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A Packing Problem in Time-Frequency Analysis

Th. Strohmer, Sc. Beaver IEEE Trans. Comm. (2003)

Abreu, Dörfler (2013)

$g \in L^2(\mathbb{R})$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$

TFS $\pi(\lambda) g(t) = e^{2\pi i \lambda_2 t} g(t - \lambda_1)$ ← time

↑ frequency

$\Lambda = M\mathbb{Z}^2 \subseteq \mathbb{R}^2$ lattice with

density $D(\Lambda) = |\det M|^{-1}$

Gabor expansion over Λ

$$c \longmapsto \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g$$

wireless comm.

↑ symbols

Quantity of interest: condition # of Gram matrix $G(g,\Lambda)$

$$G_{\lambda\mu} = \langle \pi(\mu) g, \pi(\lambda) g \rangle \qquad \lambda, \mu \in \Lambda$$

$$A \|c\|_2^2 \le \left\| \sum c_\lambda \pi(\lambda) g \right\|_2^2 = \sum_{\lambda,\mu} c_\lambda \bar{c}_\mu \langle \pi(\mu) g, \pi(\lambda) g \rangle$$

(*)

$$= \langle Gc, c \rangle \le \underset{\substack{\| \\ B(M)}}{B} \|c\|_2^2$$

$$\kappa(\Lambda) = \kappa = \frac{B_{opt}}{A_{opt}} = \underbrace{\|G\|_{op} \|G^{-1}\|_{op}}_{\text{depends on } g \text{ \& } \Lambda}$$

stability of G

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Facts (Density Thm. for Riesz sequences)

If $(*)$ with $A, B > 0$, then $D(\Lambda) \leq 1$

If $(*)$ and $g \in \mathcal{S}(\mathbb{R})$, then $D(\Lambda) < 1$

For Gaussian $g(t) = e^{-\pi t^2}$

$$(*) \iff D(\Lambda) < 1$$

capacity

spectral efficiency

Problem: Fix $g(t) = e^{-\pi t^2}$ and density $\Delta < 1$

Find

$$\inf \{ \varkappa(\Lambda) : D(\Lambda) = \Delta \}$$

Conj: The infimum is assumed for the hexagonal lattice with density Δ.

$$M = \begin{pmatrix} \left(\frac{2}{\sqrt{3}\Delta}\right)^{1/2} & \left(\frac{1}{2\sqrt{3}\Delta}\right)^{1/2} \\ 0 & \left(\frac{\sqrt{3}}{2\Delta}\right)^{1/2} \end{pmatrix}$$

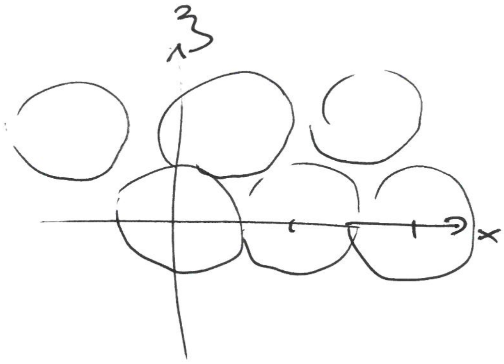
Plausibility: TF distribution of g is radially symm.

$$W(g,g)(x,\xi) = \int g\left(x+\frac{t}{2}\right) g\left(x-\frac{t}{2}\right) e^{-2\pi i \xi t}\, dt$$

for Gauss $\quad W(g,g)(x,\xi) = e^{-2\pi(x^2+\xi^2)}$

$\{(x,\xi) : W(g,g) > \varepsilon\}$ is disc

$W(\pi(\lambda)g, \pi(\lambda)g)(z) = W(g,g)(z-\lambda)$



best packing of ε-supp. yields hexag. lattice

Numerical simulation $\qquad \Delta = \frac{1}{2}$

$$\varkappa\begin{pmatrix}\sqrt{2} & 0 \\ 0 & \sqrt{2}\end{pmatrix} = \sqrt{2}$$

$$\varkappa\begin{pmatrix}\frac{2}{\sqrt[4]{3}} & \frac{1}{\sqrt[4]{3}} \\ 0 & \sqrt[4]{3}\end{pmatrix} \approx \sqrt[3]{2}$$

Math. Fact $\qquad M = \begin{pmatrix} a & c \\ 0 & b\end{pmatrix}$

$$B\begin{pmatrix} a & c \\ 0 & b\end{pmatrix} \le B\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}$$

Optimization: $\qquad \Delta = \frac{1}{2}$

$$\inf_{b,c} \frac{\max\limits_{x,\xi}}{\min\limits_{x,\xi}} \left\{ \left|\sum_k e^{-\pi(1+ibc)(x-\frac{k}{b})^2} e^{2\pi i k\xi}\right|^2 + \left|\sum_k e^{-\pi(1+ibc)(x+\frac{1}{2b}-\frac{k}{b})^2} e^{2\pi i k\xi}\right|^2 \right.$$