Titles and abstracts of the talks given at the workshop on “Sphere Packings, Lattices, and Designs”, October 27–31, 2014

Plenary talks

Eiichi Bannai: Tight Euclidean t-designs and tight relative t-designs in certain association schemes
The concept of relative t-designs in association schemes was introduced by Delsarte in his paper: Pairs of vectors in the space of an association scheme (1977). This concept, in a sense, predicted the concept of Euclidean t-designs, introduced later by Neumaier and Seidel (1988). However, it seems that the study of Euclidean t-designs has preceded the study of relative t-designs in association schemes, and that the study of the latter has just started recently in a way modeling the study of Euclidean t-designs. In this talk we first review the similarities between the studies on “spherical t-designs and Euclidean t-designs” and on “t-designs and relative t-designs in association schemes”, putting the emphasis on the study of tight t-designs in each situation. The purpose of this talk is to try to convince the reader that we should study the classification problems of tight relative t-designs in association schemes in a systematic way. More details will be available in the following papers initiating the study in this direction.

[1] E. Bannai, E. Bannai, S. Suda, and H. Tanaka: On relative t-designs in polynomial association schemes, preprint, arXiv:1303.7163.

[2] E. Bannai, E. Bannai, and H. Bannai: On the existence of tight relative 2-designs on binary Hamming association schemes, Discrete Mathematics 314 (2014), 17–37.

[3] Z. Xiang: A Fisher type inequality for weighted regular t-wise balanced designs, J. Combin. Theory Ser. A 119 (2012), 1523–1527.

[4] Y. Zhu, E. Bannai, and E. Bannai: On tight relative 2-designs on the Johnson association schemes (a tentative title, in preparation).

Andriy Bondarenko: Strongly regular graphs in metric geometry
We will give a short overview of our recent results on strongly regular graphs. First, we will show nonexistence of strongly regular graphs for certain feasible parameter sets. Representations of strongly regular graphs as two-distance sets on spheres will be widely used. Then we will prove that such a representation of G2(4) strongly regular graph implies that Borsuk's conjecture is false in all dimensions greater than 64. The relations of the mentioned problems to the group theory will be also discussed.

Renaud Coulangeon: Computing unit groups of maximal orders using Voronoi algorithm
We describe an algorithm to compute a presentation of the group of units of an order in a (semi)simple algebra over **Q**. Our method is based on a generalisation of Voronoi's algorithm for computing perfect forms, combined with Bass-Serre theory of graphs of groups. It differs essentially from previously known methods to deal with such questions, *e.g.* for units in quaternion algebras. This is a joint work with Oliver Braun, Gabriele Nebe and Sebastian Schönnenbeck (RWTH, Aachen).

Martin Ehler: Designs in unions of Grassmannians
The Grassmannian is the set of orthogonal projectors of fixed rank in the d-dimensional Euclidean space, and Grassmannian designs have already been well-studied. Here, we introduce designs in unions of Grassmannians, hence we allow the ranks to vary. This new concept of designs can be successfully applied to the so-called phase retrieval problem that has recently attracted much attention in the mathematical signal processing community. Finally, we present parametric families of highly symmetric designs in unions of Grassmannians.

Lenny Fukshansky: Well-rounded lattices from algebraic constructions
Well-rounded lattices are important in extremal lattice theory, since some classical optimization problems can usually be reduced to them without loss of generality. On the other hand, many of the standard constructions of Euclidean lattices with good properties come from various algebraic settings. This prompts a natural question: which of the lattices coming from algebraic constructions are well-rounded? We will consider some such constructions, including ideal lattices from number fields and function field lattices from curves over finite fields. In each of the cases, we provide a partial answer to the above question, as well as discuss some generalizations and directions for future research.

Oleg Musin: Extreme problems of sphere packings and irreducible contact graphs
Recently, we enumerate up to isometry, all locally rigid circle packings on the unit sphere with number of circles $N < 12$. This problem is equivalent to the enumeration of irreducible contact graphs. In this paper we show that by using the list of irreducible graphs can solve various problems of extreme packings such as the Tammes problem for the sphere and the projective plane, the maximal contacts problem, Danzer's and other problems on irreducible contact graphs.

Gabriele Nebe: Automorphisms of extremal lattices
It is a common believe that extremal even unimodular lattices in the jump dimensions $24m$ are very symmetric objects. The most famous example is the Leech lattice of dimension 24 whose automorphism group is a covering group of the sporadic simple Conway group. Following the lines of similar approaches in coding theory, I started a project to classify extremal lattices with a given automorphism in particular in dimension 48 and 72. In dimension 48 this investigation revealed a new extremal lattice with a soluble automorphism group of order 1200.

Fernando Oliveira Filho: Optimization methods for packing problems
Some of the most interesting and difficult problems in geometry are packing problems, which basically ask the question: how much of a certain space can be filled by nonoverlapping copies of given objects?

The most famous example is perhaps the sphere-packing problem: we wish to fill as much as possible of Euclidean space with nonoverlapping copies of the unit ball.

Another example with a rich history going back to Aristotle is the tetrahedra packing problem: how much of $\mathbb{R}^3$ can we fill with congruent, nonoverlapping copies of a given regular tetrahedron? A related problem, which can be seen as a packing problem in the compact space SO(3) of 3-dimensional rotations, is the following: how many nonoverlapping regular tetrahedra can have a vertex in common?

In packing problems, lower bounds for the maximum packing density are found by constructions, i.e., by showing dense packings. Finding upper bounds is a nonconstructive effort, and there are many techniques that have been explored to prove upper bounds. In this talk, I will discuss a unified approach for the computation of upper bounds. This approach is an extension of the Lovász theta number, a graph parameter that provides an upper bound, based on semidefinite programming, to the independence number of a finite graph, together with ideas from harmonic analysis. I will show how the approach applies to packings of congruent copies of a given convex body in Euclidean space and also how it applies to some interesting packing problems in compact spaces, like the packing problem on SO(3) mentioned above.

Joint work with Frank Vallentin.

Sinai Robins: Cone theta functions and rational volumes of spherical polytopes
It is natural to ask when the spherical volume defined by the intersection of a sphere at the apex of an integer polyhedral cone is a rational number. This work sets up a dictionary between the combinatorial geometry of polyhedral cones with 'rational volume' and the analytic behavior of certain associated cone theta functions. The initial motivation for this work came from a conjecture of Cheeger and Simons, asking when the volume of a spherical -simplex is rational, assuming that all of its dihedral angles are rational. We essentially discretize the volume of a spherical polytope, using lattices, and pursue number-theoretic methods to analyze the ensuing discretizations.

The particular number theoretic methods involve a new class of polyhedral functions called cone theta functions, which are closely related to classical theta functions. We show that if K is a Weyl chamber for any finite crystallographic reflection group, then its cone theta function lies in a graded ring of classical theta functions (of different weights/dimensions) and in this sense is ‘almost’ modular. It is then natural to ask whether or not the conic theta functions are themselves modular, and we prove that in general they are not. In other words, we uncover relations between the class of integer polyhedral cones that have a rational solid angle at their apex, and the class of cone theta functions that are almost modular. This is joint work with Amanda Folsom and Winfried Kohnen.

Contributed talks

Etsuko Bannai: Tight Euclidean t-designs on two concentric spheres
The concept of Euclidean t-design is a two step generalization of spherical t-designs. Namely, Euclidean t-design is a finite set on several concentric spheres in R^n which allows weight function. The natural lower bound for the size of Euclidean t-design is known and can be expressed in terms of t, n and the number p of concentric spheres. Those attaining the lower bound are called tight Euclidean t-designs on p concentric spheres. In this talk we mainly study tight t-designs on two concentric spheres for odd t. For the case of $p = 2$ and t odd satisfying $t \leq 7$, tight Euclidean t-designs are classified. For the case of $p = 2$ and t even, interesting examples are known but the classification problem is still open. The most recent results are in the two papers given below.

1) Tight 9-designs on two concentric spheres, J. of Math. Soc. Japan (64) no.4 (2011), 1359–1376. 2) Tight t-designs on two concentric spheres, Moscow J. of Combinatorics and Number Theory, 4 (2014) 52–77. (Both of them by Eiichi Bannai and Etsuko Bannai.)

Jingguo Bi: Sub-Linear Root Detection for Sparse Polynomials Over Finite Fields
We present a deterministic $2^{O(t)}q^{\frac{t-2}{t-1}+o(1)}$ algorithm to decide whether a univariate polynomial f, with exactly t monomial terms and degree $< q$, has a root in $\mathbb{F}_q$. Our method is the first with complexity *sub-linear* in q when t is fixed. We also prove a structural property for the nonzero roots in $\mathbb{F}_q$ of any t-nomial: the nonzero roots always admit a partition into no more than $2(q-1)^{\frac{t-2}{t-1}}$ cosets of two subgroups $S_1 \subseteq S_2$ of $\mathbb{F}_q^*$. This can be thought of as a finite field analogue of Descartes' Rule. A corollary of our results is the first deterministic sub-linear algorithm for detecting common degree one factors of k-tuples of t-nomials in $\mathbb{F}_q[x]$,when k and t are fixed.

Peter Boyvalenkov: Upper and lower bounds for potential energy of spherical designs
We apply linear programming techniques to derive upper and lower bounds for the potential energy of spherical designs. We improve the pure linear programming by using information for the structure of the designs under consideration. We also discuss the sharpness of our bounds.

Mathieu Dutour Sikiric: Practical Computation of Hecke operators
The computation of automorphic forms for a group Gamma is a major problem in number theory. The only known way to approach the higher rank cases is by computing the action of Hecke operators on the cohomology.

Henceforth, we consider the explicit computation of the cohomology by using cellular complexes. We then explain how the rational elements can be made to act on the complex when it originate from perfect forms. We illustrate the results obtained for the Sp4(Z) group.

Karlheinz Gröchenig: A packing problem in time-frequency analysis
I will describe a packing problem that arises in the construction of Gabor frames with a Gaussian window. Gabor frames are sets of time-frequency shifts with respect to a lattice that generate a frame (a basis-like system) for L^2. There is numerical evidence that, among all lattices with fixed density, the ratio of the frames bounds, is minimal precisely for the hexagonal lattice. This is a conjecture of Strohmer and Heath. The theory of Gabor frames leads to a new type of packing problems.

Peter Gruber: On Voronoi type results and problems for lattice packings and kissing numbers
The geometric variant of a criterion of Voronoi says, a lattice packing of balls in E^d has (locally) maximum density if and only if it is eutactic and perfect. This lecture deals with refinements of Voronoi's result and extensions to lattice packings of smooth convex bodies. For convex bodies, versions of eutaxy and perfection are used to characterize lattices with semi-stationary, stationary, maximum and ultra-maximum lattice packing density, where ultra-maximality is a sharper version of maximality. Surprisingly, for balls the lattice packings with maximum density have ultra-maximum density. We conjecture that for generic convex bodies the lattice packings of maximum density have ultra maximum density too and their kissing number is precisely $2d^2$.

Lit.:

Application of an idea of Voronoi to lattice packing, Ann Mat Pura Appl 193 (2014) 939–959
Application of an idea of Voronoi to lattice packing, supplement, in preparation Extremum properties of lattice packing and covering with circles, in preparation Lattice packing and covering of convex bodies, Proc Steklov Inst Mat 275 (2011) 229-238
Application of an idea of Voronoi, a report, Bolyai Soc Math Studies, Budapest 24 (2013) 109-157

Yoav Kallus: High-dimensional random packing lattices: Bravais new world
Bravais lattices have always been an important special case of the high-dimensional sphere packing problem, but from the statistical mechanics and random packing perspectives, they have not garnered as much attention. I will discuss the statistical mechanical phenomena exhibited by a system of one sphere under periodic boundary conditions, where the only degrees of freedom are the unit cell parameters. Equilibrium behavior includes a "crystallization" transition, but most of the interest comes from studying non-equilibrium behavior: glass transition, random packing, and hysteresis. The random-packed lattices exhibit surprising properties, including a density remarkably higher than amorphous random packing.

Wöden Kusner: A brief analysis of packing regular pentagons in the plane
It is a still an open problem to describe the densest packing of the plane by regular pentagons. Here, a general problem is described near a conjectured optimum by a nonlinear programming problem. Under certain assumptions, that optimum can be certified locally via a linear programming problem. This method can be used to computationally prove* the local optimality of the conjectured globally optimal pentagon packing.

Giovanni Lazzarini: Spherical designs and height function of lattices
If Λ is a full rank Euclidean lattice in $\mathbf{R}^n$, its height $h(\Lambda)$ is defined as $\zeta'_\Lambda(0)$, where ζ_Λ is the spectral zeta function of Λ (the zeta regularisation of the determinant of the Laplacian on the torus associated to Λ).

An open problem in the geometry of numbers is the investigation of the minima of this height function when Λ varies in the space of lattices of covolume 1 in a given dimension n. The existence of such a minimum has been shown by Chiu.

In this talk, I will show a connection between this problem and the theory of spherical designs: more precisely, I will show that the lattices, all layers of which hold a spherical 2-design, are stationary points for the height function.

Moreover, I will describe an algorithm to determine whether or not a given lattice has a 2-design on every layer.

This result is a joint work with Renaud Coulangeon.

REFERENCES

Patrick Chiu. Height of flat tori. Proc. Amer. Math. Soc., 125(3):723–730, 1997.

Renaud Coulangeon and Giovanni Lazzarini. Spherical designs and heights of Euclidean lattices. J. Number Theory, 141 :288–315, 2014.

Romanos Malikiosis: Full spark Gabor Frames in Finite Dimensions
A Gabor frame is the set of all time–frequency translates of a complex func- tion and is a fundamental tool in utilizing communications channels with wide applications in time–frequency analysis and signal processing. When the do- main of the function is a finite cyclic group of order N , then the Gabor frame forms a design on the complex sphere in N dimensions; when the N^2 unit vec- tors that constitute this Gabor frame are pairwise equiangular then the Gabor frame forms a spherical 2-design, and in addition, it has minimal coherence, an ideal property in terms of compressive sensing (whether such an equiangular set exists is also known as the SIC–POVM existence problem, which is open since 1999). In this talk, we will deal with the question of existence of a Gabor frame such that every N vectors form a basis (the discrete analogue of the HRT conjecture); such a frame is called a full spark Gabor frame. This question was posed by Lawrence, Pfander and Walnut in 2005 and was answered in the affirmative by the speaker in 2013 unconditionally. This result has applications in operator identification, operator sampling, and compressive sensing.

Juan Pablo Rossetti: Integral Lattices, the isospectral problem and the norm one
In 1941, Witt showed that the pair of 16-dimensional Euclidean lattices $E_8 \oplus E_8$ and D_{16}^+ have exactly the same lengths of vectors counted with multiplicities, while they are clearly not isometric. Twenty five years later this allowed John Milnor to show that the two flat tori obtained by the quotient of $\mathbb{R}^{16}$ by these lattices were *isospectral* but not isometric Riemannian manifolds. This was the first example of this kind, celebrated, and followed by many more in the last fifty years.

When one considers the length of a vector in $\mathbb{R}^n$, of course one uses the norm two. Is there any interest in considering the norm one (or L_1 norm) of vectors in a Euclidean lattice? In an integral lattice? In a sublattice of $I_n = \mathbb{Z}^n$? Well, our answer is that certainly there is!

While in computer science the *Manhattan norm* is considered, in the *isospectral problem* we have mentioned, there is a strong and surprising relation in which the norm one plays an essential role.

Lens spaces are very elementary Riemannian manifolds obtained by the quotient of a sphere by a cyclic finite group acting by isometries on it. To each lens space L we associate a congruence lattice $\mathcal{L}$ in $\mathbb{Z}^n$, showing that two lens spaces L and L' are isospectral on functions if and only if their associated lattices $\mathcal{L}$ and $\mathcal{L}'$ are isospectral with respect to norm one. We have found families of pairs of such congruence lattices, in dimension 3, which produce pairs of 5-dimensional lens spaces that are isospectral; moreover, they are isospectral on p-forms for every p but not *strongly isospectral*.

This is a joint work with Emilio Lauret and Roberto J. Miatello, from the same university.

Rudolf Scharlau: Configurations of short vectors in lattices: another application of a method of Bachoc and Venkov
In 2000, Christine Bachoc and Boris B. Venkov have written a paper "Modular forms, lattices and spherical designs", which combines known information about spaces of modular forms with certain facts about spherical designs to restrict the "configuration numbers" of certain lattices (of small level) by systems of linear inequalities. By solving these systems, the non-existence of certain extremal lattices could be proved. The main purpose of this talk is to review this beautiful approach. We will briefly report on a complete implementation in Magma and on rather exhaustive computations. There is one "new" application (obtained already in 2011): modular extremal lattices (in the sense of Quebbemann) of level 7 and dimension 24 do not exist.

Michael Shub: Well-conditioned Polynomials

Some decades ago Steve Smale and I proved that most homogeneous polynomials $f(x, y)$ in two complex variables are well conditioned. The paper is Complexity of Bezout's Theorem III: Condition Number and Packing Journal of Complexity Vol. 9 (1993), pp. 4-14.We did not (and still don't) know a method to produce them.

In that paper we proved that polynomials whose root are close to being Fekete points are well conditioned. Smale's 7th problem asks for a method to produce good approximations to Fekete points which then give rise to well-conditioned polynomials. But finding well conditioned polynomials is, at least, in principal easier. It may be that the work on the Fekete point problem in recent years already solves our original problem.

Maya Stoyanova: Computational algorithms for bounding potential energy of spherical codes and designs

We discuss several computational algorithms developed for using in bounding energy problems. We show how all necessary parameters are derived and linked and how different bounds are computed. Finally, a new algorithm for obtaining energy bounds will be presented.

Ziqing Xiang: Tight block designs

Tight t-designs are t-designs which achieve the Fisher type lower bound. Non-trivial tight designs are very rare. There do not exist such designs for odd t. In the case of even t which is at least 4, there are only two known examples, Witt 4-$(23, 7, 1)$ and 4-$(23, 16, 52)$. In this talk, I will discuss the current status of non-existence results.

Wei-Hsuan Yu: New bounds for spherical two-distance sets and equiangular line sets

The maximum size of spherical few-distance sets had been studied by Delsarte at al. in the 1970s. We use the semidefinite programming method to extend the known results of the maximum size of spherical two-distance sets in R^n when $n = 23$ and $40 \leq n \leq 93$ and $n \neq 46, 78$. We also find the maximum size for equiangular line sets in R^n when$24 \leq n \leq 41$ and $n = 43$. This provides a partial resolution of the conjecture set forth by Lemmens and Seidel (1973). We also derive new relative bounds for the equiangular line sets and prove the non-existence of tight spherical designs of harmonic index 4 in R^n for $n \geq 3$.

Peter Zeiner: Well-rounded sublattices in the plane

A lattice is called well-rounded, if its lattice vectors of minimal length span the underlying space. Here, we are interested in determining all well-rounded sublattices of a given planar lattice. We present a method to solve this problem and derive detailed expressions for the asymptotic growth rate of the number of well-rounded sublattices.