Tight relative 2-designs on two shells in Johnson association scheme

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The purpose of design theory is to find “good” finite subset which approximate the whole space $X$.

\[
\begin{align*}
\text{spherical } t\text{-design} & \iff \text{combinatorial } t\text{-design} \\
(X = \text{unit sphere } S^{n-1}) & \iff (X = \binom{V}{k}) \\
\downarrow & \downarrow \\
\text{weighted spherical } t\text{-design} & \iff \text{weighted combinatorial } t\text{-design} \\
\downarrow & \downarrow \\
\text{Euclidean } t\text{-design} & \iff \text{relative } t\text{-design} \\
(X = S^{n-1}(r_1) \cup \cdots \cup S^{n-1}(r_p)) & \iff (X = \binom{V}{k_1} \cup \cdots \cup \binom{V}{k_p})
\end{align*}
\]

Association scheme

Let $R_0, R_1, \ldots, R_d$ be binary relations on a finite set $X$.

- $R_0 = \{(x, x) | x \in X\}$.
- $R_0 \cup R_1 \cup \ldots \cup R_d = X \times X$, $R_i \cap R_j = \emptyset (i \neq j)$.
- $^tR_i = R_j$ for some $j \in \{0, 1, \ldots, d\}$, where $^tR_i = \{(y, x) | (x, y) \in R_i\}$.
- Given $(x, y) \in R_k$, then $\# \{z \in X | (x, z) \in R_i, (z, y) \in R_j\} = p_{i,j}^k$.

Then $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is an association scheme.

Moreover, $\mathcal{X}$ is symmetric if $^tR_i = R_i$. 
The $i$-th adjacency matrix $A_i$ of $X$ is defined by

$$A_i(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

- $A_0 = I$.
- $A_0 + A_1 + \ldots + A_d = J$.
- $^tA_i = A_j$ for some $j \in \{0, 1, \ldots, d\}$.
- $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k = A_j A_i$.

$\{A_0, A_1, \ldots, A_d\}$ generates an algebra which is called the Bose-Mesner algebra of the association scheme $X$. 
Symmetric association scheme: \( \mathcal{X} = (X, \{ R_i \}_{0 \leq i \leq d}) \).

Adjacency matrices: \( A_0, \ldots, A_d \).

Primitive idempotents: \( E_0, \ldots, E_d \).

Bose-Mesner algebra: \( \mathbb{C}[A_0, \ldots, A_d] = \mathbb{C}[E_0, \ldots, E_d] \)

\[
A_i A_j = \sum_{k=0}^{d} p_{i,j}^{k} A_k \quad \text{and} \quad E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{i,j}^{k} E_k.
\]

Eigenmatrices:

\[
(A_0, \ldots, A_d) = (E_0, \ldots, E_d) P
\]

\[
(E_0, \ldots, E_d) = \frac{1}{|X|} (A_0, \ldots, A_d) Q.
\]
$\mathcal{X}$ is called a **Q-polynomial scheme** with respect to $E_0, E_1, \ldots, E_d$, if there exist some polynomials $v_i^*(x)$ of degree $i$ such that $E_i = v_i^*(E_1)$.

$\mathcal{F}(X)$: the vector space consists of all the real valued functions on $X$.

$L_j(X)$: the subspace of $\mathcal{F}(X)$ spanned by all the columns of $E_j$.

$u_0 \in X$: a fixed point arbitrarily.

$X_i = \{x \in X \mid (u_0, x) \in R_i\}$, then $X_0, X_1, \ldots, X_d$ are called shells of $\mathcal{X}$.

$$\mathcal{F}(X) = L_0(X) \perp L_1(X) \perp \cdots \perp L_d(X).$$
Definition 1 (Ei. Bannai and Et. Bannai, 2012)

Let \((Y, w)\) be a weighted subset of \(X\) with positive function \(w\) on \(Y\). \((Y, w)\) is called a relative \(t\)-design with respect to \(u_0\) if the following condition holds

\[
\sum_{i=1}^{p} \sum_{x \in X_{r_i}} \frac{W(Y \cap X_{r_i})}{|X_{r_i}|} f(x) = \sum_{y \in Y} w(y)f(y)
\]

for any function \(f \in L_0(X) \perp L_1(X) \perp \cdots \perp L_t(X)\).

Let \(\{r_1, r_2, \ldots, r_p\} = \{r | X_r \cap Y \neq \emptyset\}\) and \(S = X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}\).

Denote \(Y_{r_i} = Y \cap X_{r_i}\) and \(N_{r_i} = |Y_{r_i}|, i = 1, 2, \ldots, p\).
Definition 2

Let \( V = \{1, 2, \ldots, v\} \) and \( X = \{T \subset V : |T| = k\} (k \leq \frac{v}{2}) \). Define

\[
(x, y) \in R_i \quad \text{if} \quad |x \cap y| = k - i.
\]

Then \( X = (X, \{R_i\}_{0 \leq i \leq k}) \) is a symmetric association scheme of class \( k \) and is called Johnson association scheme \( J(v, k) \).

Theorem 3 (Ei. Bannai and Et. Bannai, 2012)

Let \( (Y, w) \) be a relative \( 2e \)-design of a Q-polynomial scheme, then the following inequality holds.

\[
|Y| \geq \dim(L_0(S) + L_1(S) + \ldots + L_e(S)).
\] (1)

where \( L_j(S) = \{f|_S, f \in L_j(X)\}, j = 0, 1, \ldots, e. \)

If equality holds in (1), then \( (Y, w) \) is called a tight relative \( 2e \)-design with respect to \( u_0 \).
Theorem 4 (Ei. Bannai, Et. Bannai, S. Suda and H. Tanaka)

\[ \dim(L_0(S) + L_1(S) + \cdots + L_e(S)) = m_e + m_{e-1} + \cdots + m_{e-p+1} \]

for \( J(v, k) \) provided that \( e \leq r_1 < r_2 < \cdots < r_p \leq k - e \), where

\[ m_j = \binom{v}{j} - \binom{v}{j-1}. \]

Theorem 5 (Ei. Bannai, Et. Bannai and Hi. Bannai)

\( \mathcal{X} \) is a Q-polynomial scheme and \( G = \text{Aut}(\mathcal{X}) \). Let \((Y, w)\) be a tight relative 2e-design with respect to \( u_0 \). Assume that the stabilizer \( G_{u_0} \) acts transitively on every shell \( X_r \), \( 1 \leq r \leq d \). Then weight function \( w \) is constant on each \( Y_{r_i} \) \((1 \leq i \leq p)\) and denote this value by \( w_{r_i} \).
Theorem 6

Take a sequence elements from $X$ as
\[
\begin{align*}
u_0 &= \{1, 2, \ldots, k\}, \\
u_i &= \{1, 2, \ldots, k - 1, k + i + 1\}, \quad (1 \leq i \leq v - k - 1), \\
u_i &= \{1, 2, \ldots, k, k + 1\} \setminus \{i - (v - k) + 1\}, \quad (v - k \leq i \leq v - 1).
\end{align*}
\]

Then $\{\phi_{u_0}^{(0)}|_S, \phi_{u_1}^{(1)}|_S, \ldots, \phi_{u_{v-1}}^{(1)}|_S\}$ is a basis of $L_0(S) + L_1(S)$, where $S = X_{r_1} \cup X_{r_2}$.

\[
\begin{align*}
\phi_0(x) &= \phi_{u_0}^{(0)}(x) = \left|X\right|E_0(x, u_0) \equiv 1, \\
\phi_i(x) &= \phi_{u_i}^{(1)}(x) = \left|X\right|E_1(x, u_i).
\end{align*}
\]
Inner product is defined by

$$\langle f, g \rangle = \sum_{i=1}^{2} \frac{W_{r_i}}{|X_{r_i}|} \sum_{x \in X_{r_i}} f(x)g(x).$$

d_0 = \langle \phi_0, \phi_0 \rangle,
c_0 = \langle \phi_i, \phi_i \rangle, \quad \text{for} \quad 1 \leq i \leq v - 1,
c_{1,3} = \langle \phi_0, \phi_i \rangle, \quad \text{for} \quad 1 \leq i \leq v - 1,
c_{1,1} = \langle \phi_i, \phi_j \rangle = \langle \phi_i, \phi_{v-1} \rangle, \quad \text{for} \quad 1 \leq i \neq j \leq v - k - 1,
c_{1,2} = \langle \phi_i, \phi_j \rangle = \langle \phi_i, \phi_{v-1} \rangle, \quad \text{for} \quad v - k \leq i \neq j \leq v - 2,
c_2 = \langle \phi_i, \phi_j \rangle, \quad \text{for} \quad 1 \leq i \leq v - k - 1, v - k \leq j \leq v - 2.
Gram-Schmidt’s method: \( \{ \phi_1, \ldots, \phi_{v-1}, \phi_0 \} \longrightarrow \{ \varphi_1, \varphi_2, \ldots, \varphi_v \} \).

\[ \varphi_1 = \frac{\phi_1}{c_0}, \]

\[ \varphi_j = \frac{1}{\sqrt{D_{j-1} D_j}} \begin{vmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_2, \phi_1 \rangle & \ldots & \langle \phi_j, \phi_1 \rangle \\ \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \ldots & \langle \phi_j, \phi_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_1, \phi_{j-1} \rangle & \langle \phi_2, \phi_{j-1} \rangle & \ldots & \langle \phi_j, \phi_{j-1} \rangle \\ \phi_1 & \phi_2 & \ldots & \phi_i \end{vmatrix}. \]

The Gram determinant \( D_j \) is given by

\[ D_j = \begin{vmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_2, \phi_1 \rangle & \ldots & \langle \phi_j, \phi_1 \rangle \\ \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \ldots & \langle \phi_j, \phi_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_1, \phi_j \rangle & \langle \phi_2, \phi_j \rangle & \ldots & \langle \phi_j, \phi_j \rangle \end{vmatrix}. \]
Basic strategy

Let $H$ be a matrix indexed by $Y \times [v]$ whose $(y, j)$-entry is defined by $\sqrt{w(y)} \varphi_j(y)$.

Then $(^t H H)(i, j) = \delta_{i, j}$ and $(H^t H)(x, y) = \delta_{x, y}$ imply

$$
\begin{align*}
\sum_{y \in Y} w(y) \varphi_i(y) \varphi_j(y) &= \delta_{i, j}, \\
\sum_{j=1}^{v} w(y) \varphi_j(x) \varphi_j(y) &= \delta_{x, y}.
\end{align*}
$$

Case 1: $x = y \in X_{r_i} (i = 1, 2)$.

$x = \{1, 2, \ldots, k - r_i, k + 1, k + 2, \ldots, k + r_i\}$

$$
\frac{1}{w_{r_i}} = \sum_{s=1}^{v} \varphi_s^2(x). \quad (2)
$$
\[
\sum_{j=1}^{v} w(y) \varphi_j(x) \varphi_j(y) = \delta_{x,y}.
\]

Case 2: \(x, y \in X_{r_i}, (x, y) \in R_{\alpha_i}, i = 1, 2\).

\[
x = \{1, 2, \ldots, k - r_i, k + 1, k + 2, \ldots, k + r_i\},
|y \cap \{1, 2, \ldots, k - r_i\}| = a_i.
\]

\[
\sum_{s=1}^{v} \varphi_s(x) \varphi_s(y) = \frac{f(W_{r_1}, W_{r_2})}{g(W_{r_1}, W_{r_2})}.
\] (3)

Case 3: \(x \in X_{r_1}, y \in X_{r_2}, (x, y) \in R_{\gamma}, r_1 < r_2\).

\[
x = \{1, 2, \ldots, k - r_1, k + 1, k + 2, \ldots, k + r_1\},
|y \cap \{1, 2, \ldots, k - r_1\}| = a_3.
\]

\[
\sum_{s=1}^{v} \varphi_s(x) \varphi_s(y) = \frac{f'(W_{r_1}, W_{r_2})}{g'(W_{r_1}, W_{r_2})}.
\] (4)
Proposition 7

Let $w_{r_i}$ be the weight function on shell $X_{r_i}(i = 1, 2)$, then $\frac{w_{r_2}}{w_{r_1}}$ is rational for any tight relative 2-design $(Y, w)$.

\[ \sum_{s=1}^{v} \varphi_s(x)\varphi_s(y) = 0 \text{ if } x \neq y \in Y. \]

i.e., $f$ and $f'$ have a common factor $g_1 W_{r_1} - g_2 W_{r_2}$ such that $\frac{g_2}{g_1} \in \mathbb{Q}_{>0}$.

List all feasible parameters of tight relative 2-designs on two shells in $J(v, k)$ for $v \leq 100$. 
Constructing a relative 2-design on two shells is equivalent to find the $k$-element subset $u_0 \notin Y$ with the following property ($\star$):

$$\{|u_0 \cap y| : y \in Y\} \text{ consists of exactly 2 values.} \quad (\star)$$

We owe Z. Xiang for a very fast program to check whether a given design has the property ($\star$) or not.
Special results

- For 2-(16, 6, 2) designs, 2 out of 3 non-isomorphic designs have the property (⋆).

- 2-(36, 15, 6) designs are not yet completely classified, but some of them are listed on the homepage of E. Spence.
  1. 31 (out of 32548 coming from SRGs of type (36, 15, 6, 6)) have the property (⋆).
  2. 178 (out of 180 coming from SRGs of type (36, 21, 12, 12)) have the property (⋆).
  3. 339 (out of 617 coming from (known) designs with polarities with absolute points 6, 12, 18, 24, 30) have the property (⋆).

- For 2-(45, 12, 3) designs, 29 (out of 78 coming from SRGs of type (45, 12, 3, 3)) have the property (⋆).
Examples

Let \( G \) be a group of order \( v \) and \( D \) be a \( k \)-element subset of \( G \) with \( 0 < k < v \). Then \( D \) is called a \((v, k, \lambda)\)-difference set if each non-identity element \( g \) in \( G \) can be expressed in exactly \( \lambda \) ways as \( g = d_1 d_2^{-1} \) with \( d_1, d_2 \in D \).

\[ \{v, k, r_1, r_2, N_{r_1}, N_{r_2}\} = \{36, 15, 7, 10, 15, 21\} \]

- Abelian group \( G = \mathbb{Z}_6 \times \mathbb{Z}_6 \)

Difference set \( D \) and blocks set \( \mathcal{B} = \{gD | g \in G\} \).

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & \circ & \times & \times & \times & \times & \times \\
1 & \times & \times & \circ & \circ & \circ & \circ & \circ \\
2 & \times & \circ & \times & \circ & \circ & \circ & \circ \\
3 & \times & \circ & \circ & \times & \circ & \circ & \circ \\
4 & \times & \circ & \circ & \circ & \times & \circ & \circ \\
5 & \times & \circ & \circ & \circ & \circ & \times \\
\end{array}
\Rightarrow
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & \times & \times & \times & \circ & \times & \times \\
1 & \times & \circ & \circ & \times & \times & \times & \times \\
2 & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
3 & \circ & \times & \times & \circ & \circ & \circ & \circ \\
4 & \times & \circ & \circ & \circ & \circ & \circ & \circ \\
5 & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

\( D \) 

\( u_0 \)
\{v, k, r_1, r_2, N_{r_1}, N_{r_2}\} = \{36, 15, 7, 10, 15, 21\}.

- Non-Abelian group $G = S_3 \times S_3$

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$D$

$1 = Id, \quad g_1 = (12), \quad g_2 = (13), \quad g_3 = (23), \quad g_4 = (123), \quad g_5 = (132)$. 
Conjecture

There exist two families of tight relative 2-designs with the following parameters.

1. \( v = q^{s+1}(q^s + \cdots + q + 2) \),
   \( k = q^s(q^s + \cdots + q + 1) \),
   \( \lambda = q^s(q^{s-1} + \cdots + q + 1) \).
   \( r_1 = q^{s-1}(q^{s+1} - 1), \quad r_2 = q^{s-1}(q^{s+1} + q - 1), \quad N_{r_2} = k \).

2. \( v = 4u^2 \),
   \( k = 2u^2 - u \),
   \( \lambda = u^2 - u \).
   \( r_1 = u^2 - \frac{u}{2}, \quad r_2 = u^2 + \frac{u}{2}, \quad N_{r_2} = k, \) if \( u \) is even;
   \( r_1 = u^2 - \frac{u+1}{2}, \quad r_2 = u^2 + \frac{u-1}{2}, \quad N_{r_1} = k, \) if \( u \) is odd.
Thank you