Well-rounded sublattices in the plane

Peter Zeiner
joint work with
M. Baake and R. Scharlau

University of Bielefeld
Bielefeld, Germany

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Well-rounded lattices

Definition

A lattice $\Lambda \in \mathbb{R}^d$ is called *well-rounded*, if the non-zero vectors of minimal length span $\mathbb{R}^d$.

Examples (in the plane)

- square lattice
- hexagonal lattice
- rhombic lattice $\frac{\pi}{3} < \gamma < \frac{2\pi}{3}$

Well-rounded lattices

Examples (in $\mathbb{R}^3$)

- cubic lattices
- hexagonal lattice if $a = c$
- body centered orthorhombic if $\frac{1}{2}\sqrt{a^2 + b^2 + c^2} \leq \min(a, b, c)$
- rhombohedral lattice if $60^\circ < \gamma \leq \arccos\left(\frac{1}{3}\right) \approx 109.47^\circ$

Examples (in $\mathbb{R}^4$)

- hypercubic lattices: $\mathbb{Z}^4$, D_4
- A_4-lattice

From well-rounded lattices to sphere packings

well-rounded lattices

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perfect lattices

|

extreme = perfect + eutactic

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densest lattice sphere packing

Well-rounded sublattices in the plane

Questions

- Which lattices Λ have well-rounded sublattices?
- What are the well-rounded sublattices of a planar lattice Λ?
- How many well-rounded sublattices are there for a given lattice Λ?

Existence of well-rounded sublattices

Lemma

A lattice $\Lambda \subseteq \mathbb{R}^2$ has a well-rounded sublattice if and only if it contains a rectangular sublattice.

Lemma

A lattice $\Lambda \subseteq \mathbb{R}^2$ has a well-rounded sublattice if and only if Λ has a coincidence reflection.

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A lattice $\Lambda \subseteq \mathbb{R}^2$ has a well-rounded sublattice if and only if $D_2 = 2mm \subseteq OC(\Lambda)$.

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A lattice $\Lambda \subseteq \mathbb{R}^2$ *has a well-rounded sublattice if and only if* $D_2 = 2mm \subseteq OC(\Lambda)$.

Existence of well-rounded sublattices

Theorem

Let $\Lambda = \langle 1, \tau \rangle_{\mathbb{Z}}$ and $n = |\tau|^2$ and $t = \tau + \bar{\tau}$. Λ has a well-rounded sublattice if and only if one of the following conditions is satisfied:

1. *Λ is rational, i.e. both t and n are rational;*
2. *t is rational, but n is not;*
3. *t is irrational, and there exist $q, r \in \mathbb{Q}$ with $\sqrt{q + r^2} \in \mathbb{Q}$ and $n = q + rt$.*

Number of sublattices in the plane

Theorem

A planar lattice may have

1. *"many"*
2. *"few"*
3. *no*

well-rounded sublattices.

Kühnlein 2011: via matrix invariants

Baake, Scharlau, PZ 2012: via explicit expressions

Square lattice: strategy

Goal

find all well-rounded sublattices of the square lattice

- use correspondence: well-rounded sublattices $\Longleftrightarrow$ rectangular sublattices with $\frac{a}{\sqrt{3}} \leq b \leq a\sqrt{3}$
- find (primitive) similar sublattices
- find rectangular sublattices parallel to square parent lattice

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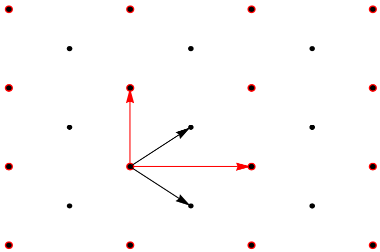

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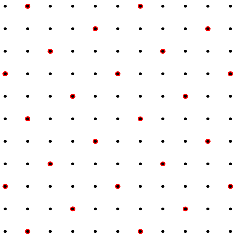

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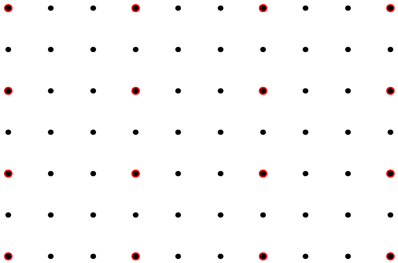
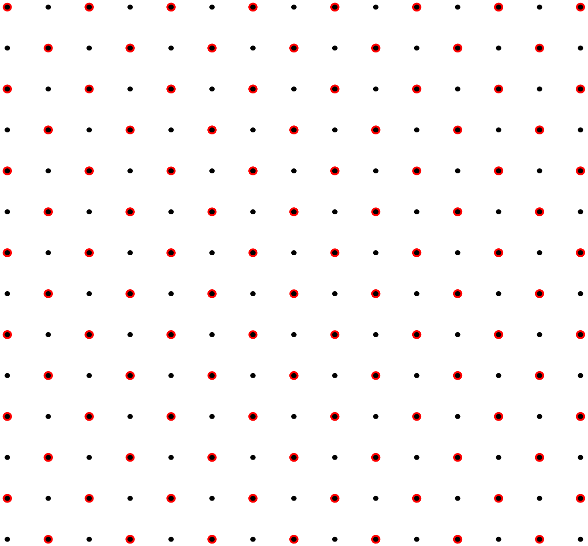


Similar sublattices



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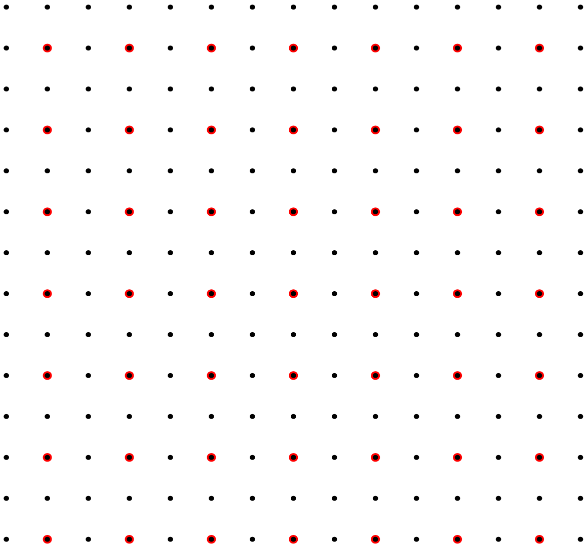

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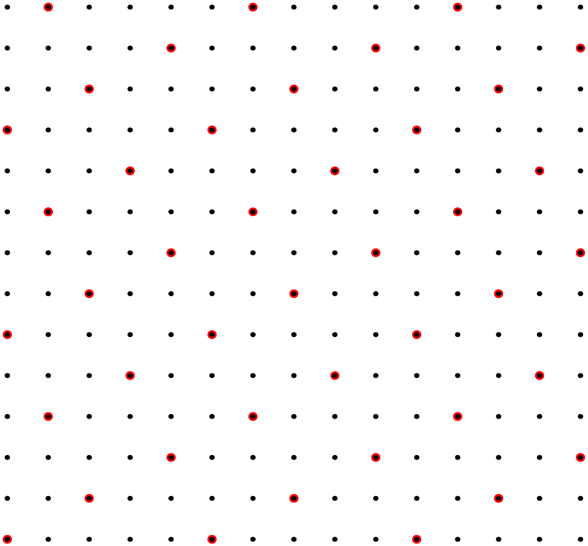

Example square lattice: number of SSLs

Theorem

$$\Phi_{\square}(s) = \sum_{n\in\mathbb{N}} \frac{b_{\square}(n)}{n^s} =$$

$$= \zeta_{\mathbb{Q}(i)}(s) = L(s,\chi_{-4})\,\zeta(s)$$

$$= \frac{1}{1-2^{-s}} \prod_{p\equiv 1(4)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 3(4)} \frac{1}{1-p^{-2s}}$$

$$= 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{2}{10^s} + \frac{2}{13^s} + \frac{1}{16^s} + \frac{2}{17^s} + \frac{1}{18^s} + \frac{2}{20^s} + \frac{3}{25^s} + \frac{2}{26^s} + \frac{2}{29^s} + \frac{1}{32^s} + \dots$$

Example square lattice: number of primitive SSLs

Theorem

$$\Phi_{\square}^{pr}(s) = \sum_{n\in\mathbb{N}} \frac{b_{\square}^{\mathrm{pr}}(n)}{n^s} =$$

$$= \frac{\Phi_{\square}(s)}{\zeta(2s)} = (1+2^{-s}) \prod_{p\equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}}$$

$$= 1 + \frac{1}{2^s} + \frac{2}{5^s} + \frac{2}{10^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{25^s} + \frac{2}{26^s} + \frac{2}{29^s}$$

$$+ \frac{2}{34^s} + \frac{2}{37^s} + \frac{2}{41^s} + \frac{2}{50^s} + \frac{2}{53^s} + \frac{2}{58^s} + \frac{2}{61^s} + \frac{4}{65^s} + \dots$$

Square lattice: generating function

$a_{\square}(n)$: number of well-rounded sublattices with index n

Theorem

$$\Phi_{\square,\mathrm{wr}}(s) = \sum_{n\in\mathbb{N}} \frac{a_{\square}(n)}{n^s} = \Phi_{\mathrm{wr},even}(s) + \Phi_{\mathrm{wr},odd}(s) + \Phi_{\square}(s)$$

where

$$\Phi_{\mathrm{wr},even}(s) = \frac{2}{2^s}\Phi_{\square}^{\mathrm{pr}}(s)\sum_{p\in\mathbb{N}}\sum_{p<q<\sqrt{3}p}\frac{1}{p^s q^s}$$

$$\Phi_{\mathrm{wr},odd}(s) = \frac{2}{1+2^{-s}}\Phi_{\square}^{\mathrm{pr}}(s)\sum_{k\in\mathbb{N}}\sum_{k<\ell<\sqrt{3}k+\frac{\sqrt{3}-1}{2}}\frac{1}{(2k+1)^s(2\ell+1)^s}$$

$$\Phi_{\square}(s) = \zeta(2s)\Phi_{\square}^{\mathrm{pr}}(s) = \zeta_{\mathbb{Q}(\mathrm{i})}(s) = L(s,\chi_{-4})\,\zeta(s)$$

Square lattice: generating function

$$\Phi_{\square,\mathrm{wr}}(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{2}{10^s} + \frac{2}{12^s} + \frac{2}{13^s} + \frac{2}{15^s} + \frac{1}{16^s} + \frac{2}{17^s} + \frac{1}{18^s} + \frac{2}{20^s} + \frac{4}{24^s} + \frac{3}{25^s} + \frac{2}{26^s} + \frac{2}{29^s} + \frac{2}{30^s} + \frac{1}{32^s} + \dots$$

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Square lattice: analyticity and some bounds

Theorem (Baake, Scharlau, PZ 2012)

$\Phi_{\square,\mathrm{wr}}(s)$ meromorphic for $s > \frac{1}{2}$, pole at $s = 1$ of order 2.

$$\Phi_{\square,\mathrm{wr}}(s) = D_{\square}(s) + \phi(s)$$

with

$$D_{\square}(s) = \frac{2+2^s}{1+2^s} \frac{1-\sqrt{3}^{1-s}}{s-1} \frac{L(s,\chi_{-4})}{\zeta(2s)} \zeta(s)\zeta(2s-1)$$

and $\phi(s)$ meromorphic for $s > \frac{1}{2}$, pole at $s = 1$ of order 1.

For $s > 1$

$$D_{\square}(s) - \Phi_{\square}(s) < \Phi_{\square,\mathrm{wr}}(s) < D_{\square}(s) + \Phi_{\square}(s).$$

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Delange's Theorem

Theorem

Let $F(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}$ *be a Dirichlet series with non-negative coefficients which converges for* $\mathrm{Re}(s) > \alpha > 0$*. Suppose that* $F(s)$ *is holomorphic at all points of the line* $\{\mathrm{Re}(s) = \alpha\}$ *except at* $s = \alpha$*. Here, when approaching* α *from the half-plane to the right of it, we assume* $F(s)$ *to have a singularity of the form* $F(s) = g(s) + h(s)/(s-\alpha)^{n+1}$ *where* n *is a non-negative integer, and both* $g(s)$ *and* $h(s)$ *are holomorphic at* $s = \alpha$*. Then, as* $x \to \infty$*, we have*

$$A(x) := \sum_{m \le x} a(m) \sim \frac{h(\alpha)}{\alpha \cdot n!} \, x^{\alpha} \, (\log x)^n \, .$$

Square lattice: asymptotics 1

Corollary (Baake, Scharlau, PZ 2012)

$$\sum_{n\leq x} a_{\square}(n) = \frac{\log(3)}{2\pi} x \log(x) + \mathcal{O}(x \log(x))$$

as $x \to \infty$.

Compare:

$$\sum_{n\leq x} b_{\square}(n) \sim \frac{\pi}{4} x$$

$$\sum_{n\leq x} b_{\square}^{\mathrm{pr}}(n) \sim \frac{3}{2\pi} x$$

Square lattice: asymptotics 2

Theorem (Baake, Scharlau, PZ 2012)

$$\sum_{n \leq x} a_{\square}(n) = \frac{\log(3)}{3} \frac{L(1, \chi_{-4})}{\zeta(2)} x(\log(x) - 1) + c_{\square} x + \mathcal{O}(x^{3/4} \log(x))$$

$$= \frac{\log(3)}{2\pi} x \log(x) + \left(c_{\square} - \frac{\log(3)}{2\pi} \right) x + \mathcal{O}(x^{3/4} \log(x))$$

where

$$c_{\square} \approx 0.6272237$$

Hexagonal lattice: asymptotics

$a_{\triangle}(n)$: number of well-rounded sublattices with index n

Theorem (Baake, Scharlau, PZ 2012)

$$\sum_{n\le x} a_{\triangle}(n) = \frac{9\log(3)}{16}\frac{L(1,\chi_{-3})}{\zeta(2)}x(\log(x)-1)+c_{\triangle}x+\mathcal{O}(x^{3/4}\log(x))$$

$$= \frac{3\sqrt{3}\log(3)}{8\pi}x\log(x)+\left(c_{\triangle}-\frac{3\sqrt{3}\log(3)}{8\pi}\right)x+\mathcal{O}(x^{3/4}\log(x))$$

where

$$c_{\triangle}\approx 0.4915036$$

CSL

Definition

Let $\Lambda \subset \mathbb{R}^d$ be a lattice, $R \in O(d)$. Then

$$\Lambda(R) := \Lambda \cap R\Lambda$$

is called a *coincidence site lattice* (CSL), if Λ and $R\Lambda$ are commensurate.

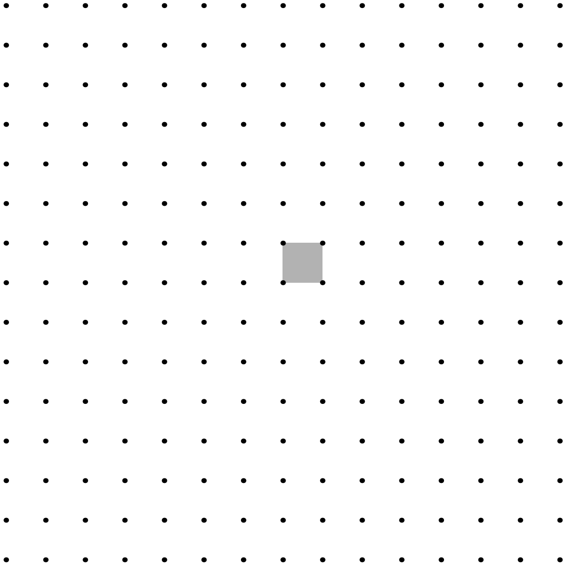
Definition

The index

$$\Sigma_\Lambda(R) := [\Lambda : \Lambda(R)] < \infty$$

is called *coincidence index*.

Example square lattice



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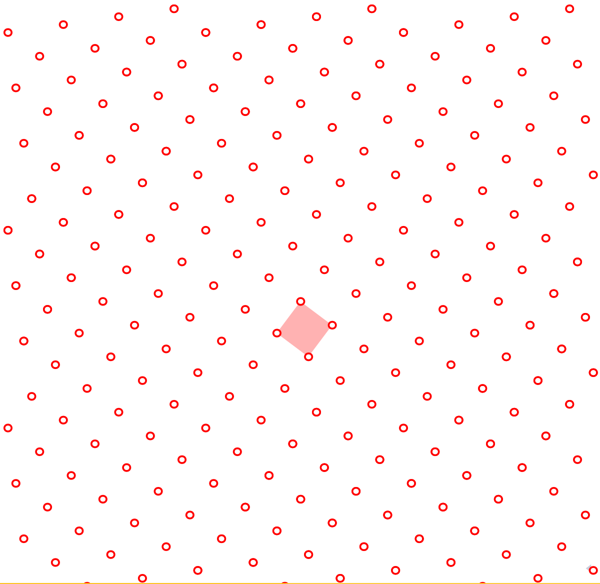

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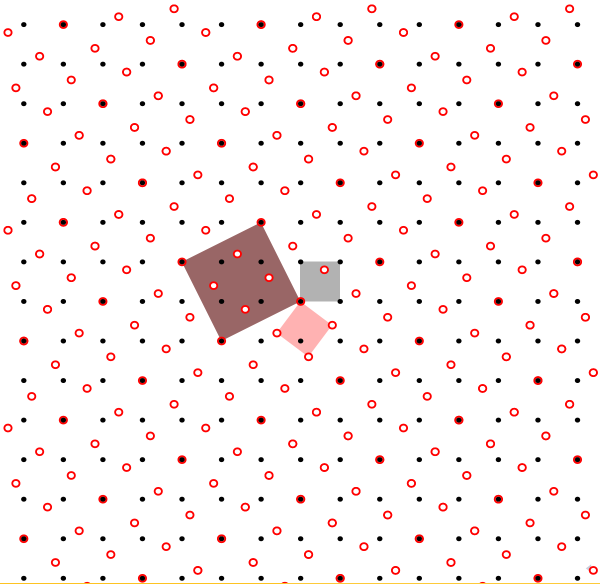

Coincidence isometries

Lemma

The set of all coincidence isometries

$$OC(\Lambda) := \{R \in O(d) \big| \Sigma_\Lambda(R) < \infty\}$$

forms a group, a subgroup of $O(d)$.

Example square lattice: Coincidence rotations

Theorem

The coincidence rotations of $\mathbb{Z}[i]$ *are given by*

$$e^{i\varphi} = \varepsilon \frac{z}{\bar{z}} = \varepsilon \prod_{p \equiv 1\,(4)} \left(\frac{\omega_p}{\bar{\omega}_p} \right)^{n_p},$$

where ε *is a unit and only finitely many* $n_p \neq 0$*.*
The corresponding coincidence index is given by

$$\Sigma(e^{i\varphi}) = \prod_{p \equiv 1\,(4)} p^{|n_p|}.$$

The spectrum is the set of all integers that contain only prime factors $p \equiv 1 \pmod 4$*.*

Example square lattice: CSLs

Theorem

The CSLs of $\mathbb{Z}[i]$ *are given by*

$$\mathbb{Z}[i] \cap e^{i\varphi}\mathbb{Z}[i] = \omega(\varphi)\mathbb{Z}[i],$$

where

$$\omega(\varphi) := \prod_{\substack{p \equiv 1\,(4) \\ n_p > 0}} \omega_p^{n_p} \prod_{\substack{p \equiv 1\,(4) \\ n_p < 0}} \bar{\omega}_p^{n_p}.$$

Example square lattice: number of CSLs

Theorem

The generating function for the number of CSLs of $\mathbb{Z}[i]$ is given by

$$\Psi_{\square}(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \prod_{p\equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}} = \frac{1}{1+2^{-s}} \frac{\zeta_{\mathbb{Q}(i)}(s)}{\zeta(s)}$$
$$= \frac{1}{1+2^{-s}} \Phi_{\square}^{pr}(s)$$
$$= 1 + \frac{2}{5^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{25^s} + \frac{2}{29^s} + \frac{2}{37^s} + \frac{2}{41^s}$$
$$+ \frac{2}{53^s} + \frac{2}{61^s} + \frac{4}{65^s} + \frac{2}{73^s} + \dots$$

Asymptotic growth rate:

$$\sum_{n\leq x} f(n) \sim \frac{1}{\pi} x$$

Lattice with exactly 2 coincidence reflections

Theorem (Baake, Scharlau, PZ 2012)

Let Λ be a lattice that has exactly two coincidence reflections. Let Σ be the coincidence index of the unique CSL and κ be the ratio of its lattice constants.

1. *If the CSL is rectangular,*

$$\Phi_{\mathrm{wr}}(s) = \frac{1}{2^s \Sigma^s} \sum_{p \in \mathbb{N}} \sum_{\frac{\kappa}{\sqrt{3}} p < q < \sqrt{3}\,\kappa\, p} \frac{1}{p^s q^s}$$

Lattice with exactly 2 coincidence reflections

Theorem

2. *If the CSL is rhombic,*

$$\Phi_{\mathrm{wr}}(s) = \frac{1}{2^s \Sigma^s} \phi_{\mathrm{wr},even}(\kappa; s) + \frac{1}{\Sigma^s} \phi_{\mathrm{wr},odd}(\kappa; s)$$

with

$$\phi_{\mathrm{wr},even}(\kappa; s) = \frac{1}{2^s} \sum_{p \in \mathbb{N}} \sum_{\frac{\kappa}{\sqrt{3}} p < q < \sqrt{3}\,\kappa\, p} \frac{1}{p^s q^s}$$

$$\phi_{\mathrm{wr},odd}(\kappa; s) = \sum_{k \in \mathbb{N}} \sum_{\frac{\kappa}{\sqrt{3}}(k+\frac{1}{2}) - \frac{1}{2} < \ell < \sqrt{3}\,\kappa\,(k+\frac{1}{2}) - \frac{1}{2}} \frac{1}{(2k+1)^s (2\ell+1)^s}$$

Lattice with exactly 2 coincidence reflections: asymptotics

$a_\Lambda(n)$: number of well-rounded sublattices of Λ with index n

Theorem (Kühnlein 2011)

$$\sum_{n \leq x} a_\Lambda(n) \sim c\, x$$

Theorem (Baake, Scharlau, PZ 2012)

Let Λ be a lattice that has exactly one non-trivial CSL. Let Σ be its coincidence index. Then

$$\sum_{n \leq x} a_\Lambda(n) = \frac{\log 3}{4\Sigma} x + \mathcal{O}(\sqrt{x}).$$

Lattice with exactly 2 coincidence reflections: asymptotics

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General rational lattice

Theorem

Let Λ *be a rational lattice and let* $\Phi_{\Lambda}^{\triangle}(s)$ *be the generating function of all hexagonal sublattices of* Λ*. Now, for any pair of coincidence reflections* $\pm R \in \mathcal{R}_{\Lambda}$*, let* $\sigma(R) = [\Lambda : \Lambda_R]$ *and let* $\kappa(R)$ *be the length ratio of orthogonal basis vectors of* Λ_R*. Then, the generating function for the number of well-rounded sublattices reads*

$$\begin{aligned}\Phi_{\Lambda,\mathrm{wr}}(s) = &\sum_{\pm R \in \mathcal{R}_1} \frac{1}{\sigma(R)^s} \phi_{\mathrm{wr,even}}(\kappa(R); s) \\ &+ \sum_{\pm R \in \mathcal{R}_2} \frac{1}{\sigma(R)^s} \left(\phi_{\mathrm{wr,even}}(\kappa(R); s) + 2^s \phi_{\mathrm{wr,odd}}(\kappa(R); s)\right) \\ &- 2\Phi_{\Lambda}^{\triangle}(s).\end{aligned}$$

General rational lattice

Theorem (Fukshansky 2012)

Let Λ be a rational lattice. Then

$$c_1 x \log(x) \leq \sum_{n \leq x} a_\Lambda(n) \leq c_2 x \log(x)$$

for sufficiently large x.

Theorem (M. Baake, R. Scharlau, PZ 2013, Kühnlein 2013)

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General rational lattice

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Thank you!