_2)$ **coincides with the set of zeros of** $Q_{n,e}(\gamma) = 0$.

(3) If $e \geq 5$, **then every** $\alpha \in A(X_i)$, $(i = 1, 2)$ **and** γ^2 **are rational numbers for any** $\gamma \in A(X_1, X_2)$

Here $Q_{n,i}(x)$ **is the Gegenbauer polynomial of degree** i, **normalized so that** $Q_{n,i}(1) = \dim(Harm_i(\mathbb{R}^n))$

Sketch of the proof for II, III and IV.

I: $t = 9$:

$$A(X_1) = \{-1, \pm\alpha_1, \pm\alpha_2\},\ A(X_2) = \{-1, \pm\beta_1, \pm\beta_2\}$$

$$A(X_1, X_2) = \left\{ \pm\sqrt{\frac{(3n+6\pm\sqrt{6(n+2)(n+1)})}{(n+4)(n+2)}} \right\}$$

$\frac{1-\alpha_1^2}{\alpha_1^2-\alpha_2^2}$ **and** $\frac{1-\beta_1^2}{\beta_1^2-\beta_2^2}$ **must be integers.**

(cf. Larman-Rogers-Seicel.)

$$\frac{1-\alpha_1^2}{\alpha_1^2-\alpha_2^2} = -\frac{1}{2} + F(n, |X_1|), \quad \frac{1-\beta_1^2}{\beta_1^2-\beta_2^2} = -\frac{1}{2} + F(n, |X_2|),$$

$$F(n, x) = \frac{(2x-n^2-3n)\sqrt{(n+2)(n+1)}}{2\sqrt{n^2(n+2)(n+1)(n+3)^2-8n(n+5)(n+1)x+24x^2}}.$$

$2F(n, |X_i|)$ **is an integer for** $i = 1, 2$.

Let $|X_1| \le |X_2|$. Then We can prove that

$$-\frac{1}{2} + \frac{\sqrt{6(n+2)(n+1)}}{12} < -\frac{1}{2} + F(n, |X_2|) < \frac{1}{2} + \frac{\sqrt{6(n+2)(n+1)}}{12}$$

Since $\sqrt{6(n+2)(n+1)} = 6k,\ k \in \mathbb{Z}_{>0}$, we get

$$\frac{k-1}{2} < -\frac{1}{2} + F(n, |X_2|) < \frac{k+1}{2}.$$

This implies that k is an even integer and

$$-\frac{1}{2} + F(n, |X_2|) = \frac{k}{2} = \frac{\sqrt{6(n+2)(n+1)}}{12}.$$

$\Rightarrow$ **solve for $|X_2|$ and then $|X_1|$ interms of n explicitly.**

$$\Rightarrow 1 + \frac{\sqrt{6(n+2)(n+1)}}{12} < -\frac{1}{2} + F(n, |X_1|) < 2 + \frac{\sqrt{6(n+2)(n+1)}}{12}$$

$\Rightarrow$ **contradiction.**

Case $t \geq 11$:

Using the propery of Q-polynomial association schemes we can show that all the elements in $A(X_1)$ and $A(X_2)$ are rational numbers for $t = 2e + 1 \geq 11$. Then the coherent configuration structure of X shows that γ^2 is a rational number for any $\gamma \in A(X_1, X_2)$.

We can show that for $e \geq 5$, $A(X_1, X_2)$ coincides the set of zeros of Gegenbauer polynomial $Q_{n,e}(x)$.

Proof for III.

Let $n \geq 3$ and $e \geq 6$. Assume $\gamma^2 \in \mathbb{Q}$ for any γ with $Q_{n,e}(\gamma) = 0$. Then there exists a constant C depending only on e and $n \leq C$ holds.

For $t = 11$, we cannot apply this property.

Thank You