ESI, "SPHERE PACKINGS, LATTICES AND DESIGNS"

COMPUTATIONAL ALGORITHMS FOR BOUNDING POTENTIAL ENERGY OF SPHERICAL CODES AND DESIGNS

Peter Boyvalenkov
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
Peter Dragnev
Department of Mathematical Sciences, IPFW, Fort Wayne, IN, USA
Douglas Hardin, Edward Saff
Department of Mathematics, Vanderbilt University, Nashville, TN, USA
Maya Stoyanova
Faculty of Mathematics and Informatics, Sofia University, Bulgaria

Vienna, Austria, October 27-31, 2014

Energy of spherical codes

- Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. We refer to a finite set $C \subset \mathbb{S}^{n-1}$ as a spherical code and, for a given *absolutely monotone* function $h(t) : [-1, 1] \to [0, +\infty]$, we define the h-energy (or potential energy) of C by

$$E(n, C; h) := \sum_{x,y \in C, x \neq y} h(\langle x, y \rangle),$$

  where $\langle x, y \rangle$ denotes the inner product of x and y.

- Problem: For fixed cardinality $|C| = N$ of C find the minimum possible potential energy, i.e., determine

$$\mathcal{E}(n, N; h) := \inf\{E(n, C; h) : |C| = N\}.$$

Spherical designs

- P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, Geom. Dedicata 6, 1977, 363-388.

- **Definition:** A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that

$$\frac{1}{\mu(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most τ.

The strength of C is the maximal number $\tau = \tau(C)$ such that C is a spherical τ-design.

Energy of spherical τ-designs

- Let $C \subset \mathbb{S}^{n-1}$ be a spherical τ-design and $E(C, n; h)$ be the h-energy of C. Denote by

$$L(N, n, \tau; h) = \inf\{E(C, n; h) : |C| = N, C \subset \mathbb{S}^{n-1}, C \text{ is } \tau\text{-design}\}$$

the minimum possible h-energy of spherical τ-designs on $\mathbb{S}^{n-1}$ of N points,

$$U(N, n, \tau; h) = \sup\{E(C, n; h) : |C| = N, C \subset \mathbb{S}^{n-1}, C \text{ is } \tau\text{-design}\}$$

the maximum possible h-energy of spherical τ-designs on $\mathbb{S}^{n-1}$ of N points.

Delsarte-Goethals-Seidel bounds for spherical designs

- P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, Geom. Dedicata 6, 1977, 363-388.

- For fixed strength τ and dimension n denote by

$$B(n,\tau) = \min\{|C| : \exists\ \tau\text{-design } C \subset \mathbb{S}^{n-1}\}$$

the minimum possible cardinality of spherical τ-designs $C \subset \mathbb{S}^{n-1}$.

$$B(n,\tau) \geq D(n,\tau) = \begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau = 2k-1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \text{if } \tau = 2k. \end{cases}$$

Gegenbauer polynomials

- For fixed dimension n, the Gegenbauer polynomials $P_i^{(n)}(t)$ are defined by $P_0^{(n)} = 1$, $P_1^{(n)} = t$, and the three-term recurrence relation

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t).$$

- In fact, $P_i^{(n)}(t) = P_i^{(n-3)/2,(n-3)/2}(t)$ is a Jacobi polynomial.
- If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree m then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as

$$f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t).$$

- The kernel $T_i(x,y) = \sum_{j=0}^{i} r_j P_j^{(n)}(x) P_j^{(n)}(y)$, where $r_0 = 1$, $r_1 = n$ and $r_j = \binom{n+j-1}{j} - \binom{n+j-3}{j-2}$.

The main identity

- The main identity:

$$|C|f(1)+\sum_{x,y\in C,x\neq y} f(\langle x,y\rangle) = |C|^2 f_0+\sum_{i=1}^{m}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} v_{ij}(x)\right)^2. \quad (1)$$

  Here $C \subset \mathbf{S}^{n-1}$ is a spherical code, $f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t)$ as above, $\{v_{ij}(x) : j = 1, 2, \ldots, r_i\}$ is an orthonormal basis of the space $\mathrm{Harm}(i)$ of homogeneous harmonic polynomials of degree i and $r_i = \dim\, \mathrm{Harm}(i)$.

- An equivalent definition of spherical designs says that

$$\sum_{x\in C} v_{ij}(x) = 0$$

  for every $i \leq \tau$ and every $j \leq r_i$.

- This suggests that polynomials of degree at most τ could be useful – the right hand side of (1) is then reduced to $|C|^2 f_0$.

Levenshtein bounds for spherical codes (1)

- For every positive integer m we consider the intervals

$$\mathcal{I}_m = \begin{cases} \left[t_{k-1}^{1,1}, t_k^{1,0}\right], & \text{if } m = 2k-1, \\ \left[t_k^{1,0}, t_k^{1,1}\right], & \text{if } m = 2k. \end{cases}$$

- Here $t_0^{1,1} = -1$, $t_i^{a,b}$, $a, b \in \{0,1\}$, $i \geq 1$, is the greatest zero of the Jacobi polynomial

$$P_i^{(a+\frac{n-3}{2}, b+\frac{n-3}{2})}(t).$$

- The intervals $\mathcal{I}_m$ define partition of $\mathcal{I} = [-1, 1)$ to countably many nonoverlapping closed subintervals.

Levenshtein bounds for spherical codes (2)

- For every $s \in \mathcal{I}_m$, Levenshtein used a polynomial $f_m^{(n,s)}(t)$ of degree m which satisfy all conditions of the linear programming bounds for spherical codes. This yields the bound

$$A(n,s) \leq \begin{cases} L_{2k-1}(n,s) = \binom{k+n-3}{k-1}\left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s)-P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)}\right] \\ \qquad\qquad \text{for } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s) = \binom{k+n-2}{k}\left[\frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s)-P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s)+P_{k+1}^{(n)}(s))}\right] \\ \qquad\qquad \text{for } s \in \mathcal{I}_{2k}. \end{cases}$$

- For every fixed dimension n each bound $L_m(n,s)$ is smooth and strictly increasing with respect to s. The function

$$L(n,s) = \begin{cases} L_{2k-1}(n,s), & \text{if } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s), & \text{if } s \in \mathcal{I}_{2k}, \end{cases}$$

is continuous in s.

Connections between DGS- and L-bounds

- The connection between the Delsarte-Goethals-Seidel bound and the Levenshtein bounds are given by the equalities

$$L_{2k-2}(n, t_{k-1}^{1,1}) = L_{2k-1}(n, t_{k-1}^{1,1}) = D(n, 2k-1),$$

$$L_{2k-1}(n, t_{k}^{1,0}) = L_{2k}(n, t_{k}^{1,0}) = D(n, 2k)$$

and the ends of the intervals $\mathcal{I}_m$.

Computing $\{\alpha_i\}$, $\{\rho_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$

- It follows from the properties of the bounds $D(n,\tau)$ and $L_m(n,s)$ that

$$N \in [D(n,\tau), D(n,\tau+1)) \iff s \in \mathcal{I}_m, \ (m=\tau),$$

where s and N are connected by the equality

$$N = L_\tau(n,s).$$

- Therefore we can always associate N with the corresponding numbers:

$$\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \rho_0, \rho_1, \ldots, \rho_{k-1} \text{ when } N \in [D(n,2k-1), D(n,2k))$$

or with

$$\beta_0, \beta_1, \ldots, \beta_k, \gamma_0, \gamma_1, \ldots, \gamma_k \text{ when } N \in [D(n,2k), D(n,2k+1)).$$

Computing $\{\alpha_i\}$, $\{\rho_i\}$ ($\{\beta_i\}$, $\{\gamma_i\}$)

- For every fixed (cardinality) $N > D(n, 2k-1)$ there exist uniquely determined real numbers $-1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ which are the roots of the equation

$$N = L_\tau(n, s), \text{ where } s = \alpha_{k-1}.$$

- The numbers α_i, $i = 0, 1, \ldots, k-1$, are the roots of the equation

$$P_k^{1,0}(t) P_{k-1}^{1,0}(s) - P_k^{1,0}(s) P_{k-1}^{1,0}(t) = 0,$$

where $s = \alpha_{k-1}$, $P_i^{1,0}(t) = P_i^{(n-1)/2,(n-3)/2}(t)$ is a Jacobi polynomial.
- In fact, α_i, $i = 0, 1, \ldots, k-1$, are the roots of the Levenshtein's polynomial $f_{2k-1}^{(n,\alpha_{k-1})}(t) = (t - \alpha_{k-1}) \left(T_{k-1}^{1,0}(t, \alpha_{k-1}) \right)^2$.

Computing $\{\alpha_i\}$, $\{\rho_i\}$ ($\{\beta_i\}$, $\{\gamma_i\}$)

- For every fixed (cardinality) $N > D(n, 2k-1)$ and already identified numbers $-1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ there exist uniquely determined weights (real positive numbers) $\rho_0, \rho_1, \ldots, \rho_{k-1}$, $\rho_i > 0$ for $i = 0, 1, \ldots, k-1$, such that the equality

$$f_0 = \frac{f(1)}{N} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i)$$

  holds for every real polynomial $f(t)$ of degree at most $2k-1$.
- We set $f(t) = t, t^3, \ldots, t^{2k-1}$ in the above formula and obtain a Vandermonde-type system with respect to ρ_i, $i = 0, 1, \ldots, k-1$. The solution is unique.
- We have $\rho_i =$

$$-\frac{(1-\alpha_0^2)(1-\alpha_1^2)\cdots(1-\alpha_{i-1}^2)(1-\alpha_{i+1}^2)\cdots(1-\alpha_{k-1}^2)}{N\alpha_i(\alpha_i^2-\alpha_0^2)(\alpha_i^2-\alpha_1^2)\cdots(\alpha_i^2-\alpha_{i-1}^2)(\alpha_i^2-\alpha_{i+1}^2)\cdots(\alpha_i^2-\alpha_{k-1}^2)},$$

  for $i = 0, 1, \ldots, k-1$.

Computing $\{\beta_i\}$, $\{\gamma_i\}$ ($\{\alpha_i\}$, $\{\rho_i\}$)

- Similarly, for every fixed (cardinality) $N > D(n, 2k)$ there exist uniquely determined real numbers $-1 = \beta_0 < \beta_1 < \cdots < \beta_k < 1$ and $\gamma_0, \gamma_1, \ldots, \gamma_k$, $\gamma_i > 0$ for $i = 0, 1, \ldots, k$, such that the equality

$$f_0 = \frac{f(1)}{N} + \sum_{i=0}^{k} \gamma_i f(\beta_i)$$

  is true for every real polynomial $f(t)$ of degree at most $2k$.
- The numbers β_i, $i = 0, 1, \ldots, k$, are the roots of the equation

$$P_k^{1,1}(t)P_{k-1}^{1,1}(s) - P_k^{1,1}(s)P_{k-1}^{1,1}(t) = 0.$$

  where $s = \beta_k$, $P_i^{1,1}(t) = P_i^{(n-1)/2,(n-1)/2}(t)$ is a Jacobi polynomial.
- The numbers β_i, $i = 0, 1, \ldots, k$, are the roots of the Levenshtein's polynomial $f_{2k}^{(n,\beta_k)}(t) = (t+1)(t-\beta_k)\left(T_{k-1}^{1,1}(t,s)\right)^2$.

Computing $\{\rho_i\}$, $\{\gamma_i\}$ ($\{\alpha_i\}$, $\{\beta_i\}$)

- V.I.Levenshtein, Designs as maximum codes in polynomial metric spaces, Acta Appl. Math. 25, 1992, 1-82.
- Formulas for ρ_i, $i = 0, 1, \ldots, k-1$:

$$\rho_i = \frac{1}{c^{1,0}(1-\alpha_i)\, T_{k-1}^{1,0}(\alpha_i,\alpha_i)}, \quad c^{1,0} = 1$$

- γ_i, $i = 1, \ldots, k$:

$$\gamma_i = \frac{1}{c^{1,1}(1-{\beta_i}^2)\, T_{k-1}^{1,1}(\beta_i,\beta_i)}, \quad c^{1,1} = \frac{n}{n-1}$$

- and for γ_0:

$$\gamma_0 = \frac{T_k^{1,1}(\beta_k,1)}{T_k^{1,1}(-1,-1)\, T_k^{1,1}(\beta_k,1) - T_k^{1,1}(-1,1)\, T_k^{1,1}(\beta_k,-1)}.$$

Computing Universal Lower Bound

- Let h be a fixed absolutely monotone potential, N and n be fixed, and $m = m(N,n) = \tau$ be such that $N \in [D(n,m), D(n,m+1))$.
  Then the Levenshtein nodes $\{\alpha_i\}$ and weights $\{\rho_i\}$, respectively $\{\beta_i\}$ and $\{\gamma_i\}$, provide the bounds

$$\mathcal{E}(n,N;h) \geq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$$

  and

$$\mathcal{E}(n,N;h) \geq N^2 \sum_{i=0}^{k} \gamma_i h(\beta_i),$$

  respectively.

Computing the test functions

- Let n and N be fixed.
- Let $N \in [D(n, 2k-1), D(n, 2k))$, $L_{2k-1}(n, s) = N$ and j be positive integer. Then we calculate the so-called test functions:

$$Q_j(n, s) = \frac{1}{N} + \sum_{i=0}^{k-1} \rho_i P_j^{(n)}(\alpha_i),$$

$j \geq 2k$.

- Let $N \in [D(n, 2k), D(n, 2k+1))$, $L_{2k}(n, s) = N$ and j be positive integer. Then we calculate:

$$Q_j(n, s) = \frac{1}{N} + \sum_{i=0}^{k} \gamma_i P_j^{(n)}(\beta_i),$$

$j \geq 2k + 1$.

Test functions

- Test functions give necessary and sufficient condition for existence of better LP bounds.
- However, the bound from the proof of the sufficiency is not good.
- We use the signs of the first four test functions $Q_j(n,s)$, $j = m+1, m+2, m+3, m+4$, to decide the form of the improving polynomial (if any).

Properties of the test functions

- We always have $Q_{m+1}(n,s) > 0$ and $Q_{m+2}(n,s) > 0$. Therefore the improving polynomial must have degree at least $m+3$.

- **Conjecture.** (P. Boyvalenkov, D. Danev, S. Bumova, Upper bounds on the minimum distance of spherical codes, IEEE Trans. Inform. Theory, 41, 1996, 1576–1581.)

  If $Q_{m+3}(n,s) \geq 0$ and $Q_{m+4}(n,s) \geq 0$ then $Q_j(n,s) \geq 0$ for every j.

- Let N, n, $m = \tau$ and h be fixed. We calculate and check the signs of $Q_{m+3}(n,s)$ and $Q_{m+4}(n,s)$ and have three main cases.

Test functions - examples

(24,4)	(40,10)	(64,14)	(128,15)	(182,7)	(120,4)
1	1	1	1	1	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0.021943574	0.013744273	0.000659722	0	0
0	0.043584477	0.023867606	0.012122396	0	0
0.085714286	0.024962302	0.015879248	0.010927837	0	0
0.16	0.015883951	0.012369147	0.005957261	0	0
-0.024	0.026086948	0.015845575	0.006751842	0.022598277	0
-0.02048	0.02824122	0.016679926	0.008493915	0.011864096	0
0.064232727	0.024663991	0.015516168	0.00811866	-0.00835109	0
0.036864	0.024338487	0.015376208	0.007630277	0.003071311	0
0.059833108	0.024442076	0.01558101	0.007746238	0.009459538	0.053050398
0.06340608	0.024976926	0.015644873	0.007809405	0.0065461	0.066587396
0.054456422	0.025919671	0.015734138	0.007817465	0.005369545	-0.046646712
-0.003869491	0.02498472	0.015637274	0.007865499	0.006137772	-0.018428319
0.008598724	0.024214119	0.015521057	0.007815602	0.005268455	0.020868837
0.091970863	0.025123445	0.01562458	0.007761374	0.005134928	-0.000422871
0.049262707	0.025449746	0.015694798	0.007812225	0.004722806	0.012656294
0.035330484	0.024905002	0.015617497	0.00784714	0.003857119	0.006371173
0.048230925	0.024837415	0.015589583	0.00781076	0.007863772	0.011244953

Finding improving polynomial $f(t)$ (1)

- An improving polynomial $f(t)$ of degree $\ell = m + 3$ or $m + 4$ must satisfy:

  (A1) $f(t) \leq h(t)$ for $-1 \leq t \leq 1$.

  (A2) the coefficients in the Gegenbauer expansion $f(t) = \sum_{i=0}^{\ell} f_i P_i^{(n)}(t)$ satisfy $f_0 > 0$, $f_i \geq 0$ for $i = 1, 2, ..., \ell$.

- We fix $f_{m+1} = 0$, $f_{m+2} = 0$.

Finding improving polynomial $f(t)$ (2)

- Case I. If $Q_{m+3}(n,s) < 0$ and $Q_{m+4}(n,s) < 0$ then we take $\ell = m+4$ and require
$$f_{m+3} > 0, \quad f_{m+4} > 0.$$
- Case II. If $Q_{m+3}(n,s) > 0$ and $Q_{m+4}(n,s) < 0$ then we take $\ell = m+4$ and require
$$f_{m+3} = 0, \quad f_{m+4} > 0.$$
- Case III. If $Q_{m+3}(n,s) < 0$ and $Q_{m+4}(n,s) > 0$ then we take $\ell = m+3$ and require
$$f_{m+3} > 0.$$

Algorithm for finding improving polynomials (Case I) (1)

- We consider the case $N \in [D(n,2k-1), D(n,2k))$. We have calculated $\{\alpha_i\}$, $\{\rho_i\}$ and the lower bound $N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$.
- BEGIN: Take $k+1$ points $c_0 < c_1 < \cdots < c_k$ (one more than the alphi's) such that

$$-1 \leq c_0 < \alpha_0 < c_1 < \alpha_1 < \cdots < \alpha_{k-2} < c_{k-1} < \alpha_{k-1} < c_k < 1.$$

- Take some constant d (we started with $d = 0.1$) and intervals (around every c_i)

$$D_i = [c_i - d, c_i + d]$$

for $i = 0, 1, \ldots, k$.

Algorithm for finding improving polynomials (Case I) (2)

- Take some constant r (we started with $r = 20$) and let $\varepsilon = d/r$. Take the points

$$d_{i,j} = c_i - d + j\varepsilon$$

in every interval D_i, here $j = 0, 1, \ldots, r$ (these are $r + 1$ points in every interval).

- For every $(k + 1)$-tuple

$$d_{j_0,j_1,\ldots,j_k} = (d_{0,j_0}, d_{1,j_1}, \ldots, d_{k,j_k})$$

find "interpolating" polynomial $f_{j_0,j_1,\ldots,j_k}(t)$ (shortly $f(t)$) of degree $m + 4$ which satisfies

$$f(d_{i,j_i}) = h(d_{i,j_i}),$$
$$f'(d_{i,j_i}) = h'(d_{i,j_i}),$$

for $i = 0, 1, \ldots, k$, and

$$f_{m+1} = f_{m+2} = 0.$$

Algorithm for finding improving polynomials (Case I) (3)

- The total number of the above conditions is $2(k+1)+2=2k+4=m+5$, equal to the number the unknowns $f_0, f_1, \ldots, f_{m+4}$.
- There are $(r+1)^{k+1}$ such polynomials $f_{j_0,j_1,\ldots,j_k}(t)$.
- For every polynomial $f_{j_0,j_1,\ldots,j_k}(t)$ check if it satisfies the LP conditions:
  – whether $f(t) \leq h(t)$ for every $t \in [-1,1]$ and
  – whether $f_i \geq 0$ for every $i=0,1,\ldots,m+4$
  (for designs only for $i>m$).
- For every admissible polynomial compute $N(f_0 N - f(1))$.

Algorithm for finding improving polynomials (Case I) (4)

- Let the polynomial $f^*(t)$ maximizes $N(f_0 N - f(1))$. Assume that $f^*(t)$ corresponds to (is obtained from) the $(k+1)$-tuple $d^*_{j_0,j_1,\ldots,j_k} = (d^*_{0,j_0}, d^*_{1,j_1}, \ldots, d^*_{k,j_k})$. Then set new values

$$c_0 := d^*_{0,j_0}, c_1 := d^*_{1,j_1}, \ldots, c_k := d^*_{k,j_k},$$

$$d := \frac{d}{2}$$

  and go to BEGIN.
- Do the above until $d > 0.000001$, say.

Algorithm for finding improving polynomials (Case II)

- If $Q_{m+3}(n,s) > 0$ and $Q_{m+4}(n,s) < 0$ then we find interpolating polynomial $f(t)$ of degree $\ell = m+4$ which satisfies

$$f(d_{i,j_i}) = h(d_{i,j_i}),$$

$$f'(d_{i,j_i}) = h'(d_{i,j_i}),$$

  for $i = 0, 1, \ldots, k$, and

$$f_{m+1} = f_{m+2} = f_{m+3} = 0$$

  in the Gegenbauer expansion $f(t) = \sum_{i=0}^{m+4} f_i P_i^{(n)}(t)$.
- The total number of the above conditions is $2(k+1)+3 = 2k+5 = m+6$, with one more than the unknowns $f_0, f_1, \ldots, f_{m+4}$.

Algorithm for finding improving polynomials (Case III)

- If $Q_{m+3}(n,s) < 0$ and $Q_{m+4}(n,s) > 0$ then we find interpolating polynomial $f(t)$ of degree $\ell = m+3$ which satisfies

$$f(d_{i,j_i}) = h(d_{i,j_i}),$$

$$f'(d_{i,j_i}) = h'(d_{i,j_i}),$$

for $i = 0, 1, \ldots, k$, and

$$f_{m+1} = f_{m+2} = 0$$

in the Gegenbauer expansion $f(t) = \sum_{i=0}^{m+3} f_i P_i^{(n)}(t)$.

- The total number of the above conditions is $2(k+1) + 2 = 2k + 4 = m + 5$, with one more than the unknowns $f_0, f_1, \ldots, f_{m+3}$.

Some remarks

- In every step we pass through the best point from the previous one, so we could not get worse result.
- Since the solution, if it exists, is unique, the method converges to the solution (its existence follows from the existence of negative test function(s)).
- When the degree ℓ of the interpolating polynomial is even we add the condition
$$f(-1) = h(-1).$$
- If the calculation is hard (takes too much time) the number r can be decreased.
- If we need relaxation of the constraints, we remove one or two of the conditions $f_{m+1} = f_{m+2} = f_{m+3} = 0$, replacing by $f_i \geq 0$ accordingly.

Examples (1a)

- $n = 3, |C| = N = 5, m = 2$, (Case III)

  $Q_3(n,s) \approx 0.1851851849$, $Q_4(n,s) \approx 0.5473251030$,
  $Q_5(n,s) \approx -0.0576131687$, $Q_6(n,s) \approx 0.1667428746$,

  $\beta_0 = -1$, $\beta_1 = -1/9$,
  $\gamma_0 = 0.125$, $\gamma_1 = 0.675$.

Potential $h(t)$	Lower Bound $LB = N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$	Lower Bound LB_{new}	Energy $E(n,C;h)$
$(2(1-t))^{-0.5}$	12.8825	12.8870	12.9493
$(2(1-t))^{-1}$	8.375	8.3834	8.5
$(2(1-t))^{-2.5}$	2.3899	2.4013	2.5687

Examples (1b)

- $n = 3$, $|C| = N = 5$, $m = 2$, (Case III)
- $h = \frac{1}{2(1-t)}$,
- $f = 0.4978192964 + 0.4532997452t + 0.1997931768t^2 - 0.05508858968t^3 + 0.04957973071t^5$,
- The Gegenbauer coefficients: $f_5 = 0.006295838820, f_4 = f_3 = f_2 = 0, f_1 = 0.4414950474,$ $f_0 = 0.5644170220$
- The interpolation points: $d_{i,j} = \{-0.9673473010, -0.1048078439\}$.

Examples (2a)

- $n = 4,\ N = 24,\ m = 5,$ (Case I)

  $Q_6(n,s) = 0.08571428562,\ Q_7(n,s) = 0.1600000000,$
  $Q_8(n,s) = -0.02399999928,\ Q_9(n,s) = -0.02048000022,$

  $\alpha_0 = -0.8173526774,\ \alpha_1 = -0.2575978126,\ \alpha_2 = 0.4749504897,$
  $\rho_0 = 0.1384365306,\ \rho_1 = 0.4339994853,\ \rho_2 = 0.3858973183.$

Potential $h(t)$	Lower Bound $LB = N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$	Lower Bound LB_{new}	Energy $E(n, C; h)$
$(2(1-t))^{-1}$	333	333.1552	334

Examples (2b)

- $n = 4,\ N = 24,\ m = 5,$ (Case I)
- $f = 0.4987995047 + 0.4853427368t + 0.4547072718t^2 + 0.5604198622t^3 + 0.9381034665t^4 + 0.8082557979t^5 - 0.3471383319t^6 - 0.7102218761t^7 + 0.1983647611t^8 + 0.3551109381t^9,$
- The Gegenbauer coefficients:
  $f_9 = 0.006935760510, f_8 = 0.006973761132, f_7 = f_6 = 0,$
  $f_5 = 0.06415737967,\ f_4 = 0.2350426572,\ f_3 = 0.5067823610,$
  $f_2 = 0.7873531072,\ f_1 = 0.9210319576,\ f_0 = 0.7134671467,$
- The interpolation points: $d_{i,j} = \{-0.8671655835, -0.5120318420, -0.1910025124, 0.4749504897\}.$
- **Remark.** In the above examples polynomials and related results are current results.

THANK YOU FOR YOUR ATTENTION !