Workshop on Sphere Packings, Lattices, and Designs, ESI

Strongly regular graphs in metric geometry

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Borsuk's conjecture

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$b(3) = 4$ (Perkal 1947)

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Kahn and Kalai ($n = 1325$), Nilli ($n = 946$), Raigorodskii ($n = 561$), Weißbach ($n = 560$), Hinrichs ($n = 323$), Pikhurko ($n = 321$), and Hinrichs, Richter ($n \geq 298$).

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Example by Kahn and Kalai

Let K_n be the set of vertices of a cube in dimension n.

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Let $\mathcal{A}$ be a subset of $\{-1, 1\}^{4n}$ with the property that no two vectors in $\mathcal{A}$ are orthogonal. Then $|\mathcal{A}| < c^{4n}$, where $c < 2$ is an absolute constant.

Upper bounds for Borsuk numbers

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Oded Schramm, 1988:

$$b(n) \leq c_\epsilon(\sqrt{3/2} + \epsilon)^n$$

Theorem 1

Theorem 1(B.) There is a two-distance subset $\{x_1, \ldots, x_{416}\}$ of the unit sphere $S^{64} \subset \mathbb{R}^{65}$ such that $(x_i, x_j) = 1/5$ or $-1/15$ for $i \neq j$ which cannot be partitioned into 83 parts of smaller diameter.

Definition of a strongly regular graph (SRG)

A strongly regular graph Γ with parameters (v, k, λ, μ) is an undirected regular graph on v vertices of valency k such that each pair of adjacent vertices has λ common neighbors, and each pair of nonadjacent vertices has μ common neighbors.

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The incidence matrix A of Γ has the following properties:

$$AJ = kJ,$$

and

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where I is the identity matrix and J is the matrix with all entries equal to 1. This then implies that

$$(v - k - 1)\mu = k(k - \lambda - 1).$$

Eigenvalues

A has only 3 eigenvalues: k of multiplicity 1, a positive eigenvalue

$$r = \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right)$$

of multiplicity

$$f = \frac{1}{2}\left(v - 1 - \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right),$$

and a negative eigenvalue

$$s = \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right)$$

of multiplicity

$$g = \frac{1}{2}\left(v - 1 + \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right).$$

f and g – integers.

Euclidean representation of SRG

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$$(x_i, x_j) = \begin{cases} 1, & \text{if } i = j, \\ p, & \text{if } i \text{ and } j \text{ are adjacent}, \\ q, & \text{else.} \end{cases}$$

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Similarly we can consider Euclidean representation in $\mathbb{R}^g$.

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Hence SRG(28,9,0,4) does not exist.

Proof of Theorem 1

Consider an Euclidean representation of $\Gamma = SRG(416, 100, 36, 20)$ in $\mathbb{R}^f = \mathbb{R}^{65}$. $(x_i, x_j) = 1/5$ if i connected to j and $(x_i, x_j) = -1/15$ if i is not connected to j. Hence, the configuration cannot be partitioned into less than v/m parts, where m is the size of the largest clique in Γ.

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To prove Theorem 1 it is enough to show that Γ has no 6-clique.

Proof

We will use the following statement

Theorem A.

(i) *For each $u \in V$ the subgraph of Γ induced on $N(u)$ is a strongly regular graph with parameters $(100, 36, 14, 12)$ (the Hall-Janko graph). In other words the Hall-Janko graph is the first subconstituent of Γ.*

(ii) *The first subconstituent of the Hall-Janko graph is the $U_3(3)$ strongly regular graph with parameters $(36, 14, 4, 6)$.*

(iii) *The first subconstituent of $U_3(3)$ is a graph on 14 vertices of regularity 4 (the co-Heawood graph).*

(iv) *The co-Heawood graph has no triangles.*

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Example: $Aut(SRG(100, 22, 0, 6)) = HS.2$.
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$Aut(SRG(77, 16, 0, 4)) = M_{22}.2$.
M_{22} – Mathieu simple group of order 443520.
$Aut(SRG(275, 112, 30, 56)) = McL.2$
McL – McLaughlin simple group of order 898128000.

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$$Suz \supset G_2(4) \supset HJ \supset U_3(3)$$

Thomas Jenrich's result

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Idea of the proof: Take a 352 point subconfiguration in $\mathbb{R}^{64}$ of our example.

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Theorem 2 (B., Radchenko) Suppose that there exists a $SRG((n^2+3n-1)^2, n^2(n+3), 1, n(n+1))$. Then $n \in \{1, 2, 4\}$.

Theorem 3 (B., Radchenko) The $SRG(729, 112, 1, 20)$ is unique up to isomorphism.

Theorem 4 (B., Prymak, Radchenko) There is no $SRG(76, 30, 8, 14)$.

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$SRG(76, 30, 8, 14)$ has one of the following subgraphs:
$SRG(40, 12, 2, 4)$, $\tilde{K}_{16}$ or $K_{6,10}$!

THANK YOU!