Pentagonal tiling generated by a domain exchange

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Let $(X, \mathbb{B}, \mu, T)$ be a discrete time measure theoretical dynamical system (MDS). By Poincaré's recurrence theorem, for a set $Y \in \mathbb{B}$ with $\mu(Y) > 0$, T-orbits of almost all points in Y are recurrent.

We can define the **induced system**: $(Y, \mathbb{B}_Y, \frac{1}{\mu(A)}\mu, \hat{T})$ by putting $\hat{T}(x) = T^{m(x)}x$ where $m(x)$ is the first return time. Induced system is quite different from the original.

MDS is **self-inducing** if it is affine isomorphic to its induced system.

In other words, there is an affine isomorphism ϕ such that

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[\hat{T}]{} & Y \end{array}$$

holds for almost all X.

A **Pisot number** is an algebraic integer > 1 whose conjugates have modulus less than 1. The scaling constant (dominant eigenvalue) of the self-inducing structure is a Pisot number in number of important dynamics related to number theory:

- Irrational rotation and continued fraction: Legendre and Gauss
- Jacobi-Perron algorithm:
- Substitutive dynamical system: Pisot conjecture
- Tiling dynamical systems
- Domain exchanges (today's talk)

Basic question:

Why do Pisot numbers appear in number of self-inducing structures?

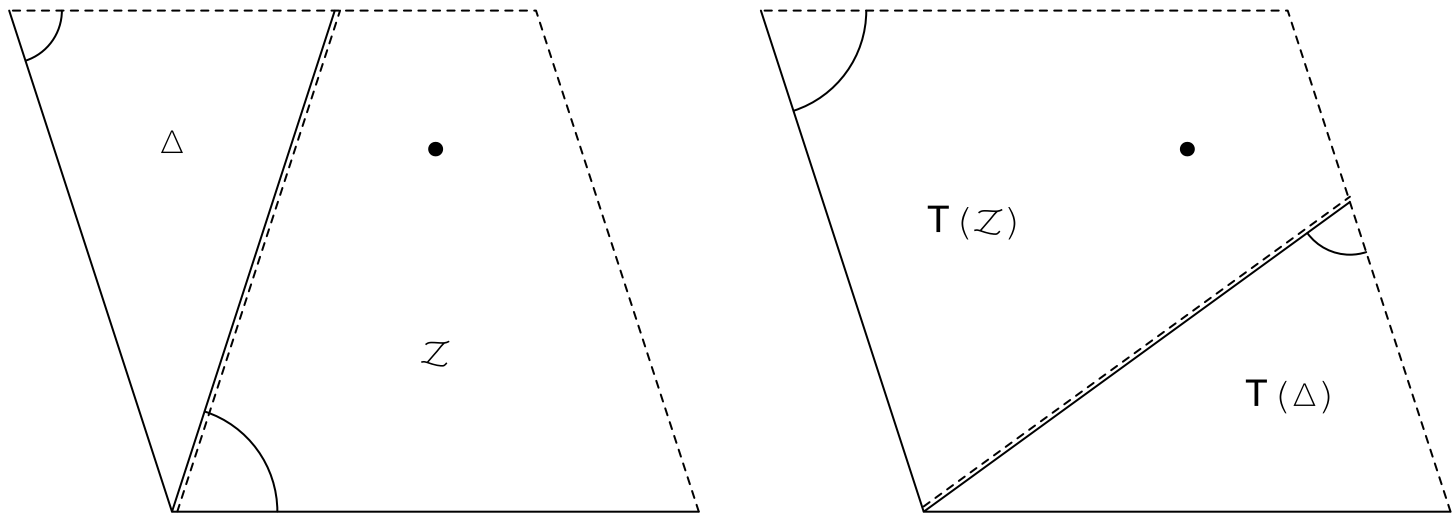
Conjecture 1. *For any* $-2 < \lambda < 2$*, the sequence defined by* $0 \leq a_{n+1} + \lambda a_n + a_{n-1} < 1$ *is periodic.*

In [1] with Brunotte, Pethő and Steiner, we proved a

Theorem 1. *The conjecture is valid in the following cases:* $\lambda = \frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}$

The case $\frac{1-\sqrt{5}}{2}$ is not new. This was shown by Lowenstein, Hatjispyros and Vivaldi [3] with heavy computer assistance. Similar problems (digital filters, rounding off error control) were studied by dynamical people: Vivaldi, Kouptsov, Lowenstein, Goetz, Poggiaspalla, Vladimirov, Bosio, Shaidenko, ...

Let us fix $\lambda = (1+\sqrt{5})/2$ and $\zeta = \exp(2\pi\sqrt{-1}/5)$. Our problem is embedded into a piecewise isometry T acting on a lozenge $L = [0,1) + (-\zeta^{-1})[0,1)$:



Our task is to prove that elements of $\mathbb{Z}[\zeta] \cap L$ has purely periodic orbits.

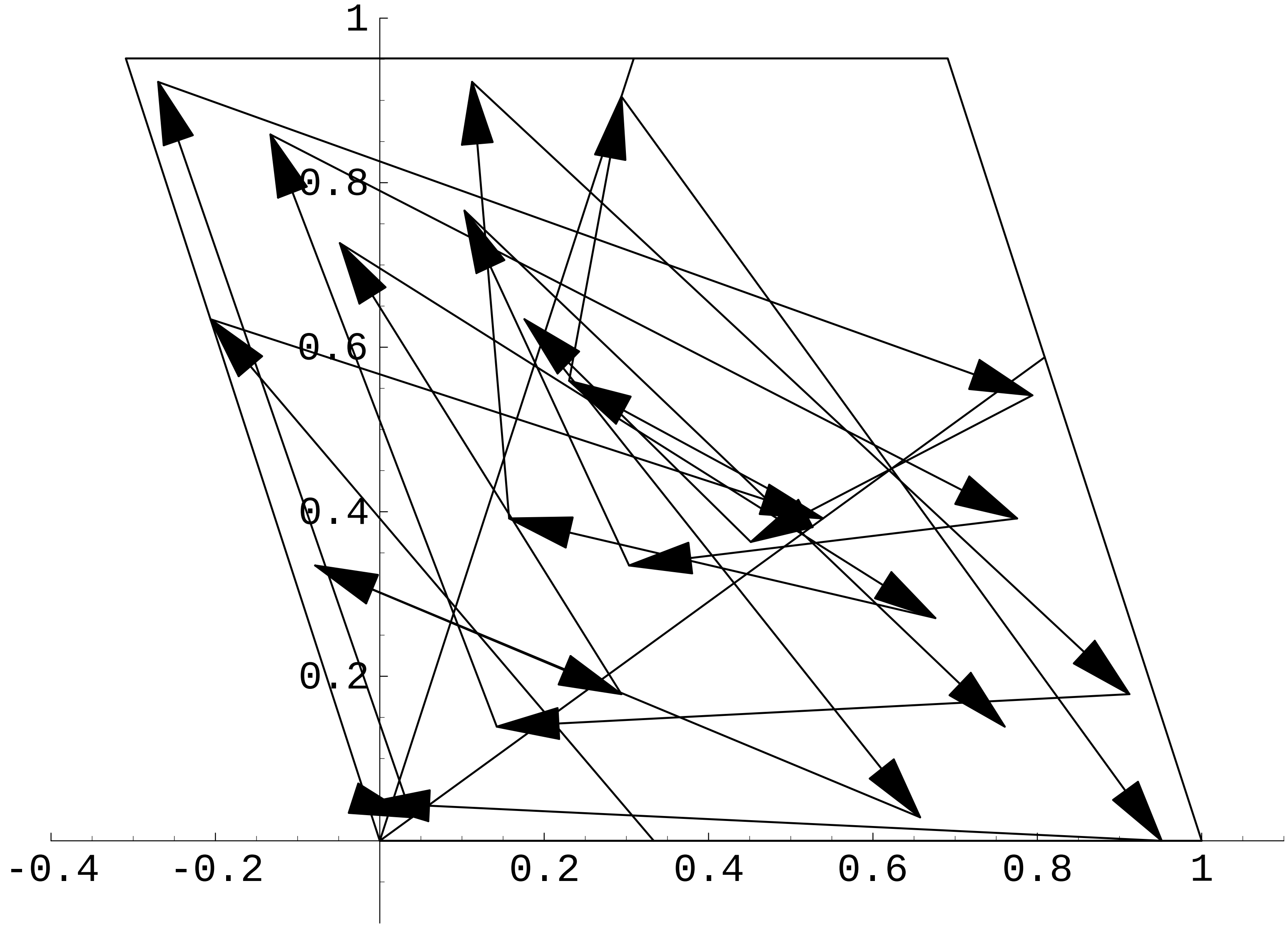


Figure 1: The orbit of $1/3$

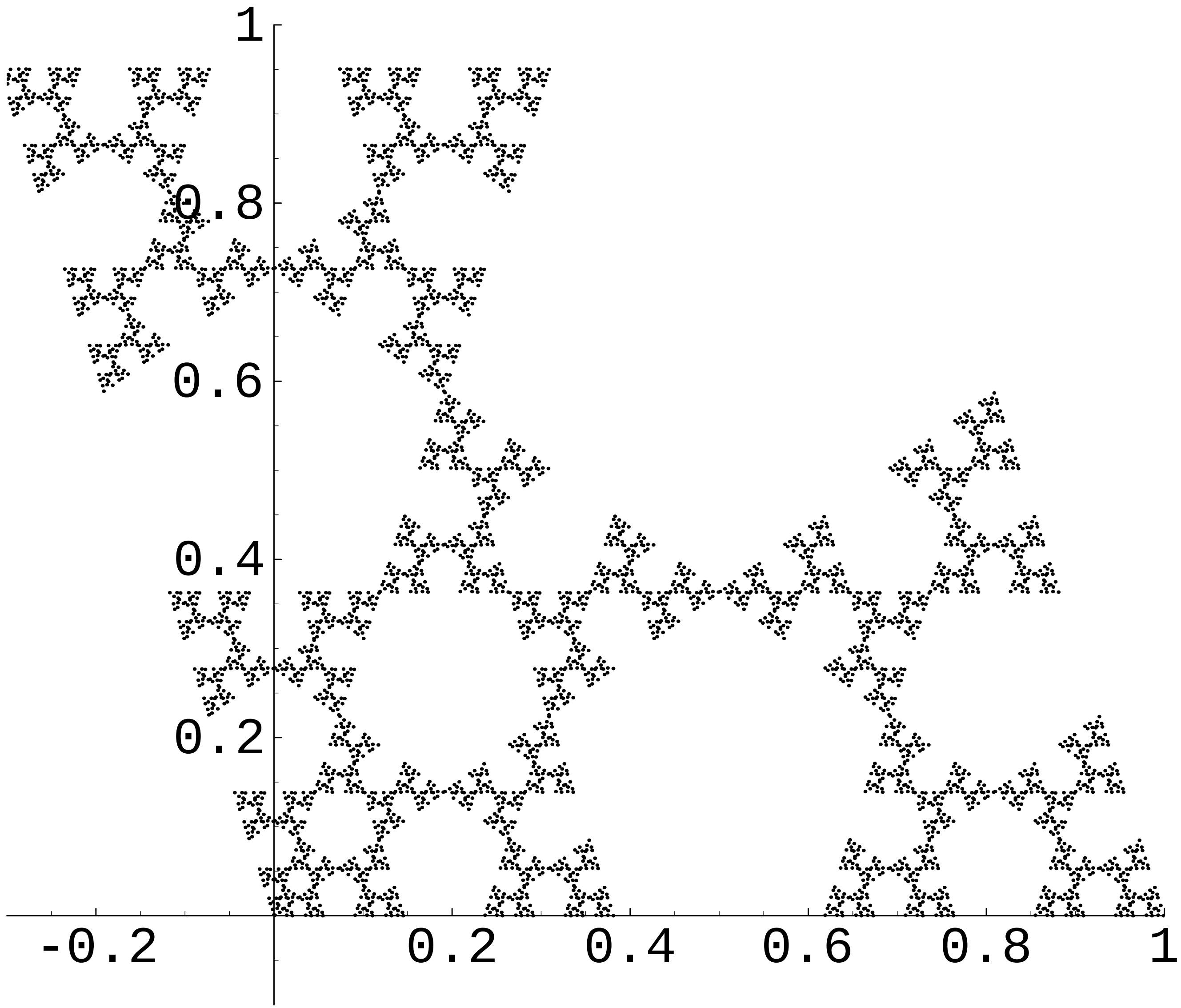


Figure 2: The orbit of $1/3$

Self-inducing structure

Consider a region $L' = \omega^{-2}L$ and the first return map

$$\hat{T}(x) = T^{m(x)}(x)$$

for $x \in L'$ where $m(x)$ is the minimum positive integer such that $T^{m(x)}(x) \in L'$. For any $x \in L'$, the value $m(x) = 1, 3$ or 6. Then

$$\omega^2 \hat{T}(\omega^{-2}x) = T(x) \tag{1}$$

for $x \in L$.

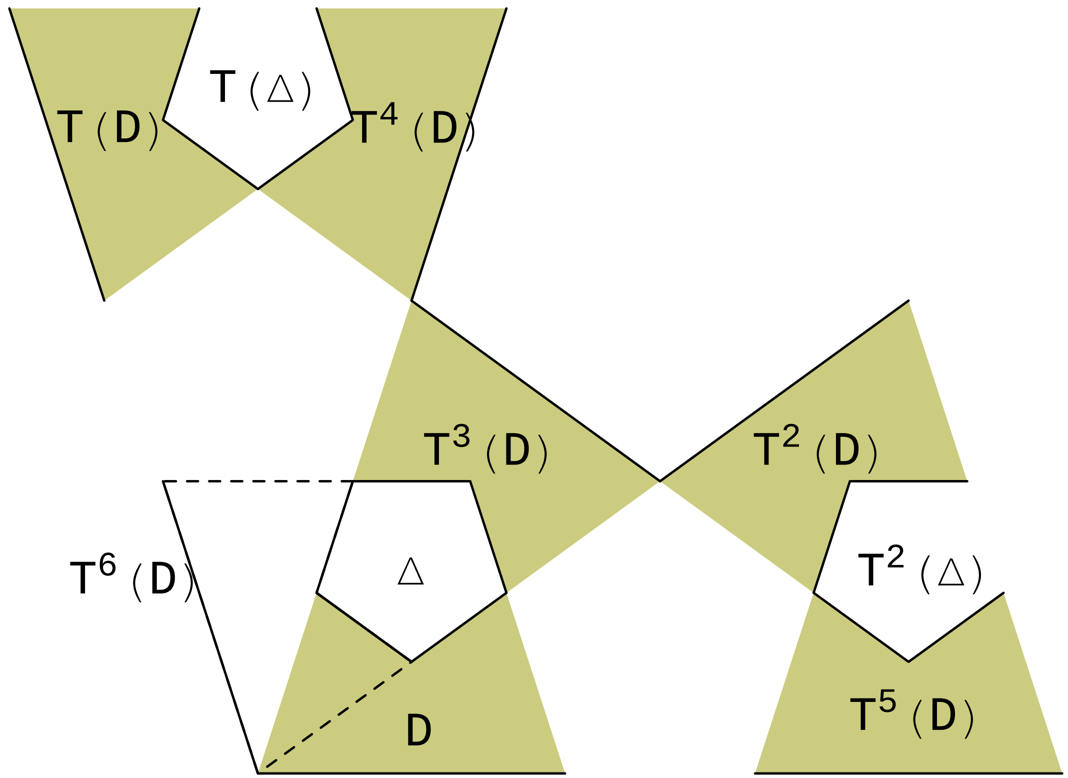


Figure 3: Self Inducing structure

Periodicity

1. Denote by S the composition of 1-st hitting map and expansion by ω^2. The map S decreases the period of T, if exists.

2. For a point in $\mathbb{Z}[\zeta]$, we have a clear distinction:
   S-orbit is finite $\Longleftrightarrow$ T-orbit is periodic
   S-orbit is periodic $\Longleftrightarrow$ T-orbit is aperiodic

3. Finite candidates to be examined to the required periodicity. Further we can show that $\frac{1}{2}\mathbb{Z}[\zeta]$ are periodic but $1/3$ is aperiodic.

Pisot scaling conjecture

Are there self-inducing structures in domain exchanges of other angles?

Yes, for ζ_n with $n = 5, 7, 8, 9, 10, 12$. The case $n = 7, 9$ cubic Pisot numbers appear.

Aperiodic points

Because of the discontinuity of the system, the 'minimal' set is not closed nor open. To study the set $\mathbf{A}$ of aperiodic points, another self-inducing structure plays an important role.

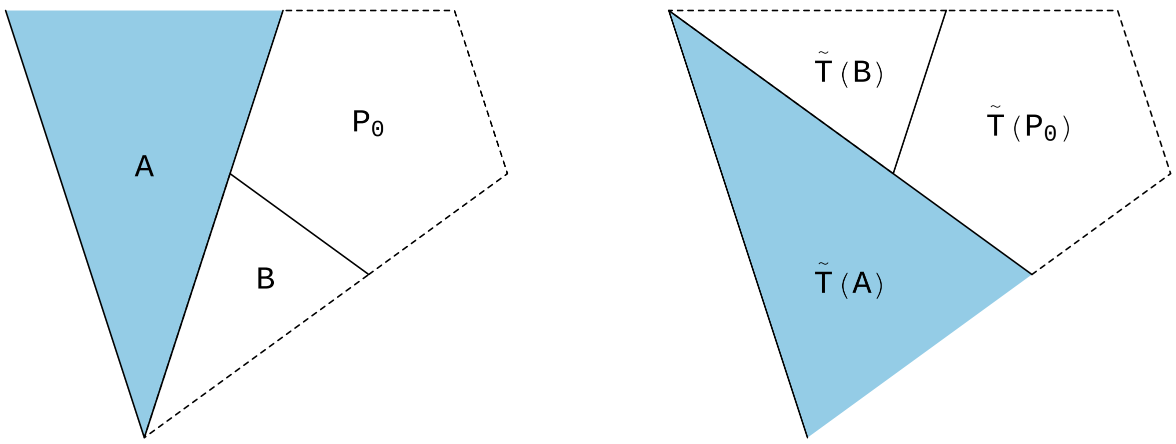


Figure 4: Induced Rotation $\widetilde{T}$ on $T(\mathcal{Z})$

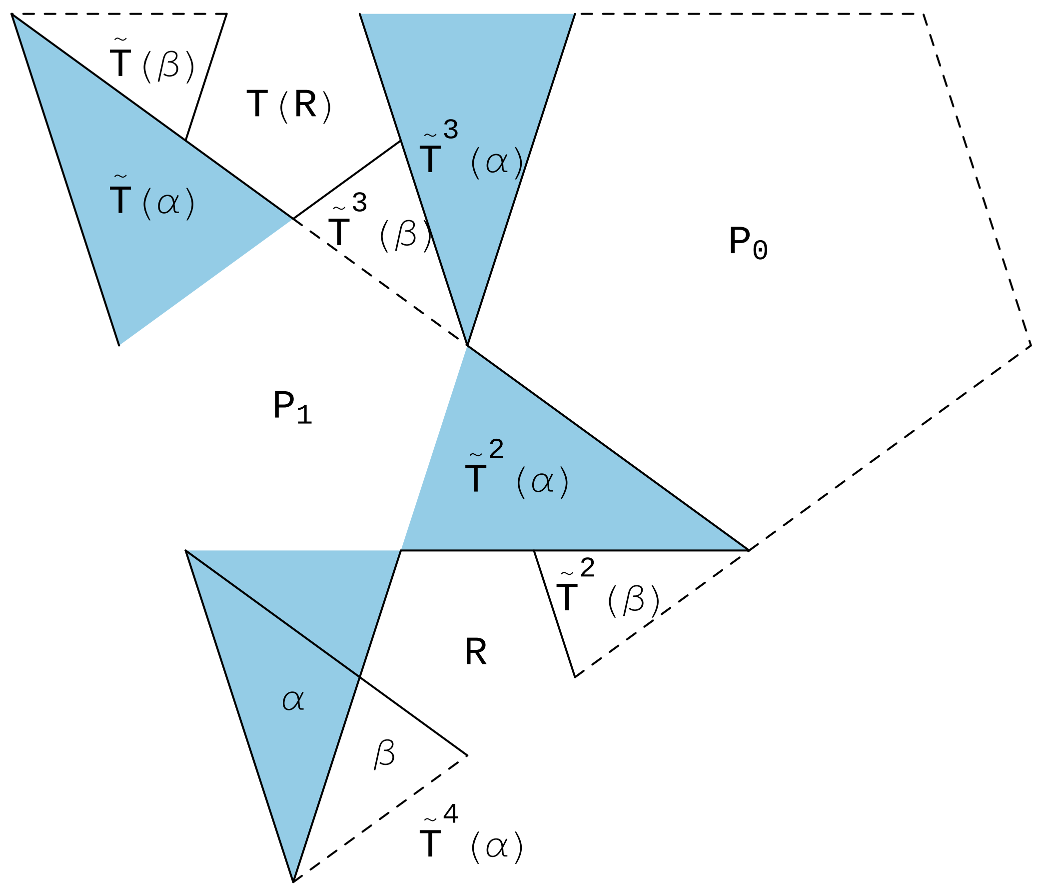


Figure 5: Self-inducing structure of $(T(\mathcal{Z}), \widetilde{T})$

Beta expansion with rotation

Each point $x \in T(\mathcal{Z})$ is expanded into:

$$x_1 = d_{m_1} + \frac{\zeta^{m_1}}{\omega^2}\left(d_{m_2} + \frac{\zeta^{m_2}}{\omega^2}\left(d_{m_3} + \frac{\zeta^{m_3}}{\omega^2}\left(d_{m_4} + \frac{\zeta^{m_4}}{\omega^2}\dots\right.\right.\right. \tag{2}$$

with digits

$$\{d_0, d_2, d_3, d_5\} = \left\{0,\ \zeta,\ \frac{\zeta}{\omega},\ -\frac{1}{\omega\zeta}\right\}.$$

Therefore $\mathbf{A} \cap T(\mathcal{Z})$ must be contained in the attractor of an

IFS:

$$Y' = \left(\frac{1}{\omega^2}Y' + d_0\right) \cup \left(\frac{\zeta^2}{\omega^2}Y' + d_2\right) \cup \left(\frac{\zeta^3}{\omega^2}Y' + d_3\right) \cup \left(\frac{\zeta^5}{\omega^2}Y' + d_5\right) \tag{3}$$

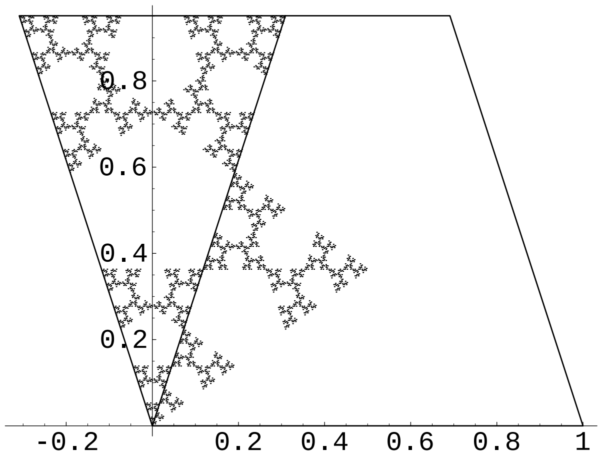


Figure 6: Attractor containing $\mathbf{A} \cap T(\mathcal{Z})$

The first return map is coded by a substitution σ_0:

$$a \to aaba, \quad b \to baba.$$

on $\{a, b\}^*$. One can confirm that

$$\widetilde{\mathbf{d}}\left(\frac{\zeta^m}{\omega^2}x + d_m\right) = \begin{cases} \sigma_0(\widetilde{\mathbf{d}}(x)) & m = 0 \\ a \oplus \sigma_0(\widetilde{\mathbf{d}}(\widetilde{T}(x))) & m = 2 \\ ba \oplus \sigma_0(\widetilde{\mathbf{d}}(\widetilde{T}^2(x))) & m = 3 \\ aba \oplus \sigma_0(\widetilde{\mathbf{d}}(\widetilde{T}^3(x))) & m = 5 \end{cases} \tag{4}$$

where $\oplus$ is the concatenation of letters.

Using conjugate substitutions:

$$\begin{aligned} \sigma_0(a) &= aaba, & \sigma_0(b) &= baba \\ \sigma_1(a) &= aaab, & \sigma_1(b) &= abab \\ \sigma_2(a) &= baaa, & \sigma_2(b) &= baba \\ \sigma_3(a) &= abaa, & \sigma_3(b) &= abab \end{aligned}$$

$$\widetilde{\mathbf{d}}\left(\frac{\zeta^m}{\omega^2}x + d_m\right) = \begin{cases} \sigma_0(\widetilde{\mathbf{d}}(x)) & m = 0 \\ \sigma_1(\widetilde{\mathbf{d}}(\widetilde{T}(x))) & m = 2 \\ \sigma_2(\widetilde{\mathbf{d}}(\widetilde{T}^2(x))) & m = 3 \\ \sigma_3(\widetilde{\mathbf{d}}(\widetilde{T}^3(x))) & m = 5. \end{cases}$$

Each element $x \in T(\mathcal{Z}) \cap \mathbf{A}$ is written as:

$$\widetilde{\mathbf{d}}(x) = \sigma_{m_1}(\sigma_{m_2}(\sigma_{m_3}(\ldots \sigma_\ell(\widetilde{\mathbf{d}}(x_\ell)) \ldots))).$$

This shows that $\widetilde{\mathbf{d}}(x)$ is an S-adic limit of σ_i $(i = 0, 1, 2, 3)$.

Theorem 2. *Within Y', the addresses of $\mathbf{A} \cap T(\mathcal{Z})$ are recognized by a Büchi automaton.*

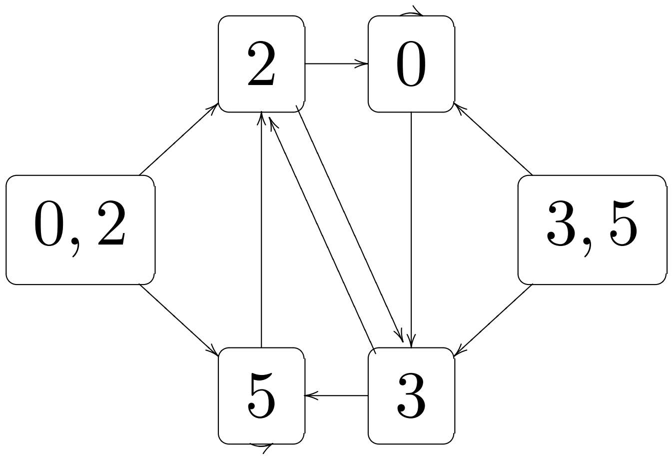


Figure 7: Tails of periodic expansions in Y'

The substitution σ_0:

$$a \longrightarrow aaba, \quad b \longrightarrow baba$$

satisfies the coincidence condition in the sense of Dekking [2, 4]. The substitutive dynamical system has pure discrete spectrum. Its equicontinuous factor is conjugate to the odometer on $\mathbb{Z}_2$. Our coding makes explicit this connection in the following:

Theorem 3. $(Y', \mathbb{B}_{Y'}, \nu, \widetilde{T})$ *is isomorphic to the 2-adic odometer* $(\mathbb{Z}_2, x \mapsto x+1)$*:*

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{+1} & \mathbb{Z}_2 \\ \phi \big\downarrow & & \phi \big\downarrow \\ Y' & \xrightarrow{\widetilde{T}} & Y' \end{array} \tag{5}$$

Moreover the division map

$$\rho : x \mapsto \frac{x - (x \bmod 4)}{4}$$

from $\mathbb{Z}_2$ *to itself gives an isomorphism of multiplicative* *dynamics:*

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{\rho} & \mathbb{Z}_2 \\ \phi \Big\downarrow & & \phi \Big\downarrow \\ Y' & \xrightarrow{S} & Y'. \end{array} \tag{6}$$

Corollary 4. *Each aperiodic point* $x \in \mathbf{A} \cap T(\mathcal{Z})$*, the* $\widetilde{T}$*-orbit of* x *is uniformly distributed in* Y' *with respect to the self similar measure* ν*.*

It is possible to construct a natural extension of the multiplicative system by using the dual numeration system. We use a conjugate map $\phi : \zeta \to \zeta^2$ in $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. We confirm $\phi(\omega) = -1/\omega$. Let us denote by $u_i = \phi(v_i)$. Each point is written as

$$\frac{\zeta^{-2i_1}}{\omega^2}\left(\frac{\zeta^{-2i_2}}{\omega^2}\left(\frac{\zeta^{-2i_3}}{\omega^2}((\dots) - u_{i_3})\right) - u_{i_2}\right) - u_{i_1}$$

Combining dual number system with the original one, we can construct a natural bi-infinite extension of this multiplicative system. The dynamics on the dual is described as base 4 adding machine on the fractal simple arc:

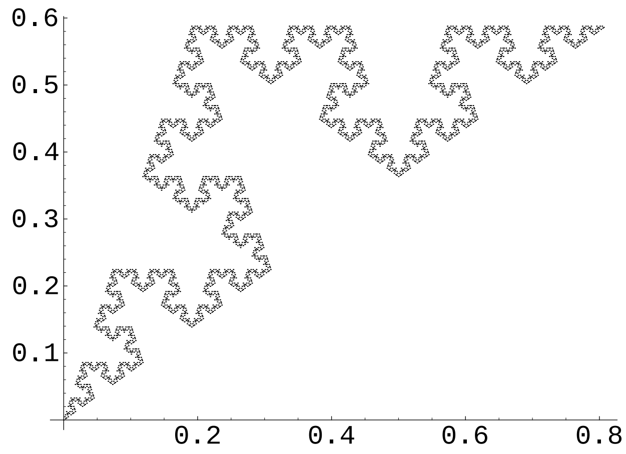


Figure 8: Dual of the aperiodic points

References

[1] S. Akiyama, H. Brunotte, A. Pethő, and W. Steiner, *Periodicity of certain piecewise affine planar maps*, Tsukuba J. Math. **32** (2008), no. 1, 1–55.

[2] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002.

[3] J.H. Lowenstein, S. Hatjispyros, and F. Vivaldi, *Quasi-periodicity, global stability and scaling in a model of hamiltonian round-off*, Chaos **7** (1997), 49–56.

[4] M. Queffélec, *Substitution dynamical systems—Spectral analysis*, Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 1987.