# Tight relative $2$-designs on two shells in Johnson association scheme

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The purpose of design theory is to find "good" finite subset which approximate the whole space $X$.

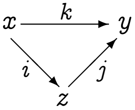
| | | |
|---|---|---|
| spherical $t$-design <br> ($X =$ unit sphere $S^{n-1}$) | $\longleftrightarrow$ | combinatorial $t$-design <br> ($X = \binom{V}{k}$) |
| $\downarrow$ | | $\downarrow$ |
| weighted spherical $t$-design | $\longleftrightarrow$ | weighted combinatorial $t$-design |
| $\downarrow$ | | $\downarrow$ |
| Euclidean $t$-design <br> ($X = S^{n-1}(r_1) \cup \cdots \cup S^{n-1}(r_p)$) | $\longleftrightarrow$ | relative $t$-design <br> ($X = \binom{V}{k_1} \cup \cdots \cup \binom{V}{k_p}$) |

Ei. Bannai, Et. Bannai, Hi. Bannai, em On the existence of tight relative 2-designs on binary Hamming association schemes, Discrete Math. 314 (2014), 17–37.

# Association scheme

Let $R_0, R_1, \ldots, R_d$ be binary relations on a finite set $X$.

- $R_0 = \{(x,x) | x \in X\}$.
- $R_0 \cup R_1 \cup \ldots \cup R_d = X \times X$, $R_i \cap R_j = \emptyset (i \neq j)$.
- ${}^tR_i = R_j$ for some $j \in \{0, 1, \ldots, d\}$ ,where ${}^tR_i = \{(y,x) | (x,y) \in R_i\}$.
- Given $(x,y) \in R_k$, then $\#\{z \in X | (x,z) \in R_i, (z,y) \in R_j\} = p^k_{i,j}$.




Then $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is an association scheme.

Moreover, $\mathfrak{X}$ is symmetric if ${}^tR_i = R_i$.

# Adjacency matrix

The $i$-th adjacency matrix $A_i$ of $\mathfrak{X}$ is defined by

$$A_i(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

- $A_0 = I$.
- $A_0 + A_1 + \ldots + A_d = J$.
- ${}^t A_i = A_j$ for some $j \in \{0, 1, \ldots, d\}$.
- $A_i A_j = \sum_{i=0}^{d} p_{i,j}^k A_k = A_j A_i$.

$\{A_0, A_1, \ldots, A_d\}$ generates an algebra which is called the Bose-Mesner algebra of the association scheme $\mathfrak{X}$.

# Matrix version

- Symmetric association scheme: $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$.
- Adjacency matrices: $A_0, \ldots, A_d$.
- Primitive idempotents: $E_0, \ldots, E_d$.
- Bose-Mesner algebra: $\mathbb{C}[A_0, \ldots, A_d] = \mathbb{C}[E_0, \ldots, E_d]$

$$A_i A_j = \sum_{i=0}^{d} p_{i,j}^k A_k \quad \text{and} \quad E_i \circ E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_{i,j}^k E_k.$$

- Eigenmatrices:

$$(A_0, \ldots, A_d) = (E_0, \ldots, E_d) P$$

$$(E_0, \ldots, E_d) = \frac{1}{|X|} (A_0, \ldots, A_d) Q$$

$\mathfrak{X}$ is called a Q-polynomial scheme with respect to $E_0, E_1, \ldots, E_d$, if there exist some polynomials $v_i^*(x)$ of degree $i$ such that $E_i = v_i^*(E_1)$.

$\mathcal{F}(X)$: the vector space consists of all the real valued functions on $X$.

$L_j(X)$: the subspace of $\mathcal{F}(X)$ spanned by all the columns of $E_j$.

$u_0 \in X$: a fixed point arbitrarily.

$X_i = \{x \in X \mid (u_0, x) \in R_i\}$, then $X_0, X_1, \ldots, X_d$ are called shells of $\mathfrak{X}$.

$$\mathcal{F}(X) = L_0(X) \perp L_1(X) \perp \cdots \perp L_d(X).$$

**Definition 1 (Ei. Bannai and Et. Bannai, 2012)**

Let $(Y, w)$ be a weighted subset of $X$ with positive function $w$ on $Y$. $(Y, w)$ is called a relative $t$-design with respect to $u_0$ if the following condition holds

$$\sum_{i=1}^{p} \sum_{x \in X_{r_i}} \frac{W(Y \cap X_{r_i})}{|X_{r_i}|} f(x) = \sum_{y \in Y} w(y) f(y)$$

for any function $f \in L_0(X) \perp L_1(X) \perp \cdots \perp L_t(X)$.

Let $\{r_1, r_2, \ldots, r_p\} = \{r | X_r \cap Y \neq \emptyset\}$ and $S = X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$.

Denote $Y_{r_i} = Y \cap X_{r_i}$ and $N_{r_i} = |Y_{r_i}|$, $i = 1, 2, \ldots, p$.

**Definition 2**

Let $V = \{1, 2, \ldots, v\}$ and $X = \{T \subset V : |T| = k\}(k \leq \frac{v}{2})$. Define

$$(x, y) \in R_i \quad \text{if} \quad |x \cap y| = k - i.$$

Then $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq k})$ is a symmetric association scheme of class $k$ and is called Johnson association scheme $J(v, k)$.

**Theorem 3 (Ei. Bannai and Et. Bannai, 2012)**

*Let $(Y, w)$ be a relative $2e$-design of a Q-polynomial scheme, then the following inequality holds.*

$$|Y| \geq \dim(L_0(S) + L_1(S) + \ldots + L_e(S)). \tag{1}$$

*where $L_j(S) = \{f|_S, f \in L_j(X)\}, j = 0, 1, \ldots, e$.*

If equality holds in $(1)$, then $(Y, w)$ is called a tight relative $2e$-design with respect to $u_0$.

**Theorem 4 (Ei. Bannai, Et. Bannai, S. Suda and H. Tanaka)**

$$\dim(L_0(S) + L_1(S) + \cdots + L_e(S)) = m_e + m_{e-1} + \cdots + m_{e-p+1}$$

*for* $J(v, k)$ *provided that* $e \leq r_1 < r_2 < \cdots < r_p \leq k - e$, *where* $m_j = \binom{v}{j} - \binom{v}{j-1}$.

**Theorem 5 (Ei. Bannai, Et. Bannai and Hi. Bannai)**

$\mathfrak{X}$ *is a Q-polynomial scheme and* $G = \mathrm{Aut}(\mathfrak{X})$. *Let* $(Y, w)$ *be a tight relative* $2e$*-design with respect to* $u_0$. *Assume that the stabilizer* $G_{u_0}$ *acts transitively on every shell* $X_r$, $1 \leq r \leq d$. *Then weight function* $w$ *is constant on each* $Y_{r_i}$ $(1 \leq i \leq p)$ *and denote this value by* $w_{r_i}$.

## Theorem 6

*Take a sequence elements from $X$ as*

$u_0 = \{1, 2, \dots, k\}$,

$u_i = \{1, 2, \dots, k-1, k+i+1\}, \quad (1 \le i \le v-k-1)$,

$u_i = \{1, 2, \dots, k, k+1\} \setminus \{i-(v-k)+1\}, \quad (v-k \le i \le v-1)$.

*Then* $\{\phi_{u_0}^{(0)}|_S, \phi_{u_1}^{(1)}|_S, \dots, \phi_{u_{v-1}}^{(1)}|_S\}$ *is a basis of* $L_0(S) + L_1(S)$*, where* $S = X_{r_1} \cup X_{r_2}$.

$$\phi_0(x) = \phi_{u_0}^{(0)}(x) = |X| E_0(x, u_0) \equiv 1,$$

$$\phi_i(x) = \phi_{u_i}^{(1)}(x) = |X| E_1(x, u_i).$$

Inner product is defined by

$$\langle f, g\rangle = \sum_{i=1}^{2} \frac{W_{r_i}}{|X_{r_i}|} \sum_{x \in X_{r_i}} f(x)g(x).$$

$$\begin{aligned}
d_0 &= \langle \phi_0, \phi_0\rangle, & & \\
c_0 &= \langle \phi_i, \phi_i\rangle, & \text{for} \quad & 1 \le i \le v-1, \\
c_{1,3} &= \langle \phi_0, \phi_i\rangle, & \text{for} \quad & 1 \le i \le v-1, \\
c_{1,1} &= \langle \phi_i, \phi_j\rangle = \langle \phi_i, \phi_{v-1}\rangle, & \text{for} \quad & 1 \le i \ne j \le v-k-1, \\
c_{1,2} &= \langle \phi_i, \phi_j\rangle = \langle \phi_i, \phi_{v-1}\rangle, & \text{for} \quad & v-k \le i \ne j \le v-2, \\
c_2 &= \langle \phi_i, \phi_j\rangle, & \text{for} \quad & 1 \le i \le v-k-1, v-k \le j \le v-2.
\end{aligned}$$

# Orthonormal basis

Gram-Schmidt's method: $\{\phi_1, \ldots, \phi_{v-1}, \phi_0\} \longrightarrow \{\varphi_1, \varphi_2, \ldots, \varphi_v\}$.

$$\varphi_1 = \tfrac{\phi_1}{c_0},$$

$$\varphi_j = \frac{1}{\sqrt{D_{j-1} D_j}} \begin{vmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_2, \phi_1 \rangle & \ldots & \langle \phi_j, \phi_1 \rangle \\ \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \ldots & \langle \phi_j, \phi_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \phi_1, \phi_{j-1} \rangle & \langle \phi_2, \phi_{j-1} \rangle & \ldots & \langle \phi_j, \phi_{j-1} \rangle \\ \phi_1 & \phi_2 & \ldots & \phi_i \end{vmatrix}.$$

The Gram determinant $D_j$ is given by

$$D_j = \begin{vmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_2, \phi_1 \rangle & \ldots & \langle \phi_j, \phi_1 \rangle \\ \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \ldots & \langle \phi_j, \phi_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \phi_1, \phi_j \rangle & \langle \phi_2, \phi_j \rangle & \ldots & \langle \phi_j, \phi_j \rangle \end{vmatrix}.$$

# Basic strategy

Let $H$ be a matrix indexed by $Y \times [v]$ whose $(y, j)$-entry is defined by $\sqrt{w(y)}\varphi_j(y)$.

Then $({}^tHH)(i,j) = \delta_{i,j}$ and $(H\,{}^tH)(x,y) = \delta_{x,y}$ imply

$$\begin{cases} \sum_{y \in Y} w(y)\varphi_i(y)\varphi_j(y) = \delta_{i,j}, \\ \sum_{j=1}^{v} w(y)\varphi_j(x)\varphi_j(y) = \delta_{x,y}. \end{cases}$$

Case 1: $x = y \in X_{r_i} (i = 1, 2)$.

$x = \{1, 2, \ldots, k - r_i, k + 1, k + 2, \ldots, k + r_i\}$

$$\frac{1}{w_{r_i}} = \sum_{s=1}^{v} \varphi_s^2(x). \tag{2}$$

$$\sum_{j=1}^{v} w(y)\varphi_j(x)\varphi_j(y) = \delta_{x,y}.$$

Case 2: $x, y \in X_{r_i},\ (x, y) \in R_{\alpha_i}, i = 1, 2.$

$x = \{1, 2, \ldots, k - r_i, k + 1, k + 2, \ldots, k + r_i\}$,
$|y \cap \{1, 2, \ldots, k - r_i\}| = a_i$.

$$\sum_{s=1}^{v} \varphi_s(x)\varphi_s(y) = \frac{f(W_{r_1}, W_{r_2})}{g(W_{r_1}, W_{r_2})}. \quad (3)$$

Case 3: $x \in X_{r_1}, y \in X_{r_2}, (x, y) \in R_\gamma, r_1 < r_2$.

$x = \{1, 2, \ldots, k - r_1, k + 1, k + 2, \ldots, k + r_1\}$,
$|y \cap \{1, 2, \ldots, k - r_1\}| = a_3$.

$$\sum_{s=1}^{v} \varphi_s(x)\varphi_s(y) = \frac{f'(W_{r_1}, W_{r_2})}{g'(W_{r_1}, W_{r_2})}. \quad (4)$$

## Proposition 7

*Let $w_{r_i}$ be the weight function on shell $X_{r_i}(i = 1, 2)$, then $\frac{w_{r_2}}{w_{r_1}}$ is rational for any tight relative 2-design $(Y, w)$.*

$$\sum_{s=1}^{v} \varphi_s(x)\varphi_s(y) = 0 \text{ if } x \neq y \in Y.$$

i.e., $f$ and $f'$ have a common factor $g_1 W_{r_1} - g_2 W_{r_2}$ such that $\frac{g_2}{g_1} \in \mathbb{Q}_{>0}$.

List all feasible parameters of tight relative 2-designs on two shells in $J(v, k)$ for $v \leq 100$.

| $v$ | $k$ | $r_1$ | $r_2$ | $\alpha_1$ | $a_1$ | $\alpha_2$ | $a_2$ | $\gamma$ | $a_3$ | $N_{r_1}$ | $\frac{w_{r_2}}{w_{r_1}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 6 | 3 | 5 | 4 | 0∼2 | 4 | 0, 1 | 4 | 0 | 10 | 1 | ○ |
| 36 | 15 | 7 | 10 | 9 | 1∼6 | 9 | 0 ∼ 4 | 9 | 0∼5 | 15 | 1 | ○ |
| 45 | 12 | 8 | 11 | 9 | 0∼3 | 9 | 0,1 | 9 | 0,1 | 33 | 1 | ○ |
| 64 | 28 | 14 | 18 | 16 | 0∼12 | 16 | 0∼10 | 16 | 0∼10 | 36 | 1 | ○ |
| 96 | 20 | 15 | 19 | 16 | 0∼4 | 16 | 0,1 | 16 | 0,1 | 76 | 1 | ? |
| 100 | 45 | 22 | 27 | 25 | 1∼20 | 25 | 0∼18 | 25 | 0∼18 | 45 | 1 | ? |

Constructing a relative 2-design on two shells is equivalent to find the $k$-element subset $u_0 \notin Y$ with the following property $(\star)$:

$$\{|u_0 \cap y| : y \in Y\} \text{ consists of exactly 2 values.} \qquad (\star)$$

We owe Z. Xiang for a very fast program to check whether a given design has the property $(\star)$ or not.

# Special results

- For $2\text{-}(16, 6, 2)$ designs, 2 out of 3 non-isomorphic designs have the property $(\star)$.
- $2\text{-}(36, 15, 6)$ designs are not yet completely classified, but some of them are listed on the homepage of E. Spence.
  1. 31 (out of 32548 coming from SRGs of type $(36, 15, 6, 6)$) have the property $(\star)$.
  2. 178 (out of 180 coming from SRGs of type $(36, 21, 12, 12)$) have the property $(\star)$.
  3. 339 (out of 617 coming from (known) designs with polarities with absolute points $6, 12, 18, 24, 30$) have the property $(\star)$.
- For $2\text{-}(45, 12, 3)$ designs, 29 (out of 78 coming from SRGs of type $(45, 12, 3, 3)$) have the property $(\star)$.

Let $G$ be a group of order $v$ and $D$ be a $k$-element subset of $G$ with $0 < k < v$. Then $D$ is called a $(v, k, \lambda)$-difference set if each non-identity element $g$ in $G$ can be expressed in exactly $\lambda$ ways as $g = d_1 d_2^{-1}$ with $d_1, d_2 \in D$.

$\{v, k, r_1, r_2, N_{r_1}, N_{r_2}\} = \{36, 15, 7, 10, 15, 21\}$.

- Abelian group $G = Z_6 \times Z_6$

Difference set $D$ and blocks set $\mathfrak{B} = \{gD | g \in G\}$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $\circ$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 1 | $\times$ | $\times$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ |
| 2 | $\times$ | $\circ$ | $\times$ | $\circ$ | $\circ$ | $\circ$ |
| 3 | $\times$ | $\circ$ | $\circ$ | $\times$ | $\circ$ | $\circ$ |
| 4 | $\times$ | $\circ$ | $\circ$ | $\circ$ | $\times$ | $\circ$ |
| 5 | $\times$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ | $\times$ |

$D$

$\Longrightarrow$

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $\times$ | $\times$ | $\times$ | $\circ$ | $\times$ | $\times$ |
| 1 | $\times$ | $\circ$ | $\circ$ | $\times$ | $\times$ | $\times$ |
| 2 | $\times$ | $\circ$ | $\times$ | $\circ$ | $\circ$ | $\times$ |
| 3 | $\circ$ | $\times$ | $\times$ | $\circ$ | $\circ$ | $\circ$ |
| 4 | $\times$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ |
| 5 | $\circ$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ |

$u_0$

$\{v, k, r_1, r_2, N_{r_1}, N_{r_2}\} = \{36, 15, 7, 10, 15, 21\}$.

- Non-Abelian group $G = S_3 \times S_3$

| | 1 | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|---|
| 1 | ∘ | × | × | × | × | × |
| $g_1$ | × | × | ∘ | ∘ | ∘ | ∘ |
| $g_2$ | × | ∘ | × | ∘ | ∘ | ∘ |
| $g_3$ | × | ∘ | ∘ | × | ∘ | ∘ |
| $g_4$ | × | ∘ | ∘ | ∘ | × | ∘ |
| $g_5$ | × | ∘ | ∘ | ∘ | ∘ | × |

$D$

$\Longrightarrow$

| | 1 | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|---|
| 1 | × | × | × | × | × | × |
| $g_1$ | ∘ | × | × | × | ∘ | ∘ |
| $g_2$ | ∘ | × | × | × | ∘ | ∘ |
| $g_3$ | ∘ | × | × | × | ∘ | ∘ |
| $g_4$ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ |
| $g_5$ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ |

$u_0$

$$1 = Id, \quad g_1 = (12), \quad g_2 = (13), \quad g_3 = (23), \quad g_4 = (123), \quad g_5 = (132).$$

# Conjecture

There exist two families of tight relative 2-designs with the following parameters.

- $v = q^{s+1}(q^s + \cdots + q + 2)$,
  $k = q^s(q^s + \cdots + q + 1)$,
  $\lambda = q^s(q^{s-1} + \cdots + q + 1)$.
  $r_1 = q^{s-1}(q^{s+1} - 1),\ r_2 = q^{s-1}(q^{s+1} + q - 1),\ N_{r_2} = k$.
- $v = 4u^2$,
  $k = 2u^2 - u$,
  $\lambda = u^2 - u$.
  $r_1 = u^2 - \frac{u}{2},\ r_2 = u^2 + \frac{u}{2},\ N_{r_2} = k$, if $u$ is even;
  $r_1 = u^2 - \frac{u+1}{2},\ r_2 = u^2 + \frac{u-1}{2},\ N_{r_1} = k$, if $u$ is odd.

Thank you