Extreme problems of sphere packings and irreducible contact graphs

Oleg R. Musin

University of Texas at Brownsville

ESI, October 29, 2014

Papers

O. R. Musin and A. S. Tarasov, *The strong thirteen spheres problem,* Discrete Comput. Geom., **48** (2012) 128–141.

O. R. Musin and A. S. Tarasov, *Enumeration of irreducible contact graphs on the sphere,* Fundam. Prikl. Mat., **18:2** (2013), 125–145.

O. R. Musin and A. V. Nikitenko, *Optimal packings of congruent circles on a square flat torus,* arXiv:1212.0649.

O. R. Musin and A. S. Tarasov, *Extreme problems of circle packings on a sphere and irreducible contact graphs,* arXiv:1410.0744

O. R. Musin and A. S. Tarasov, *The Tammes problem for N=14,* arXiv:1410.2536

Contact graphs

Let X be a finite subset of a metric space M. Here we consider $M = \mathbb{S}^2$. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x,y)\}, \text{ where } x \neq y.$$

The *contact graph* $\mathrm{CG}(X)$ is the graph with vertices in X and edges (x,y), $x, y \in X$ such that

$$\mathrm{dist}(x,y) = \psi(X)$$

Shift of a single vertex

Let X be a finite set in M. Let $x \in X$ be a vertex of CG(X) We say that there exists a shift of x if x can be slightly shifted to x' such that $\mathrm{dist}(x', X \setminus \{x\}) > \mathrm{dist}(x, X \setminus \{x\})$.

Irreducible contact graph

We say that the graph $\mathrm{CG}(X)$ is *irreducible* [Schütte - van der Waerden, Fejes Tóth] if there are no shift of vertices.

Properties of spherical irreducible contact graphs

Theorem

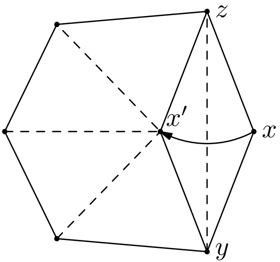
Let $X \subset \mathbb{S}^2$ with $|X| = N$ is such that the graph $\mathrm{CG}(X)$ is irreducible. Then $G := \mathrm{CG}(X)$ satisfies the following properties:

1. *G is a planar graph;*
2. *Any vertex of G is of degree $0, 3, 4,$ or 5;*
3. *If $N > 10$ and G contains an isolated vertex v, then v lies in a face with $m \geq 6$ vertices. Moreover, a hexagonal face of G cannot contain two or more isolated vertices.*

D–flip and D–irreducible contact graphs

Danzer [1963] defined the following flip. Let x, y, z be vertices of $\mathrm{CG}(X)$ with $\mathrm{dist}(x, y) = \mathrm{dist}(x, z) = \psi(X)$. We say that x is flipped over yz if x is replaced by its mirror image x' relative to the great circle yz. We say that this flip is *D (Danzer's)–flip* if $\mathrm{dist}(x', X \setminus \{x, y, z\}) > \psi(X)$.

If there are neither D–flips nor shifts of vertices, then we call $\mathrm{CG}(X)$ as a *D–irreducible graph.*



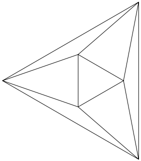
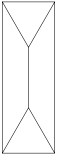
Danzer's work

In the Habilitationsschrift of Ludwig Danzer

"Endliche Punktmengen auf der 2-sphäre mit möglichst großem Minimalabstand", Universität Göthingen, 1963,

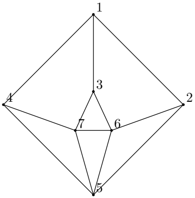
are given all D-irreducible graphs for $6 \leq N \leq 10$.

Danzer's work: D–irreducible contact graphs for $N = 6$

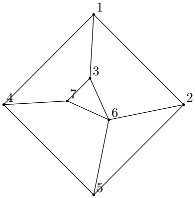
maximal  $G_I = \mathcal{M}_6(t)$ 

Danzer's work: D–irreducible contact graphs for $N = 7$

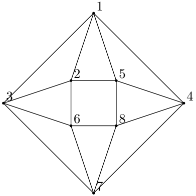
maximal $\mathcal{M}_7$



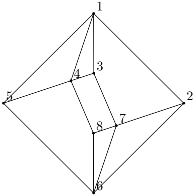
$\mathcal{M}_7(t)$



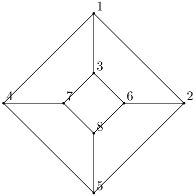
Danzer's work: D–irreducible contact graphs for $N = 8$



maximal $\mathcal{M}_8$

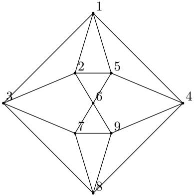


$\mathcal{M}_8(t)$

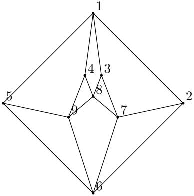


$\mathcal{M}_8(u, v)$

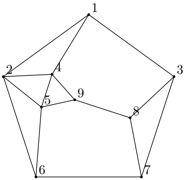
Danzer's work: D–irreducible contact graphs for $N = 9$



maximal

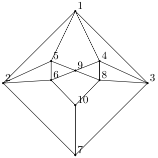


$\mathcal{M}_9(t)$

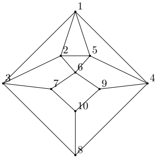


$\mathcal{M}_9^*$

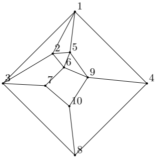
Danzer's work: D–irreducible contact graphs for $N = 10$



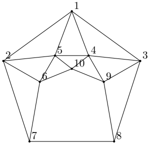
maximal $\mathcal{M}_{10}$



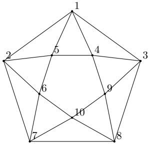
$\mathcal{M}_{10,2}(t)$



$\mathcal{M}_{10,3}(t)$

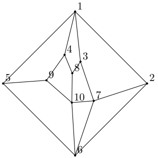


$\mathcal{M}^*$

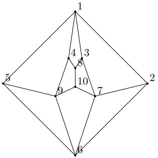


$\mathcal{M}^{**}$

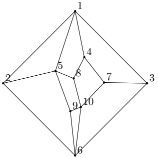
Danzer's work: D–irreducible contact graphs for $N = 10$



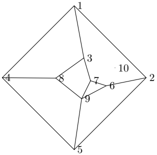
$\mathcal{M}_{10}^{3}(t)$



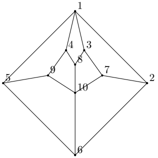
$\mathcal{M}_{10}^{1,2}(t)$



$\mathcal{M}_{10}^{1,3}(t)$



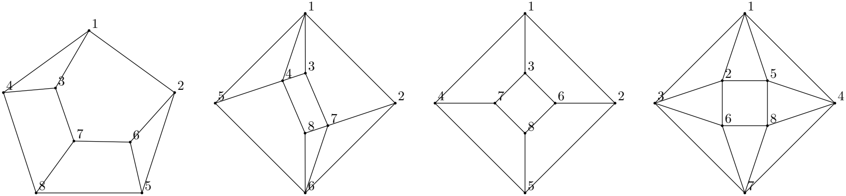
$\tilde{\mathcal{M}}_{10}^{1,2,3}(t) \cup \mathcal{M}$



$\mathcal{M}_{10}^{3}(u, v)$

M. & Tarasov: Irreducible graphs for N=8

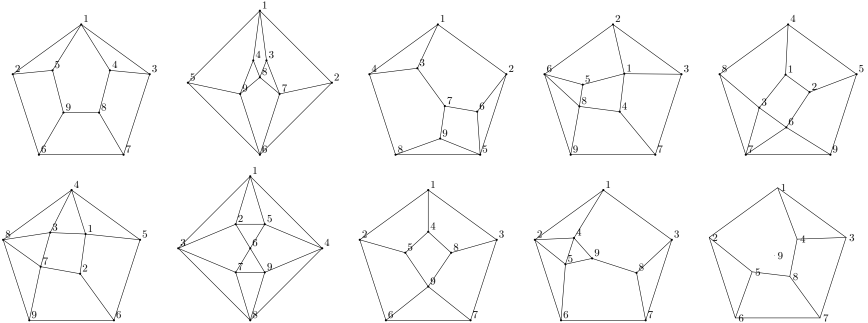
N	d_{min}	d_{max}
1	1.17711	1.18349
2∗	1.28619	1.30653
3∗	1.23096	1.30653
4 ∗ ∗	1.30653	1.30653



M. & Tarasov: Irreducible graphs for N=9

N	d_{min}	d_{max}
1	1.14099	1.14143
2∗	1.22308	1.23096
3	1.10525	1.14349
4	1.17906	1.18106
5	1.15448	1.17906
6	1.17906	1.17906
7 ∗ ∗	1.23096	1.23096
8	1.15032	1.18106
9∗	1.10715	1.14342
10	1.17906	1.18428

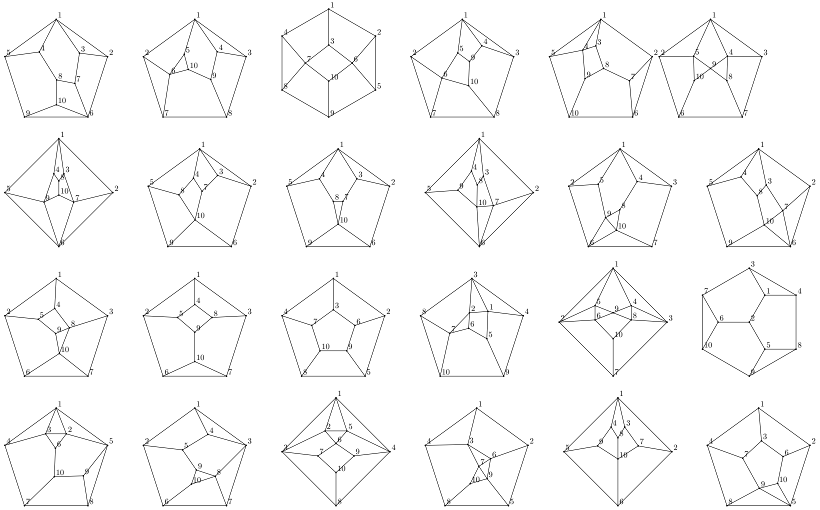
M. & Tarasov: Irreducible graphs for N=9



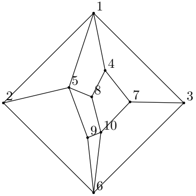
M. & Tarasov: Irreducible graphs for N=10

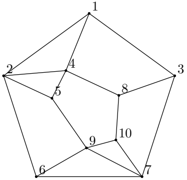
N	d_{min}	d_{max}	N	d_{min}	d_{max}
1	1.0839	1.09751	2	1.08161	1.08439
3	1.03067	1.04695	4	1.10715	1.0988
5	1.07529	1.09431	6	1.09386	1.12285
7*	1.15278	1.15448	8	1.10012	1.10801
9	1.06344	1.07834	10*	1.15074	1.15191
11	1.0843	1.08442	12	1.10055	1.10889
13	1.09504	1.10429	14	1.06032	1.09604
15	1.06278	1.1098	16	1.09567	1.10715
17 **	1.15448	1.15448	18	0.99865	1.0467
19	1.0843	1.0844	20	1.08334	1.09547
21*	1.15341	1.15341	22	1.0988	1.10608
23*	1.14372	1.15191	24	1.09249	1.1098
25*	1.15191	1.15245	26	1.09658	1.10977
27*	1.15191	1.15191	28*	1.10715	1.10715
29*	1.10715	1.10715	30	1.15103	1.15341

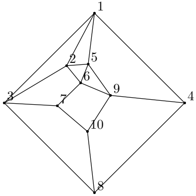
M. & Tarasov: Irreducible graphs for N=10

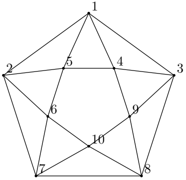


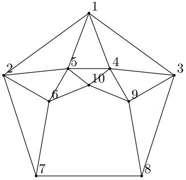
M. & Tarasov: Irreducible graphs for N=10

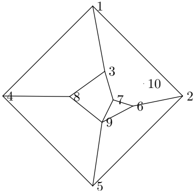




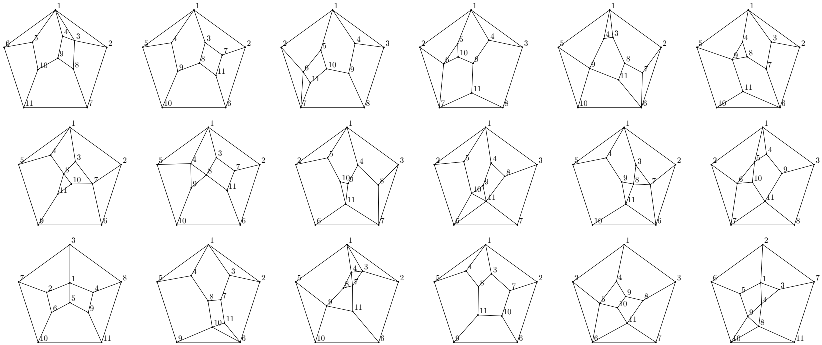




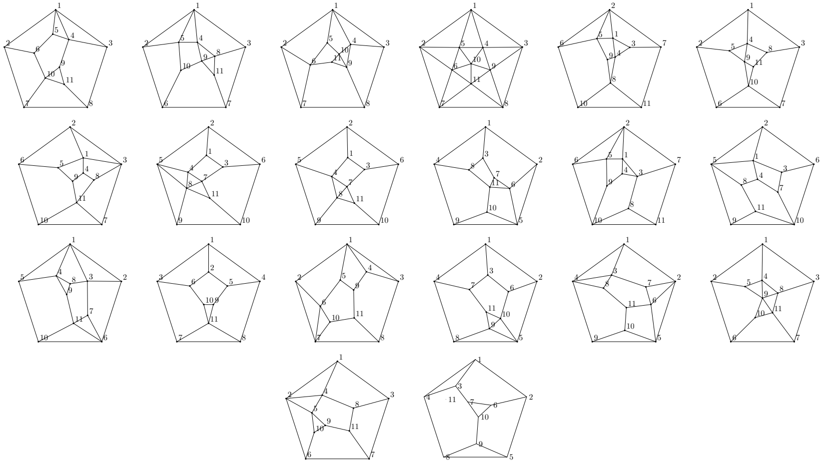




M. & Tarasov: Irreducible graphs for N=11



M. & Tarasov: Irreducible graphs for N=11



The Tammes problem

How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible

Tammes PML (1930). "On the origin of number and arrangement of the places of exit on pollen grains". Diss. Groningen.

The Tammes problem

Let X be a finite subset of $\mathbb{S}^2$. Recall that

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x, y)\}, \text{ where } x \neq y.$$

Then X is a spherical $\psi(X)$-code.

Denote by d_N the largest angular separation $\psi(X)$ with $|X| = N$ that can be attained in $\mathbb{S}^2$, i.e.

$$d_N := \max_{X \subset \mathbb{S}^2} \{\psi(X)\}, \text{ where } |X| = N.$$

The Tammes problem

L. Fejes Tóth (1943): $N = 3, 4, 6, 12, \infty$

K. Schütte, and B. L. van der Waerden (1951): $N = 5, 7, 8, 9$

L. Danzer (1963): $N = 10, 11$

R. M. Robinson (1961): $N = 24$

M. & Tarasov: $N = 13$ and $N = 14$

N	d_N
4	109.4712206
5	90.0000000
6	90.0000000
7	77.8695421
8	74.8584922
9	70.5287794
10	66.1468220
11	63.4349488
12	63.4349488
13	57.1367031
14	55.6705700
....	....
15	53.6578501
16	52.2443957
17	51.0903285

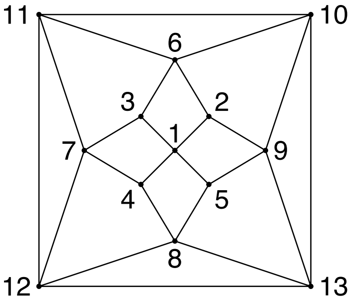
Maximal graphs G_N

Let X be a subset of $\mathbb{S}^2$ with $|X| = N$. We say that $\mathrm{CG}(X)$ is *maximal* if $\psi(X) = d_N$ and its number of edges is minimum. We denote this graph by G_N.
Actually, this definition does not assume that G_N is unique. We use this designation for some $\mathrm{CG}(X)$ with $\psi(X) = d_N$.

Proposition. Let $\mathrm{CG}(X)$ be a maximal graph G_N. Then for $N \geq 6$ the graph $\mathrm{CG}(X)$ is irreducible.

Tammes' problem for $N = 13$

The contact graph $\Gamma_{13} :=\mathrm{CG}(P_{13})$ with $\psi(P_{13}) \approx 57.1367^\circ$



Tammes' problem for $N = 13$

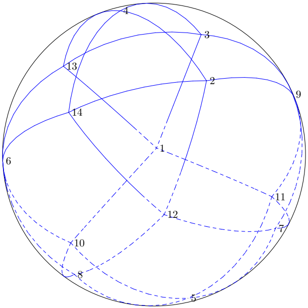
Theorem (M. & A. Tarasov). The arrangement of 13 points P_{13} in $\mathbb{S}^2$ is the best possible, the maximal arrangement is unique up to isometry, and $d_{13} = \psi(P_{13})$.

Tammes' problem for $N = 13$

Theorem. The arrangement of 13 points P_{13} in $\mathbb{S}^2$ is the best possible, the maximal arrangement is unique up to isometry, and $d_{13} = \psi(P_{13})$.

Tammes' problem for $N = 14$

Theorem (M. & A. Tarasov). The arrangement of 14 points P_{14} in $\mathbb{S}^2$ is the best possible and the maximal arrangement is unique up to isometry.



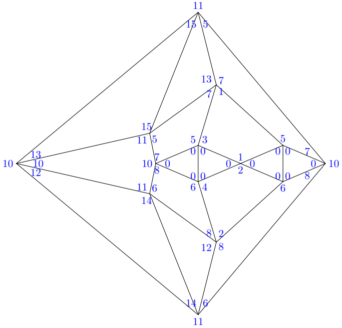


Figure: An arrangement of 14 points P_{14} and its contact graph Γ_{14} with $\psi(P_{14}) \approx 55.67057°$.

Tammes problem for antipodal configurations

Consider antipodal (centrally symmetric) sets X on $\mathbb{S}^2$, i. e. $X = -X$. In other words, X/A, *where* $A(x) = -x$, is a projective code on $\mathbb{RP}^2$.

$$a_M := \max_{X=-X\subset\mathbb{S}^2} \{\psi(X)\}, \ \text{ for } \ |X| = 2M.$$

It is clear that $a_M \leq d_{2M}$. Therefore, if $\psi(X) = d_{2M}$, $|X| = 2M$, and X is antipodal then $a_M = d_{2M}$.

Tammes problem for antipodal configurations

Theorem

Let $P_M \subset \mathbb{S}^2$ be a maximal set for Tammes' problem for antipodal configurations, i. e. $\psi(P_M) = a_M$. Then

1. *P_2 is the set of vertices of a square on the equator, $a_2 = 90^\circ$;*
2. *P_3 is the set of vertices of a regular octahedron, $a_3 = 90^\circ$;*
3. *P_4 is the set of vertices of a cube, $a_4 = \arccos(1/3)$;*
4. *P_5 consists of five pairs of antipodal vertices of a regular icosahedron, $a_5 = \arccos(1/\sqrt{5})$.*
5. *P_6 is the set of vertices of a regular icosahedron, $a_6 = \arccos(1/\sqrt{5})$.*

Maximum contacts problem

Let $X \subset \mathbb{S}^2$

$e(X) :=$ number of edges of the contact graph $\mathrm{CG}(X)$

$\mathrm{K}_N(d) :=$ max contact number of N spherical caps with diameter d

$\mathrm{K}_{12}(60^\circ) = 24$ (Flatley, Tarasov, Taylor, and Theil, 2013)

$$\mathrm{K}_N := \max_{d \le d_N} \mathrm{K}_N(d) = \max_{X \in \mathbb{S}^2, |X| = N} e(X)$$

$$\mathrm{K}_N^* := \max_{X \in \mathbb{S}^2, |X| = N} e(X), \text{ where } \mathrm{CG}(X) \text{ is irreducible}$$

Maximum contacts problem

It is clear, $\mathrm{K}_2 = 1$.

Theorem

Let $N > 2$. Then $\mathrm{K}_N \leq 3N - 6$ and the equality holds only for $N = 3, 4, 6, 12$.

$\mathrm{K}_5 = 8$ (square pyramid)

Maximum contacts problem

Theorem

1. $K_7^* = K_7 = 12;$
2. $K_8^* = K_8 = 16;$
3. $K_9^* = K_9 = 18;$
4. $K_{10}^* = 20,\ K_{10} = 21;$
5. $K_{11}^* = K_{11} = 25.$

Maximum contacts problem: two lemmas

Lemma (1)

Let X be a finite set in $\mathbb{S}^2$. If any face of the contact graph $\mathrm{CG}(X)$ is a triangle or a quadrilateral then this graph is irreducible.

Lemma (2)

Let $X \subset \mathbb{S}^2$ and $|X| = N$, $N > 6$. Suppose that $e(X) \geq 3N - 8$. Then $\mathrm{CG}(X)$ is irreducible.

Maximum contacts: open problem

$$\mathrm{K}_{inf} := \lim_{N\to\infty} \inf \frac{\mathrm{K}_N}{N}, \quad \mathrm{K}_{sup} := \lim_{N\to\infty} \sup \frac{\mathrm{K}_N}{N}$$

We have

$$2 \leq \mathrm{K}_{inf} \leq \mathrm{K}_{sup} \leq 5/2$$

Open problem: *Find better bounds for* K_{inf} *and* K_{sup}*.*
Do we have the equality: $\mathrm{K}_{inf} = \mathrm{K}_{sup}$*?*

Minimum contact problem

Let J_N denote the all sets X in $\mathbb{S}^2$ such that $|X| = N$ and $\mathrm{CG}(X)$ is irreducible.

$$\kappa_N := \min_{X \in J_N} e(X).$$

Theorem

1. $\kappa_6 = 9$;
2. $\kappa_7 = 11$;
3. $\kappa_8 = 12$;
4. $\kappa_9 = 12$;
5. $\kappa_{10} = 14$;
6. $\kappa_{11} = 15$.

Danzer's problems on irreducible graphs

In *L. Danzer, Finite point-sets on* $\mathbb{S}^2$ *with minimum distance as large as possible, Discrete Math.,* **60** *(1986), 3–66*, there are a lot of open problems that are related to irreducible graphs. One of them:

"Is there X *on* $\mathbb{S}^2$ *such that* $|X| < 12$, $\mathrm{CG}(X)$ *is irreducible, and* $\psi(X) < d_{12} = \arccos(1/\sqrt{5}) \approx 1.1071$*?"* [Question 5, p. 65].

The answer is positive - there is such set with 9 points – graph 9.3, where $d \in [1.10525, 1.14349]$.

Danzer's open problems on irreducible graphs

Let $X \subset \mathbb{S}^2$ and $\mathrm{CG}(X) = G$. There are two possibilities:
(i) X is unique up to isometry,
(ii) there is a degree of freedom for X, i. e. there is a k–parametric family of X with $k \geq 1$.

Problem 1. *Is it true that all maximal graphs lie in class* (*i*)*?*

It is true for $N \leq 14$ and $N = 24$.

Optimal packings of circles on a square flat torus

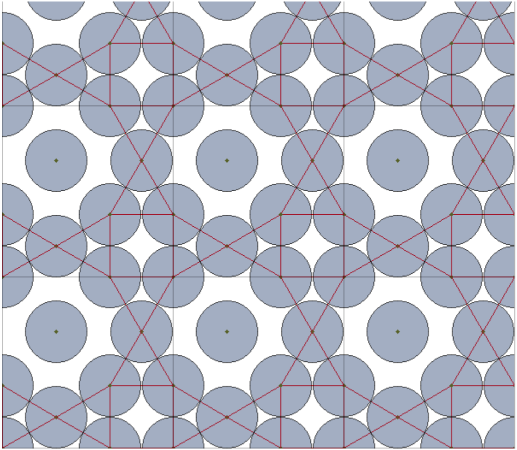


Figure: The first optimal configurations for N=7

Optimal packings of circles on a square flat torus

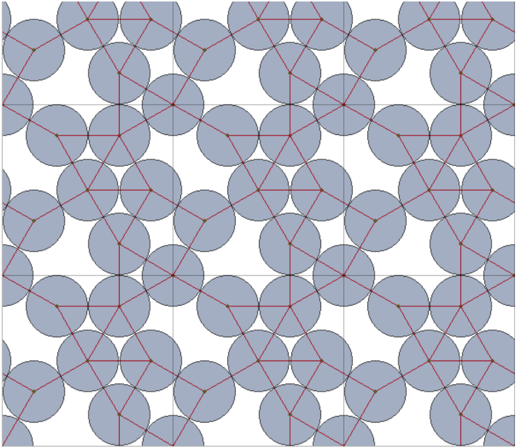


Figure: The second optimal configurations for N=7

Optimal packings of circles on a square flat torus

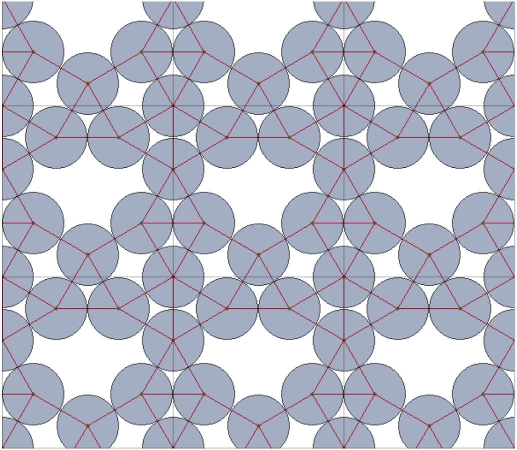


Figure: The third optimal configurations for N=7

Danzer's open problems on irreducible graphs

Problem 2. *Is it true that for $N > 5$ on $\mathbb{S}^2$ there is unique (up to isomorphism) maximal graph G_N?*

Danzer's open problems on irreducible graphs

For $6 \leq N \leqslant 10$, $N = 13$ and $N = 14$ maximal graphs can be obtain from graphs in class (ii) by adding edges. However, for $N = 11, 12$ maximal graphs are "isolated". It arises a question:

Problem 3. *Are the cases* $N = 11, 12$ *exceptional and all other maximal graphs can be obtained from class* (ii)*?*

Danzer's open problems on irreducible graphs

Problem 4. *Let a graph G belongs to class (ii). Suppose that $X \subset \mathbb{S}^2$ such that $\mathrm{CG}(X) = G$. Is it true that you can change a little bit X to X' such that $\mathrm{CG}(X') = G$ and $\psi(X') > \psi(X)$?*

Danzer's open problems on irreducible graphs

Let points $X = \{x_1, \dots, x_N\}$ are defined by spherical coordinates (θ_i, φ_i). We can assume that $\theta_1 = \pi/2$, $\varphi_1 = 0$ and $\theta_2 = \pi/2$. Then spherical coordinates define the configuration space Π_N of dimension $2N - 3$. We have $\psi : \Pi_N \to \mathbb{R}$.

Problem 5. *Find conditions under which a maximal graph G_N is the maximum of the function ψ on Π_N.*

THANK YOU