Workshop on Fractals and Tilings 2009

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Abstracts

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Pentagonal tiling generated by a domain exchange

Shigeki Akiyama
Niigata University

Rational rotation on a certain lozenge gives a domain exchange dynamics. This dynamics is highly non-ergodic with respect to the Lebesgue measure and it is conjectured that almost all points have periodic orbits. It is interesting that all self-inducing structures found so far have Pisot numbers as scaling constants. This fact allows us to define number systems of their minimal systems, as well as dual systems. The case of angle $\pi/5$ can be studied in detail, which gives a pentagonal tiling. I hope to discuss other self-inducing dynamics as well in a similar manner.

Extensions of substitutions

Pierre Arnoux
University Aix-Marseille II

We will discuss a new tool, the extensions of substitutions and their duals, and show a few applications.

Diffraction of Tiling Dynamical Systems

Michael Baake
University of Bielefeld

The dynamical spectrum is a well-known concept in the study of dynamical systems. The diffraction spectrum is a related quantity that is often easier to calculate. It has interesting applications in structure determination, crystallography and physics, with a challenging inverse problem.

This talk gives a survey with special emphasis on important aperiodic examples, both with pure point and singular continuous spectra. The related talk by Uwe Grimm will complement this by a discussion of absolutely continuous spectra.

Three-dimensional self-similar sets and tilings

Christoph Bandt
University of Greifswald

Most known examples of fractals are subsets of the plane. We shall present a number of self-similar constructions in three dimensions which have a simple, mathematically tractable structure. We also discuss new self-similar tilings, their visualization, and algebraic techniques to explore their geometry.

Fundamental groups of wild spaces

Greg Conner
Brigham Young University

Connectedness via Jordan and rational normal form

Eva Curry and Avra Laarakker
Acadia University

We consider the problem of finding digits sets D for multidimensional radix representations with base (radix) an expanding matrix $A \in M_n(\mathbb{Z})$ such that the associated iterated function system attractor $T(A, D)$ is a connected set that tiles $\mathbb{R}^n$ by translations by $\mathbb{Z}^n$ or some sublattice of $\mathbb{Z}^n$. We show that such digit sets can be found via a similarity transform to the Jordan or rational normal form of the matrix A for two-dimensional matrices, and present experimental evidence suggesting that this approach will generalize straightforwardly to arbitrary dimensions.

Fourier series on fractals

Dorin Dutkay
University of Central Florida

We show that for certain measures associated to iterated function systems it is possible to construct orthogonal families of complex exponentials. Such measures are called spectral. For example, the triadic Cantor set is not spectral, but if we change the scale to 4 we do obtain a spectral measure. We present some examples in higher dimensions and several open problems.

Symmetry and enumeration of self-similar fractals

Kenneth Falconer
University of St Andrews

We describe a general method for enumerating the distinct self-similar sets that arise as attractors of certain families of iterated function systems, using a little group theory to analyse the symmetries of the attractors. The talk will be illustrated by a range of pictorial examples.

Systems with continuous diffraction

Uwe Grimm
The Open University Milton Keynes

Pure point diffraction is a characteristic feature of crystals and quasicrystal, reflecting the perfect order of these structures. In constrast, continuous diffraction is less straightforward to interpret, and often linked to the presence of disorder. Simple model systems, based on the Rudin-Shapiro and Bernoulli random sequences, demonstrate that, in general, it may be impossible to draw conclusions on the degree of order in the system from its diffraction. In particular, we construct a family of one-dimensional binary systems which cover the entire entropy range but still share the same purely continuous diffraction spectrum.

(Joint work with Michael Baake)

Riesz s-equilibrium measures on fractal sets

Doug Hardin
Vanderbilt University

Let A be a compact set in $\mathbf{R}^p$ of Hausdorff dimension d and let $M(A)$ denote the collection of Borel probability measures supported in A. For $0 < s < d$, there is a unique measure $\mu^{s,A} \in M(A)$ called the *Reisz s-equilibrium measure* that minimizes

$$I_s(\mu) := \iint |x-y|^{-s} d\mu(y) d\mu(x)$$

over all $\mu \in M(A)$. However, $I_s(\mu) = \infty$ for $s \geq d$ and any $\mu \in M(A)$. In joint work with M. Calef, we consider the limiting behavior of μ^s as $s \to d^-$. We show that if A is d-rectifiable with finite and positive d-dimensional Hausdorff measure then $\mu^{s,A}$ converges to normalized d-dimensional Hausdorff measure restricted to A as $s \to d^-$. In separate work, Calef shows this is also true if A is self-similar.

Tree-adjoined spaces and the Hawaiian earring

Wolfram Hojka
Vienna University of Technology

Attaching a binary tree to a topological space along a Cantor set gives rise to a surprising link between compact metric spaces and subgroups of the fundamental group of the Hawaiian earring.

Let f be a map from the Cantor set onto a topological space X. It is possible to attach the 'leaves' of an infinite binary tree to X along f, in such a way that there is a naturally induced homomorphism φ from the fundamental group of the adjunction space to the fundamental group $\pi_1(H)$ of the Hawaiian earring (a shrinking wedge of circles). This construction yields a correspondence between the locally one-connected compact metric spaces and the subgroups of $\pi_1(H)$, and remarkably, the original space X can be reconstructed up to homeomorphism from the subgroup determined by the image of φ.

Hausdorff dimension and non-degenerate families of projections

Esa Järvenpää
University of Jyväskylä

We study families of projections in Euclidean spaces for which the dimension of the parameter space is strictly less than that of the Grassmann manifold. We answer the natural question of how much the Hausdorff dimension may decrease by verifying the best possible lower bound for the dimension of a typical projection of a finite measure. We also show that the similar result is valid for smooth families of maps from n-dimensional Euclidean space to m-dimensional Euclidean space.

(Joint work with Maarit Järvenpää and Tamás Keleti)

Visibility

MAARIT JÄRVENPÄÄ
University of Jyväskylä

The visible part of a compact set $E \subset \mathbb{R}^2$ from an affine line ℓ consists of those points $x \in E$ where one first hits the set E when looking perpendicularly from ℓ. We consider dimensional properties of visible parts of plane sets. Special emphasis is given to fractal percolation. We show that, conditioned on non-extinction, almost surely almost all visible parts from lines are 1-dimensional and have positive and finite Hausdorff measure provided that the dimension of the fractal percolation is larger than 1. We also verify an analogous result for visible parts from points.

(Joint work with I. Arhosalo, E. Järvenpää, M. Rams and P. Shmerkin)

Projections of self-affine carpets

THOMAS JORDAN
University of Bristol

Let $E \subset \mathbb{R}^2$ and let $P_\theta(E)$ be the orthogonal projection of E onto a line with an angle θ with the origin. A classical result of Marstand states that the Hausdorff dimension ($\dim_H$) of $P_\theta(E)$ is given by $\min(\dim_H E, 1)$ for Lebesgue almost all θ. For general sets E very little is known about the zero measure set for which this result does not hold. In this talk we will show that for several self-affine sets it is possible to determine this exceptional set explicitly. The main class of self-affine sets considered will be the self-affine carpets studied by Bedford and McMullen and some more recent generalisations.

(Joint work with Andrew Ferguson (Warwick) and Pablo Shmerkin (Manchester))

Short time asymptotics of the heat kernel on the harmonic Sierpiński gasket

NAOTAKA KAJINO
Kyoto University

The harmonic Sierpiński gasket is a self-affine fractal in $\mathbb{R}^2$ which is given as the image of an injective harmonic map Φ from the usual Sierpiński gasket into $\mathbb{R}^2$. Since it is homeomorphic to the usual Sierpiński gasket, we have the standard Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the harmonic gasket, and the energy of the harmonic map Φ defines a measure ν, called the Kusuoka measure, on it. Several known results indicate that the analytic behavior of the harmonic Sierpiński gasket is often quite close to that of Riemannian manifolds: for example, Kigami (2008) has shown that the heat kernel associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\nu)$ is subject to the two-sided Gaussian estimate.

In this talk, we show that the local behavior of this heat kernel at a junction point is approximately the same as that of the reflecting Brownian motion on a 1-dimensional interval. In particular, we see that there is no oscillation in the on-diagonal short time behavior of the heat kernel at a junction point, just as in the case of the heat kernel on a Riemannian manifold.

Dirichlet forms and heat kernels on the Cantor sets as the traces of random walks on trees

Jun Kigami
Kyoto University

Let T is an contably infinite set and let $C : T\times T \to [0, \infty)$ which satisfies $C(x, y) = C(y, x)$ and $C(x, x) = 0$ for any $x \in T$. We call $(x_0, \ldots, x_n)$, where $x_i \in T$, a path between x_0 and x_n if and only if $C(x_i, x_{i+1}) > 0$ for any $i = 0, 1, \ldots, n-1$ and $x_i \neq x_j$ for $i \neq j$. Assume that there exists a unique path between x and y for any $x \neq y \in T$. Then (T, C) is called a **weighted tree**. Let

$$P(x, y) = \frac{C(x, y)}{\sum_{y\in T} C(x, y)}.$$

Then $\{P(x, y)\}_{x,y\in T}$ is the transition probability of the natural random walk $(\{X_n\}_{n\geq 0}, \{P_x\}_{x\in T})$ on T. The associted Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{x,y\in T, C(x,y)>0} \frac{(u(x) - u(y))(v(x) - v(y))}{C(x, y)}$$
$$\mathcal{F} = \{u|u : T \to \mathbb{R}, \mathcal{E}(u, u) < +\infty\}.$$

Assume that (T, C) is transitent. By the classical result of Cartier, it is known that the **Martin boundary** of the transient weighted tree coinsides with the set of infinite paths

$$\Sigma = \{(\phi, x_1, x_2, \ldots)|(\phi, x_1, \ldots, x_n) \text{ is the path between } \phi \text{ and } x_n \text{ for any } n \geq 1\},$$

where ϕ is a fixed reference point in T. Morevoer, Σ is a Cantor set, i.e. totally disconnected, uniformly perfect and compact.

Define the trace $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$ of $(\mathcal{E}, \mathcal{F})$ on the Martin boundary by

$$\mathcal{E}|_\Sigma(f, f) = \mathcal{E}(H(f), H(f)) \quad \text{and} \quad \mathcal{F}|_\Sigma = \{f|f : \Sigma \to \mathbb{R}, \mathcal{E}|_\Sigma(f, f) < +\infty\}$$

for $f : \Sigma \to \mathbb{R}$, where $H(f)$ is the **harmonic function** on T with the boundary value f on the Martin boundary Σ. In this talk, we will show that

$$\begin{aligned}\mathcal{E}_\Sigma(f, f) &= \sum_{x,y\in T, C(x,y)>0} \alpha_{x,y}((f)_{\nu,x} - (f)_{\nu,y})^2 \\ &= \int_{\Sigma\times\Sigma} J(\omega, \tau)(u(\omega) - u(\tau))^2 \nu(\omega)\nu(\tau),\end{aligned}$$

with the expicite formula of $\alpha_{x,y}$ and $J(\omega, \tau)$, where $(f)_{\nu,x}$ is the mean of f on Σ_x. Moreover, under the volume doubling condition, we obtain a full on- and off-diagonal estimate of the heat kernel assicated with the jump process derived from $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$. As an application, the Martin boundary of (the Sierpinski gasket) \ (the line segment between two boundary points) is shown to be the Cantor set.

On the self-affine tiles with polytope convex hulls and their Lebesgue measure

Ibrahim Kirat
Istanbul Technical University

Let F be a self-affine tile generated by a finite number of affine maps. The problem of determining the convex hull of F is of geometrical interest. In regard to this problem, we give necessary and sufficient conditions for the convex hull of F to be a polytope. Additionally, we determine the vertices of such polytopes. Our constructive proofs lead us to upper bounds for the Lebesgue measure of F. We also test our technique on well-known examples.

Bernoulli systems and ONBs of orthogonal exponentials I

Keri Kornelson
Grinnell College

We examine a class of questions surrounding Bernoulli measures and their associated orthogonal exponential families. In particular, we consider whether infinite orthogonal families form orthonormal bases in their respective Hilbert spaces. We discuss both conjectures and new results.

(Joint work between Palle Jorgensen, Keri Kornelson, and Karen Shuman)

Self-affine tiles and connectedness

Ka-Sing Lau
Chinese University of Hong Kong

The class of self-affine tiles are generated by expanding matrix atogether with certain digit sets. It has very rich algebraic, geometric and topological properties. In this talk we will consider the connectedness of such tiles. We concentrate on those tiles that the digit sets are collinear and consecutive. The other cases are largely unknown, we will report on some instructive examples in our recent attempt.

Parametrization of the boundary of self-affine tiles

Benôit Loridant
Niigata University

We are interested in self-affine tiles whose boundary is the attractor of a graph directed construction. A number system is associated to the graph with fundamental domain the unit interval. This number system is used to parametrize the boundary of the tile. Properties of the parametrization in relation with the topology of the tile will be presented.

Dimensions of individual points in self-similar fractals and their random subfractals

Jack Lutz
Iowa State University

Random fractals occurring in algorithmic randomness

Dan Mauldin
University of North Texas

In recent years there has been a focused research effort, mainly from computer scientists and logicians, on random subsets of the Cantor space generated by some specific algorithm. I will discuss these and show how they can be sometimes be considered as random recursive constructions and one may apply the results of Graf, Williams and myself to obtain some information about these sets. Some open problems will be mentioned.

Quantized calculus on Julia sets

Volodymyr Nekrashevych
Texas A&M University

We will discuss a possible approach to construction of quantized calculus on Julia sets of hyperbolic dynamical systems (coming from the theory of iterated monodromy groups). Among examples and applications we will discuss dense p-summability, Hausdorff dimension and Dixmier traces.

Conformal iterated function systems with overlaps

Sze-Man Ngai
Georgia Southern University

We study the invariant set and invariant measures defined by a finite iterated function systems of conformal contractions that satisfy the bounded distortion property and the weak separation condition. Results concerning the dimensions of the attractor and singularity of the associated self conformal measures and will be discussed.

(Joint work with Q.-R. Deng, K.-S. Lau and X.-Y. Wang)

Tube formulas and self-similar tilings

Erin Pearse
Cornell University

Let A be a bounded open subset of d-dimensional Euclidean space. A tube formula for A is a function of ϵ that gives the volume of the region lying outside of A but within ϵ of A. An inner tube formula for A is a function that gives the volume of the subset of A which lies within ϵ of the boundary of A.

A type of self-similar tiling is naturally associated to iterated function systems satisfying a strengthened version of the open set condition. I will describe the conditions under which the inner tube formula of the tiling allows one to compute the tube formula for the corresponding self-similar set. The resulting formula is an extension of both the Steiner formula of convex geometry and the tube formulas for fractal subsets of the real line from the theory of fractal strings and complex dimensions as developed by Lapidus and van Frankenhuijsen.

(Joint work with Steffen Winter)

Self-similar energies on finitely ramified fractals

Roberto Peirone
University of Rome Tor Vergata

A homology theory for basic sets

Ian Putnam
University of Victoria

David Ruelle introduced Smale spaces as topological dynamical systems with canonical coordinates of contracting and expanding directions as an axiomatic description of the basic sets for Smale's Axiom A systems. The totally disconnected irreducible Smale spaces are shifts of finite type. In the 1970's, Krieger introduced an invariant for shifts of finite type which is a dimension group. The goal of the talk is to show that this invariant may be extended to a type of homology theory for all irreducible Smale spaces.

Resolvent kernel estimates on p.c.f. self-similar fractals

Luke Rogers
Cornell University

I will explain how an explicit formula for the resolvent kernel of the Laplacian on a p.c.f. self-similar fractal may be used to obtain estimates on the resolvent, as well as some applications of these estimates. Among the applications are a resolvent kernel formula for blowups of the fractal and upper bounds on the heat kernel. Sharp estimate for the latter are known from the work of Hambly, Kumagai and others; the estimates we obtain are sharp only for affine nested fractals, but the method is new.

Tilings without finite local complexity

Lorenzo Sadun
University of Texas at Austin

A standard assumption in tiling theory is that a tiling can only have a finite number of local patterns, up to rigid motion. However, there are many interesting substitution tilings that violate this assumption. I will discuss how many of the standard tools of tiling theory can be modified to handle these tilings, and how the topological invariants (e.g., Cech cohomology groups) of the corresponding tiling spaces can be computed.

Bernoulli systems and ONBs of orthogonal exponentials II

Karen Shuman
Grinnell College

We examine a class of questions surrounding Bernoulli measures and their associated orthogonal exponential families. In particular, we consider whether infinite orthogonal families form orthonormal bases in their respective Hilbert spaces. We discuss both conjectures and new results.

(Joint work between Palle Jorgensen, Keri Kornelson, and Karen Shuman)

Rational numbers with purely periodic beta-expansion

Anne Siegel
University of Rennes

Our aim is to have a better understanding of expansions of positive real numbers in greedy numeration systems (beta-expansions), especially the properties of rational numbers with purely periodic expansion. Several results exist in the litterature on this subject : in the unit quadratic case, since Schmidt, we know that their exists a dichotomy with respect to the so-called property (F); either all rational numbers in]0,1[have a purely periodic beta-expansion, or no rational has. In the cubic case, from Akiyama, the property (F) implies that a non empty small interval around zero contain only rational with purely periodic expansion, but the lenght of this interval might be strictly smaller than 1 (as in the smallest Pisot number case).

In this talk, we prove that for Pisot unit cubic, the property (F) still governs a dichotomy since non-(F) implies that 0 is approximated by rational with non-purely periodic expansions. We also prove that, when (F) is satisfied, the length of the largest interval containing purely periodic rationals is an irrational number. The proofs are based on the spiral properties of the fractal structure of the natural expansion associated with the beta-transformation.

(Joint work with B. Adamczewski, C. Frougny and W. Steiner)

The intersection of the Sierpiński Carpet with straight lines

Karoly Simon
Technical University of Budapest

It is well known that most lines intersect the Sierpiński carpet in a set which has Hausdorff dimension equal to the dimension of the carpet minus one. However, as was observed by Q.H. Liu, L.F. Xi and Y.F. Zhao (2007), if we confine ourselves to lines with rational slope, we are in a much more interesting situation. I will give an account of our most recent results about this problem.

(Joint work with A. Manning, University of Warwick)

On the k-bonacci Mandelbrot set

Víctor F. Sirvent
Simón Bolívar University Caracas

In 1985 Barnsley and Harrington [3] introduced a variant of the classical Mandelbrot set, the so-called "Mandelbrot set for pair of linear maps". We recall the definition of this set. Let β be a complex number of modulus smaller than one and define the iterated function system $\{h_0, h_1\}$ by the contraction mappings

$$h_0(z) = \beta z \quad \text{and} \quad h_1(z) = \beta z + 1.$$

Let A_β be the attractor of this set. Then the "Mandelbrot set for pair of linear maps" is the set of all β for which A_β is connected. Various properties of this set have been studied by different authors (see for instance [1, 2, 4, 7, 8, 9, 10].

In the present talk we want to study variants of this set as shall be illustrated by the following example. Let $\{f_0, f_1\}$ be the iterated function system (IFS) on $\mathbb{C}$ given by the maps $f_0(z) = \beta z$ and $f_1(z) = \beta^2 z + 1$ with $\beta \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. If $\beta = \tau = (\sqrt{5} - 1)/2$ its attractor $\mathfrak{R}(\tau)$ is the interval $[-1, -\frac{1}{\tau}]$ which can be described as

$$\mathfrak{R}(\tau) = \left\{ \sum_{i=0}^{\infty} a_i \tau^i \ : a_n \dots a_0 \in \mathcal{L},\ \forall n \right\}$$

where $\mathcal{L}$ is the *Fibonacci or golden-mean language*, *i.e.*, the language of all finite words in the alphabet $\{0, 1\}$ with no occurrence of the word 11.

We consider the fixed points of the IFS for different values of β, which are of the form

$$\mathfrak{R}(\beta) := \left\{ \sum_{i=0}^{\infty} a_i \beta^i \ : a_n \dots a_0 \in \mathcal{L},\ \forall n \right\}.$$

Let

$$\mathcal{M} = \mathcal{M}(2) := \{\beta \in \mathbb{D} \ : \ \mathfrak{R}(\beta) \text{ is connected}\}$$

be the connectivity locus of the attractors of the IFS. We shall call this set the *Fibo-Mandelbrot set*.

We consider the generalization to the "k-bonacci" case. Let us consider the IFS given by the maps

$$f_0(z) = \beta z,\ f_1(z) = 1 + \beta^2 z,\ \dots,\ f_k(z) = 1 + \beta + \cdots + \beta^{k-1} + \beta^{k+1} z.$$

The definition of the Julia sets $\mathfrak{R}(\beta)$ and the Mandelbrot set $\mathcal{M}(k)$ extends in a straight forward manner. The main result that we present here is:

Theorem. *The set $\mathcal{M}(k)$ is a locally connected continuum, for each $k \geq 2$.*

We shall end the talk with a list of open problems.

(Joint work with Jörg M. Thuswaldner)

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Spacings and pair correlations for finite Bernoulli convolutions

Boris Solomyak
University of Washington

We consider the sequence of (generically, 2^N) points which can be represented as $\sum_{n=1}^{N} a_n\lambda^n$, with $a_n = 0$ or 1, for a fixed parameter $\lambda \in (0.5, 1)$. These sequences are uniformly distributed with respect to the infinite Bernoulli convolution measure ν_λ, as $N \to \infty$. Numerical evidence suggests that for a generic λ, the distribution of spacings between appropriately rescaled points is Poissonian. We obtain some partial results in this direction; for instance, we show that, on average, the pair correlations do not exhibit attraction or repulsion in the limit. On the other hand, for certain algebraic λ the behavior is totally different.

(Joint work with Itai Benjamini)

Analysis of fractals, image compression and entropy encoding

Myung-Sin Song
Southern Illinois University Edwardsville

In this talk we show that algorithms in a diverse set of applications may be cast in the context of relations on a finite set of operators in Hilbert space. The Cuntz relations for a finite set of isometries form a prototype of these relations. Such applications as entropy encoding, analysis of correlation matrices (Karhunen-Loève), fractional Brownian motion, and fractals more generally, admit multi-scales. In signal/image processing, this may be implemented with recursive algorithms using subdivisions of frequency-bands; and in fractals with scale similarity.

Multiple tilings defined by generalized beta-transformations

Wolfgang Steiner
LIAFA, University Paris 7

The β-transformation is defined for $\beta > 1$ by $T(x) = \beta x \mod 1$. If β is a Pisot unit of degree d, then we can define tiles in $\mathbb{R}^{d-1}$ by considering the algebraic conjugates of all points $\beta^k z$, $z \in \mathbb{Z}[\beta] \cap [0, 1)$, with equal value $T^k(z)$. This construction is due to Thurston (1989), and produces a multiple tiling of $\mathbb{R}^{d-1}$. It is conjectured that this multiple tiling is always a tiling. If $d > 2$, then the tiles have fractal boundary.

We extend this construction to more general piecewise linear transformations with slope β, where β is a Pisot unit, and obtain again multiple tilings. In many cases, these multiple tilings are still tilings, e.g. for the symmetric β-transformation $T(x) = \beta x - \lfloor \beta x + 1/2 \rfloor$ when β is the golden mean. In other cases, the covering degree is larger than 1. Surprisingly, the symmetric β-transformation produces a double tiling when β is the Tribonacci number or the smallest Pisot number.

(Joint work with Charlene Kalle)

Spectrum of a Laplacian on Laakso spaces

Ben Steinhurst
University of Connecticut

Laakso presented a family of space in 2000 as examples of spaces supporting Poincare inequalities of arbitrary dimension. In his work he used minimal generalized upper gradients. We take the work of Barlow and Evans on projective limit spaces to offer another construction of Laakso spaces. Then we construct a Dirichlet form from the minimal generalized upper gradients and calculate the spectrum of the associated Laplacian.

Fractal tiles associated to generalised radix representations and shift radix systems

Paul Surer
University of Leoben

For $\mathbf{r} \in \mathbb{R}^d$ define the mapping

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} \mapsto (x_1, \dots x_{d-1}, -\lfloor \mathbf{r} \cdot \mathbf{x} \rfloor) \quad (\mathbf{x} = (x_0, \dots, x_{d-1})).$$

$\tau_{\mathbf{r}}$ is called a shift radix system (SRS for short) if for every $\mathbf{z} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that $\tau_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}$. It is well known that SRS can be used to describe beta expansions and canonical number systems. In our research we associate to every $\mathbf{r} \in \mathbb{R}^{\mathbf{d}}$ a family of fractal tiles $(\mathcal{T}_{\mathbf{r}}(\mathbf{z}))_{\mathbf{z} \in \mathbb{Z}^d}$ (SRS tiles) with

$$\mathcal{T}_{\mathbf{r}}(\mathbf{z}) := \lim_{n \to \infty} R_{\mathbf{r}}^n \tau_{\mathbf{r}}^{-n}(\mathbf{z})$$

where $R_{\mathbf{r}}$ is the $n \times n$ companion matrix with $-\mathbf{r}$ as last row vector. We present several properties of these SRS tiles and show the relation to tiles induced by Pisot numbers and self affine tiles associated to expanding polynomials.

(Joint work with Valérie Berthé, Anne Siegel, Wolfgang Steiner, Jörg M. Thuswaldner)

Structure of planar self-affine tilings

Tai-Man Tang
Xiangtan University

We investigate the structure of the planar tiling by a self-affine tile T(A,D), where A is an expanding integral ma- trix and D a consecutive collinear digit set. In terms of the characteristic polynomial of A, we give the number and the positions of the neighbors of a tile, and criteria for the existence of vertex neighbors and Cantor set neighbors in the tiling. The case of disklike T(A,D) is already well understood from the work of Bandt and Wang, and Leung and Lau. (Joint work with Tao Jiang and Sze-Man Ngai)

Uniqueness of locally invariant Laplacian, Dirichlet form and Brownian motion on Sierpiński carpets

Alexander Teplyaev
University of Connecticut

We prove that, up to scalar multiples, there exists only one Dirichlet form on a generalized Sierpiński carpet that is invariant with respect to the local symmetries of the carpet. Consequently for each such fractal the law of Brownian motion is uniquely determined and the notion of Laplacian is well defined.

(Joint work with M.T. Barlow, R.F. Bass and T. Kumagai)

Fundamental groups of fractals via projective limits of semigroups of words

Reinhard Winkler
Vienna University of Technology

Fundamental groups of many fractals naturally embed into the projective limit of free groups, i.e. of groups which are understood quite well. But it seems quite difficult to fully understand this embedding. In a cooperation with Shigeki Akiyama, Gerhard Dorfer and Jörg Thuswaldner we have found a description in terms of projective limits of semigroups of words and their combinatorics.

(Research supported by the Austrian Science Foundation FWF, project S9612-N13.)

Curvature densities of self-similar sets

Martina Zähle
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In [2] and [3] for large classes of self-similar (random) sets with dimension D fractal curvatures have been introduced by means of approximations with small parallel neighborhoods. For the latter the corresponding classical curvature notions go back to convex geometry and geometric measure theory, resp. (cf. [1] and the references in [2], [3]). Because of the self-similarity property in the deterministic case the limit curvature measures are all constant multiples of D-dimensional Hausdorff measure on the fractals.

Here we will present a local interpretation of these results in terms of fractal Lipschitz-Killing curvature densities. They are determined at a.a. points of the self-similar sets by the average limit of appropriately rescaled local values of the (signed) curvature measures of their parallel sets.

References

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