Extensions of substitutions

Pierre Arnoux

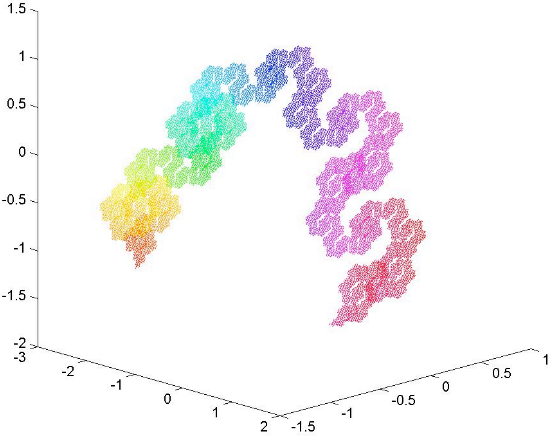
Strobl, July 9, 2009

Joint work with

Julien Bernat
Xavier Bressaud
Hiromi Ei
Maki Furukado
Edmund O. Harriss
Shunji Ito

Aim of the talk

Generate nice fractal sets and self-similar tilings from substitutions and automorphisms of free groups.
Define related dynamical systems.


1.5
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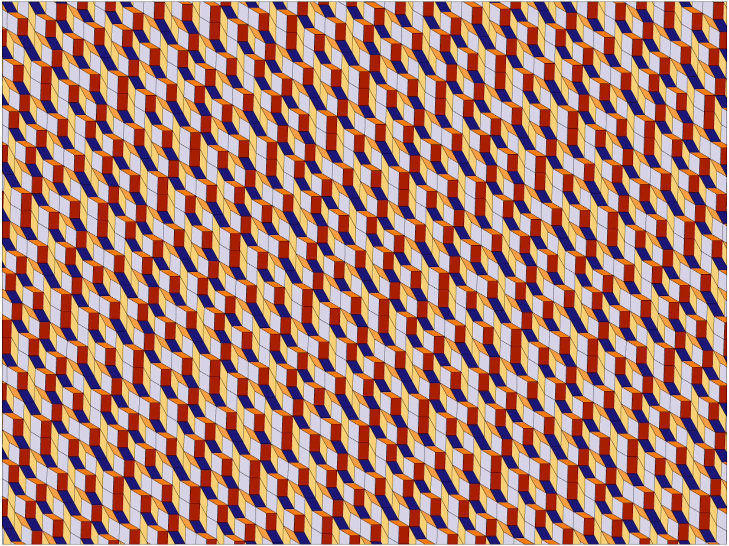
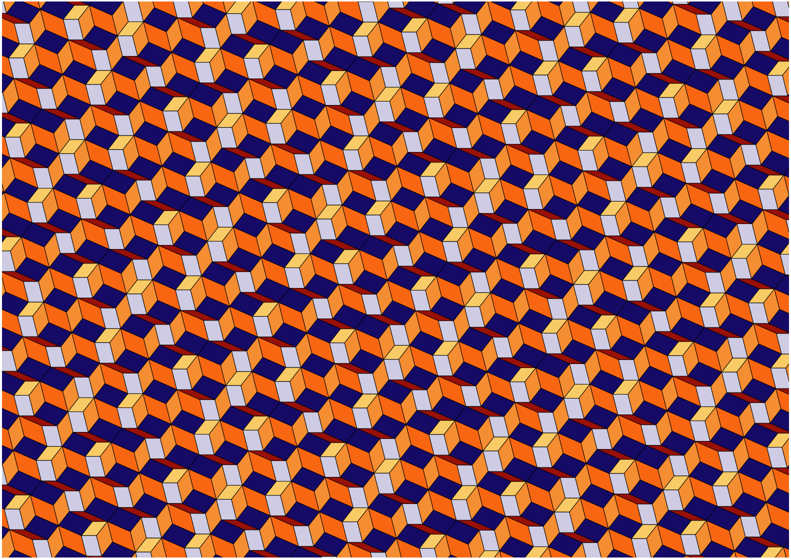
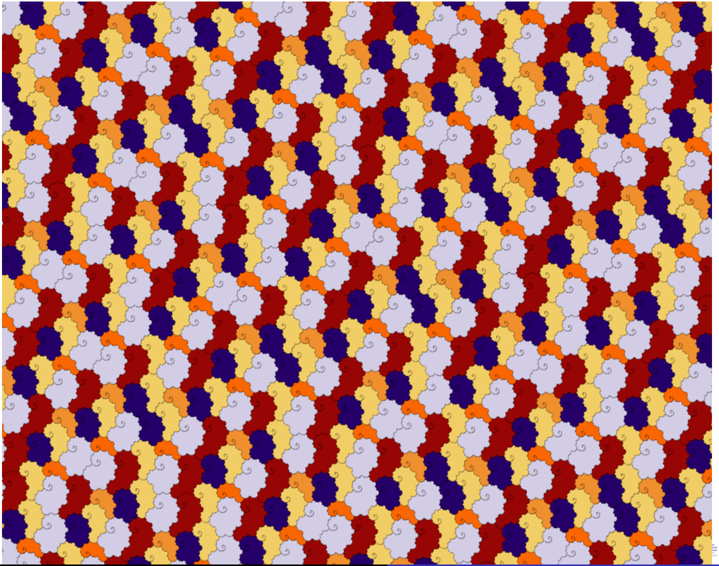
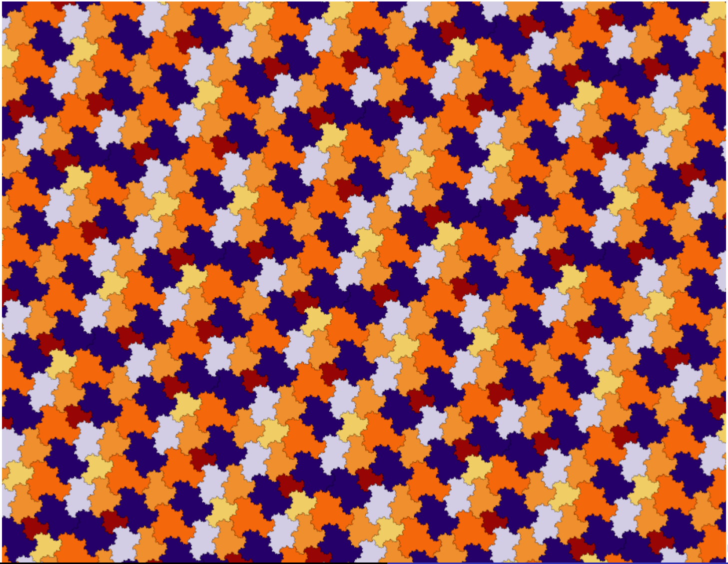




Nautilus_fract_patch_03.gif 932x738 pixels 5/09/06 1:40



Conch_fract_patch_03.gif 926x734 pixels 5/09/06 1:41



some of these pictures, and many other tilings, can be found on the site:

http://saturn.math.uni-bielefeld.de/tilings/index

maintained by E. Harriss and D. Frettloeh

Primitive substitutions

- σ substitution on $\mathcal{A}$ alphabet of cardinal d
- For $\mathbf{i} \in \mathcal{A}$:

$$\begin{aligned}\sigma(\mathbf{i}) &= W^{(\mathbf{i})} \\ &= W_1^{(\mathbf{i})} \dots W_{l_\mathbf{i}}^{(\mathbf{i})} \\ &= P_k^{(\mathbf{i})} W_k^{(\mathbf{i})} S_k^{(\mathbf{i})}\end{aligned}$$

- *Abelianization* $f : \mathcal{A}^* \to \mathbb{Z}^d$. $f(W) = (|W|_\mathbf{1}, |W|_\mathbf{2}, \dots, |W|_\mathbf{d})$
- *Abelianization* of σ: matrix A such that: $f(\sigma(W)) = A.f(W)$.
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1-dimensional tiling

- $u \in \mathcal{A}^{\mathbb{N}}$ periodic point of σ.
- $\Omega_\sigma = \overline{\{S^n u\}}$ dynamical system associated with σ.
- $L = (l_a)$ Perron eigenvector of A.
- 1-dim tiling $\mathcal{T}$ from u and L; this tiling is self-similar. ϕ_t tiling flow.
- (Ω_σ, S) is a section of the tiling flow.

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Stepped lines

Definition

Stepped line $\mathcal{V} = (x_n)_{n \geq 0}$ in $\mathbb{R}^d$: the *steps* $x_{n+1} - x_n$ belong to the canonical basis $\mathcal{B}$ of $\mathbb{R}^d$.

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Definition

Stepped line associated to a fixed point $u = (a, \ldots)$ of σ : $\mathcal{V}^u = \cup_n \mathcal{V}^{\sigma^n(a)}$.

Self-similarity of stepped lines I

Proposition

For any letter $a \in \mathcal{A}$, we have a disjoint decomposition:

$$\begin{aligned}\mathcal{V}^{\sigma^{n+1}(a)} &= \coprod_{P,b|Pb \text{ prefix of } \sigma(a)} \mathcal{V}^{\sigma^n(b)} + e_{\sigma^n(P)} \\ &= \coprod_{P,b;Pb \text{ prefix of } \sigma(a)} \mathcal{V}^{\sigma^n(b)} + A^n e_P\end{aligned}$$

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Proof. $\sigma^{n+1}(a) = \sigma^n(\sigma(a))$ □

Self-similarity of stepped lines II

Definition

$x_n \in \mathcal{V}^u$ is of *type* a if $u_{n+1} = a$.
$\mathcal{V}^{u,a}$: set of vertices of type a of $\mathcal{V}^u$.

Self-similarity of stepped lines II

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Proposition

Let u be an infinite fixed point of the substitution σ. Then we have the disjoint decomposition:

$$\mathcal{V}^{u,a} = \coprod_{P,b|Pa \text{ prefix of } \sigma(b)} A\mathcal{V}^{u,b} + e_P$$

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Projection on the Perron-Frobenius space

Stepped line of the fixed point:
Natural projection on the Perron-Frobenius space: renormalizable tiling.

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Natural projection on the Perron-Frobenius space: renormalizable tiling.
It also defines a number system, and a parametrization of the curve underlying the canonical stepped line

Projection on contracting spaces and Rauzy fractals

Proposition

$E \oplus F$ invariant splitting of $\mathbb{R}^{\mathcal{A}}$ for A. If the restriction of A to E is strictly contracting, the projection of the vertices of a canonical line on E along F is a bounded set.

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Proposition

Let $\mathcal{R}_a$ be the closure of the projection of the vertices of type a, for all $a \in \mathcal{A}$. The sets $\mathcal{R}_a$ satisfy the relation:

$$\mathcal{R}_a = \coprod_{P,b \mid Pa \text{ prefix of } \sigma(b)} A\mathcal{R}_b + e_P$$

Projection on expanding spaces

Let $E \oplus F$ be an invariant splitting of $\mathbb{R}^{\mathcal{A}}$, such that the restriction of A is strictly expanding, and denote by Π_F the projection on F along E.

Proposition

For all $a \in \mathcal{A}$, the set $\overline{\cup_{n \geq 0} A^{-n} \Pi_F(\mathcal{V}^{\sigma^n(a)})}$ is a compact curve $\mathbb{C}^a$.

Proposition

We have:

$$\mathbb{C}^a = \bigcup_{P,b | Pb \text{ prefix of } \sigma(a)} A^{-1}(\mathbb{C}^b + e_P)$$

An example

$$\begin{aligned}\sigma: \quad 1 &\longrightarrow 35\\ 2 &\longrightarrow 45\\ 3 &\longrightarrow 46\\ 4 &\longrightarrow 17\\ 5 &\longrightarrow 18\\ 6 &\longrightarrow 19\\ 7 &\longrightarrow 29\\ 8 &\longrightarrow 2\\ 9 &\longrightarrow 3.\end{aligned}$$

Invariant subspaces

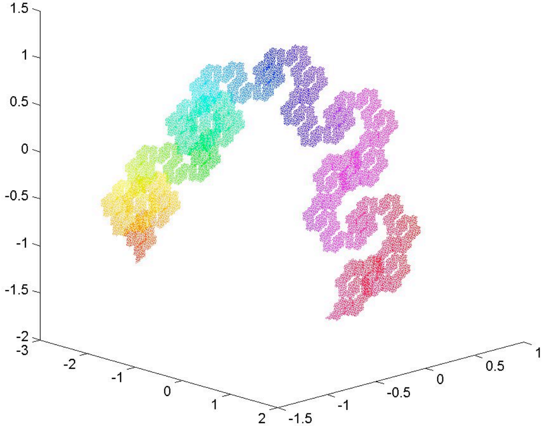
The characteristic polynomial is
$P(X) = X^9 - X^7 - 5X^6 - X^5 + X^4 + 5X^3 + X^2 - 1,$

$$P(X) = (X-1)(X^2+X+1)(X^3+X^2+X-1)(X^3-X^2-X-1).$$

The space splits in a natural way:
$\mathbb{R}^9 = E_n \oplus E_T \oplus E_{\overline{T}}$

$$\begin{aligned} \mathbb{R}^9 &= E_n \oplus E_{\overline{T}} \oplus E_T = E_n \oplus E_s \oplus E_u && (1) \\ &= E_l \oplus E_j \oplus E_{\overline{T},s} \oplus E_{\overline{T},u} \oplus E_{T,u} \oplus E_{T,s} && (2) \end{aligned}$$

projection on the contracting space



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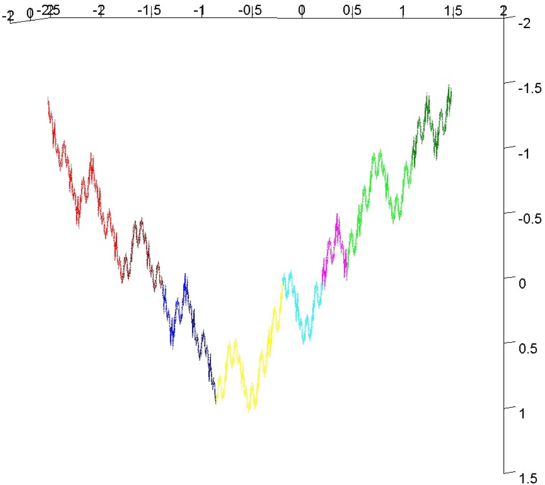

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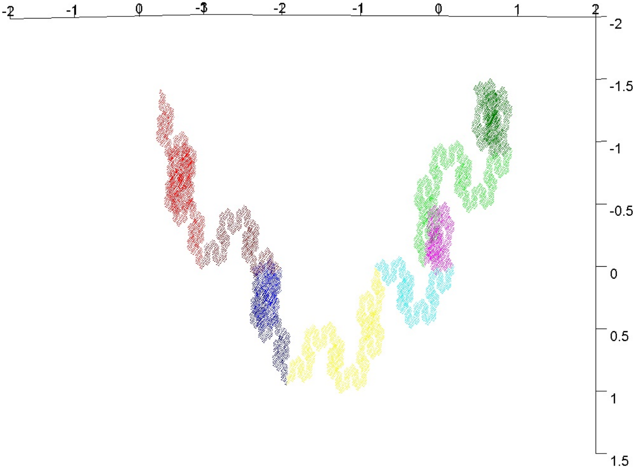

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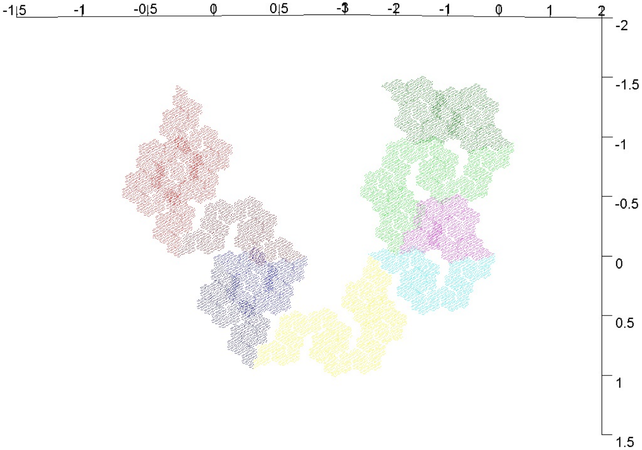

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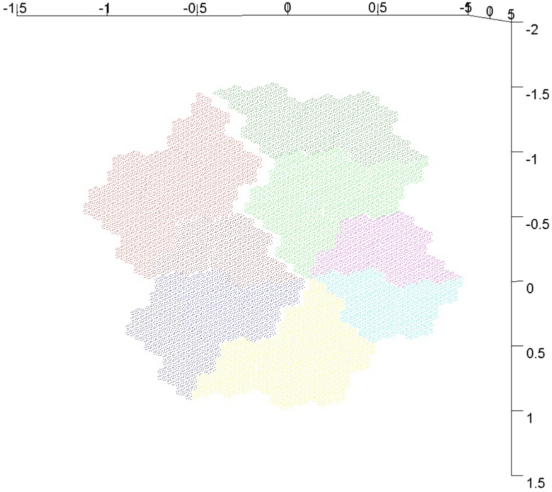

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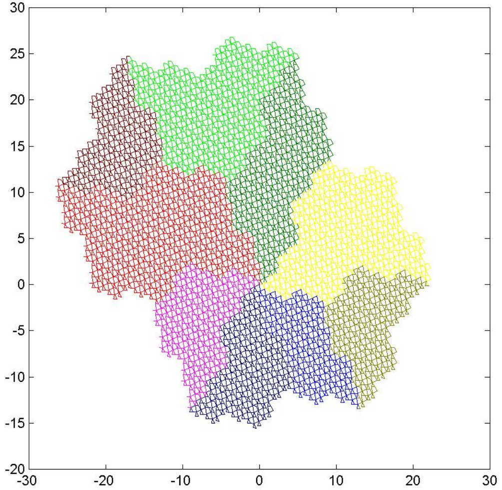

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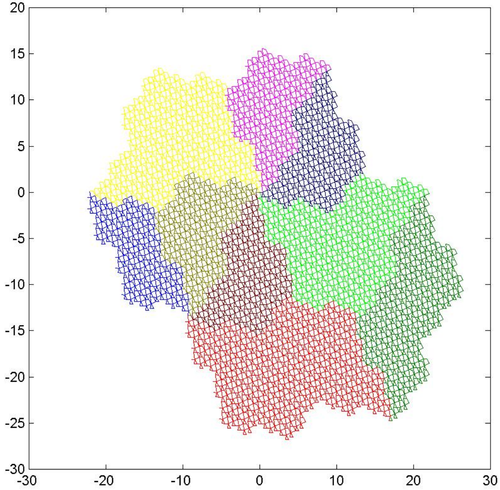

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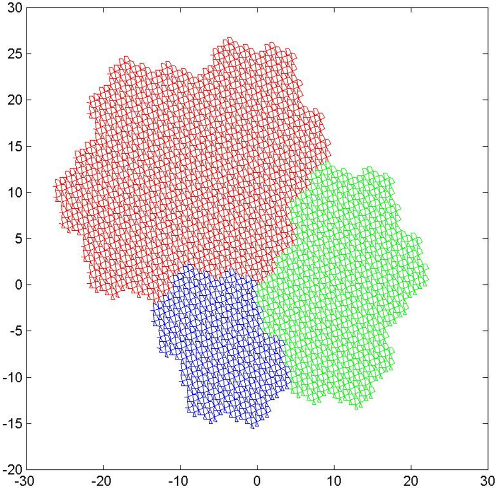

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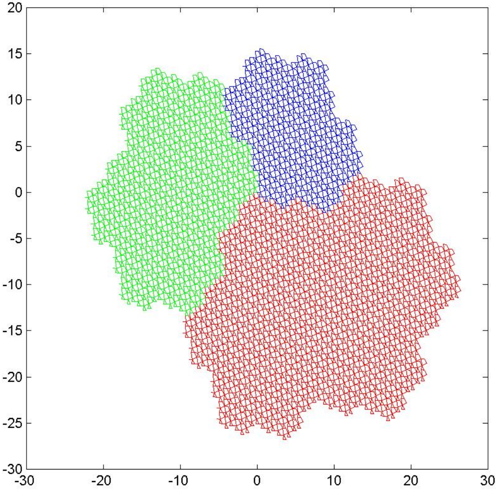

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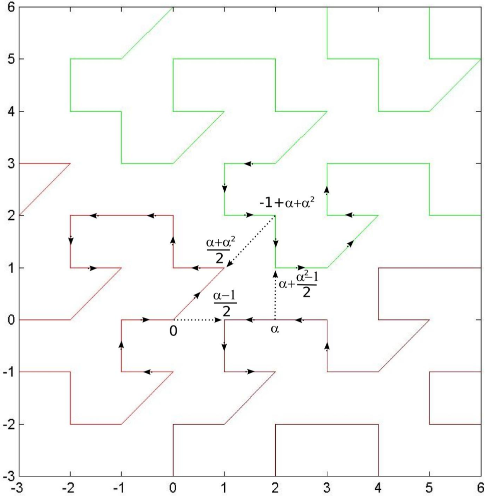

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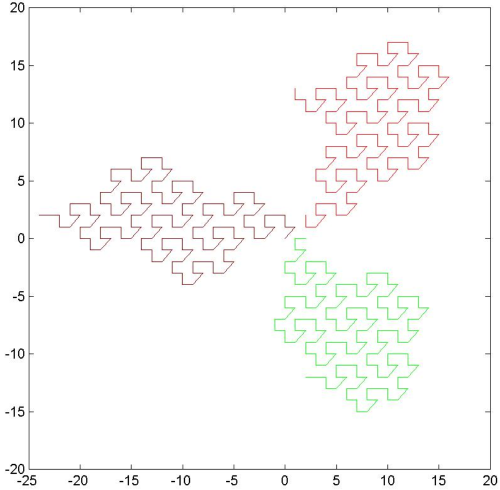

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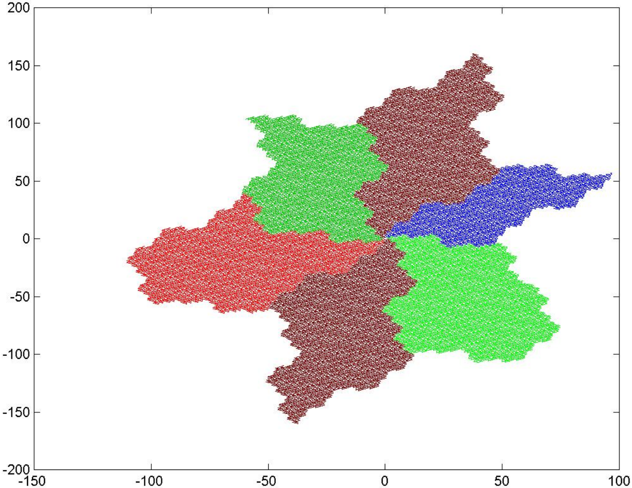

Formalism I: geometric models of words and weighted paths

- For $W \in \mathcal{A}^*$:
  $f(W)$ = abelianization of W.
- Example: $f(\mathbf{21121211}) = (5, 3)$.
- $(x, \mathbf{i})$: segment $(x, x + e_{\mathbf{i}})$
  $\mathcal{G}_1$:
  set of formal sums of weighted paths
- This weighted path is
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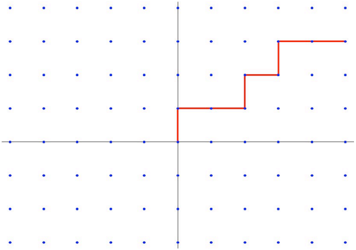

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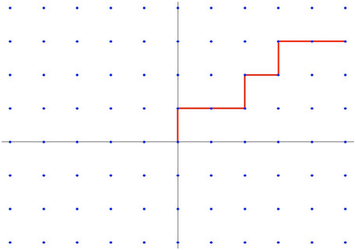

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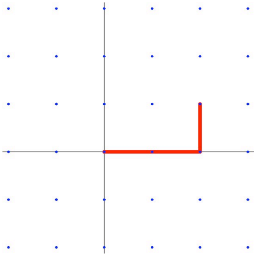

Formalism II: the geometric model for the substitution

- Associate to σ a geometric map $E_1(\sigma)$
- To a segment $\mathbf{i}$, we associate the path $\sigma(\mathbf{i})$
- Example:

$$1 \mapsto 112$$

$$2 \mapsto 12$$

- $E_1(\sigma)(x, \mathbf{i}) = \sum_{n=1}^{l_{\mathbf{i}}} \left(A(x) + f(P_n^{(\mathbf{i})}), W_n^{(\mathbf{i})}\right).$
Shift of the origin: $x \rightarrow A.x$, needed for connexity of the image.

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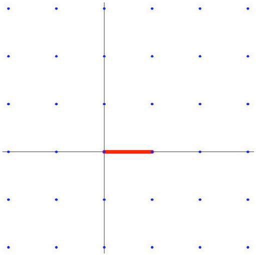

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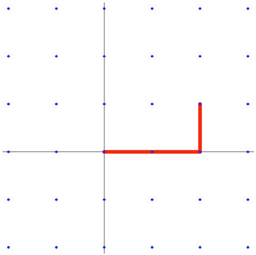

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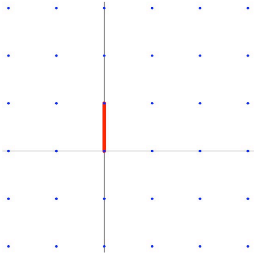

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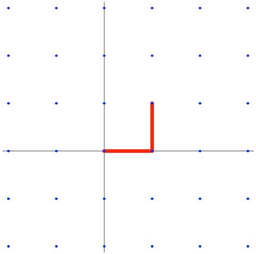

Formalism II: the geometric model for the substitution

- Associate to σ a geometric map $E_1(\sigma)$
- To a segment $\mathbf{i}$, we associate the path $\sigma(\mathbf{i})$
- Example:

$$1 \mapsto 112$$

$$2 \mapsto 12$$

- $E_1(\sigma)(x, \mathbf{i}) = \sum_{n=1}^{l_\mathbf{i}} \left(A(x) + f(P_n^{(\mathbf{i})}), W_n^{(\mathbf{i})} \right).$
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Rauzy fractals and dual substitutions

- Rauzy fractal: projection of the fixed line of $E_1(\sigma)$ on the contracting space
- Solution of an IFS coming from the substitution
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Formalism III: the dual map for the substitution

- $E_1(\sigma)$ is a linear map on a vector space
- We can define formally its dual map.
- If A is invertible, this dual map is easily computed:
- $$E_1^*(\sigma)(\mathbf{x}, \mathbf{i}^*) = \sum_{W_n^{(\mathbf{j})}=\mathbf{i}} \left(A^{-1}\left(\mathbf{x} - f(P_n^{(\mathbf{j})})\right), \mathbf{j}^*\right).$$

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- Geometric interpretation of this map: represent $(\mathbf{x}, \mathbf{i}^*)$ by the upper face of the unit cube at $\mathbf{x}$ orthogonal to $(\mathbf{x}, \mathbf{i})$.
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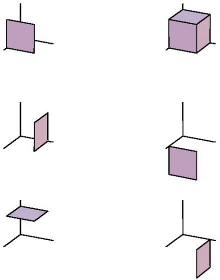

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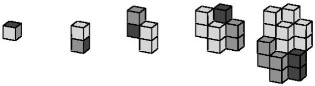

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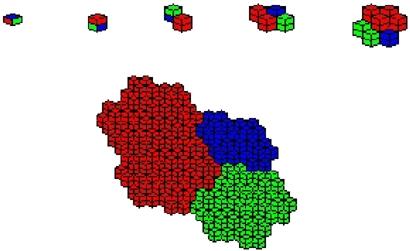

Properties of the dual map

- Generates a polygonal tiling which approximates the contracting plane
- By renormalization, generates the Rauzy fractal
- We can obtain a self-similar tiling with fractal tiles
- There is a related $\mathbb{R}^{d-1}$ action on this tiling space, dual to the previous tiling flow.

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A free group automorphism

σ automorphism of the free group F_4:

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 3 \\ 3 &\mapsto 4 \\ 4 &\mapsto 41^{-1} \end{aligned}$$

$$\text{Matrix } M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

A free group automorphism

Characteristic polynomial $X^4 - X^3 + 1$
Eigenvalues $1.01891 \pm 0.602565i$, $-0.518913 \pm 0.66661i$
Non Pisot!
Expanding plane $P_e \equiv \mathbb{C}$
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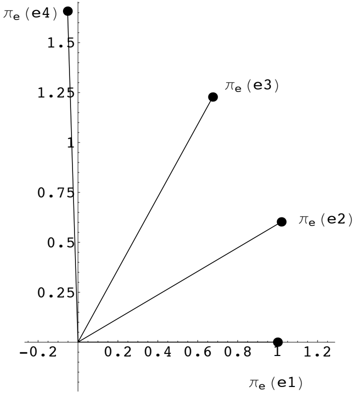
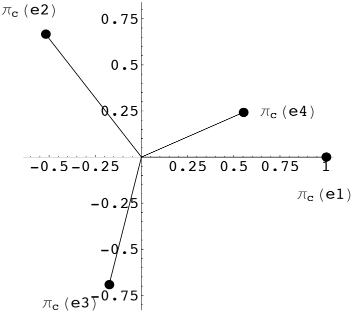
Projections of the canonical basis





Geometric extensions of free group automorphisms I

- Words as discrete lines in $\mathbb{R}^4$
- σ acts naturally on discrete lines in $\mathbb{R}^4$
- Map $E_1(\sigma)$ defined on space $\mathcal{G}_1$ of weighted sum of discrete lines
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Geometric extensions of free group automorphisms II

- $i : s \mapsto i(s)$ and $j : t \mapsto j(t)$ segments; define the oriented face $i \wedge j$ as the oriented surface $(s, t) \mapsto i(s) + j(t)$.
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- Map $E_2(\sigma)$ defined on space $\mathcal{G}_2$ of weighted sum of discrete faces
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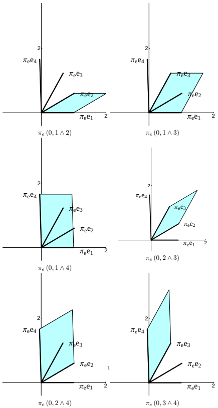

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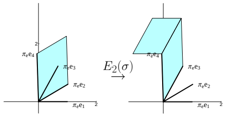

$\pi_e\,(0, 2 \wedge 4)$ $\quad$ $\pi_e\,(E_2(\sigma)(0, 2 \wedge 4))$

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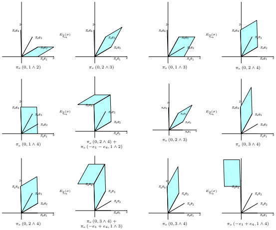

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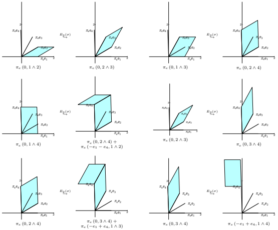

Geometric extensions III: formalism

- We define the space of formal finite sums of weighted 2-faces $(\mathbf{x}, i \wedge j)$, with $\mathbf{x} \in \mathbb{Z}^4$.
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$E_2(\sigma)(\mathbf{x}, \mathbf{i} \wedge \mathbf{j}) :=$
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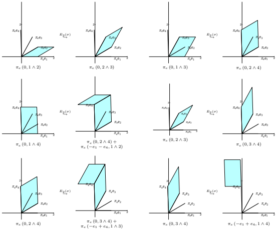

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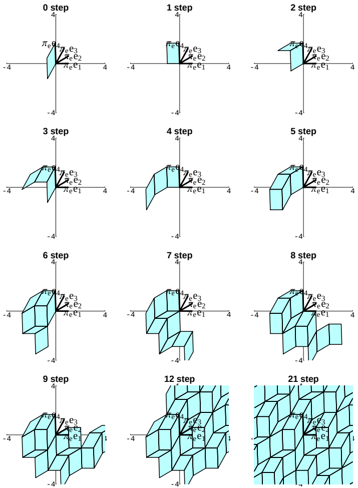

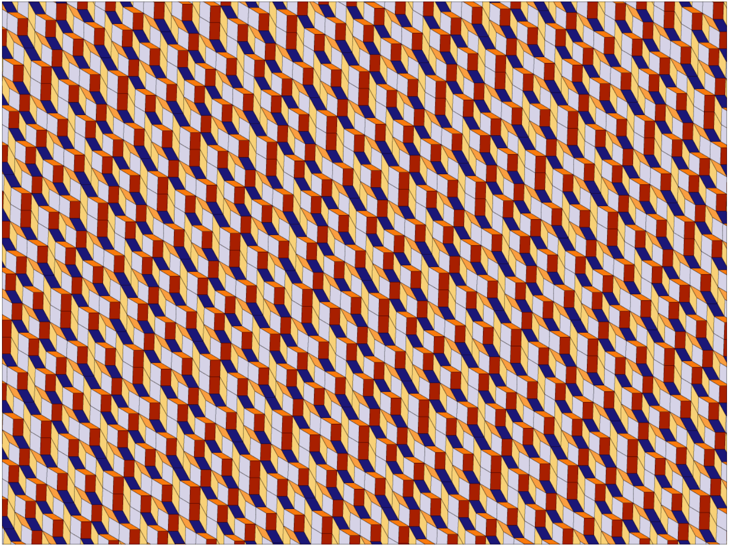

An exact substitution tiling

- By replacing each face by the limit of its renormalization, one obtains an exactly self-similar tiling, with fractal tiles.
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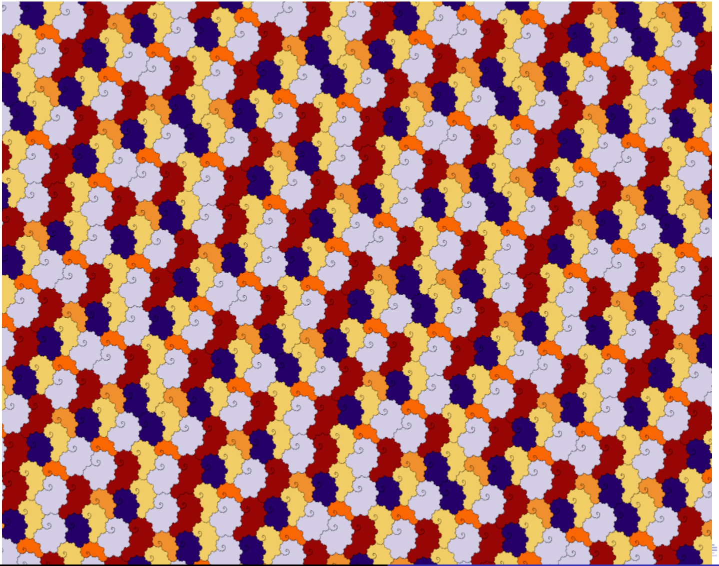
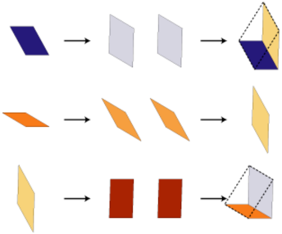
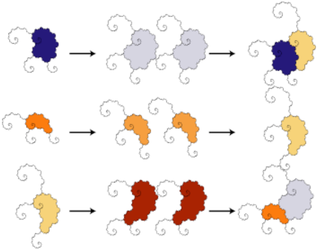
Nautilus_fract_patch_03.gif 932x738 pixels 5/09/06 1:40







A discrete surface in $\mathbb{R}^4$

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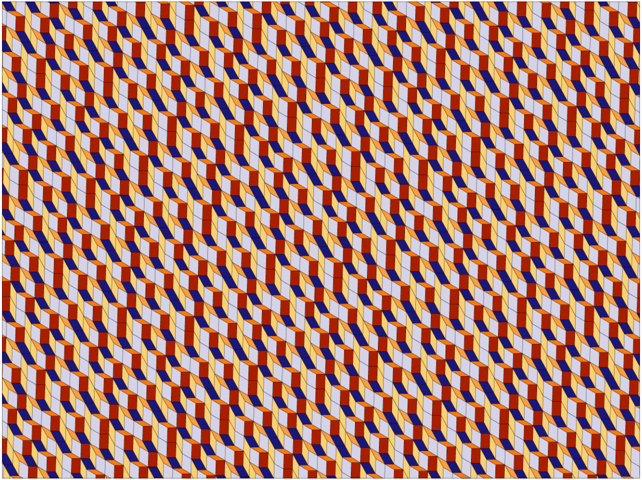
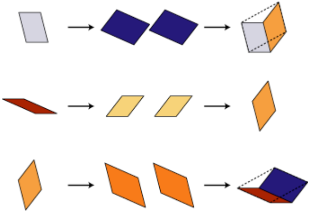
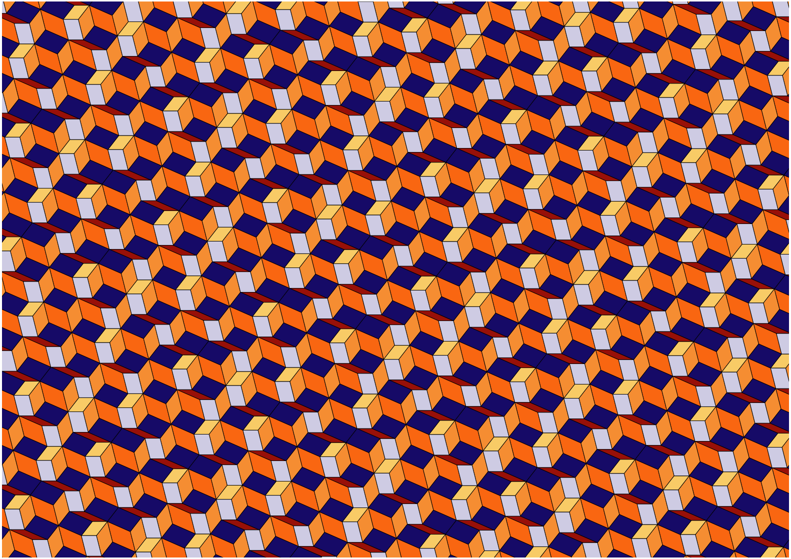
The discrete surface



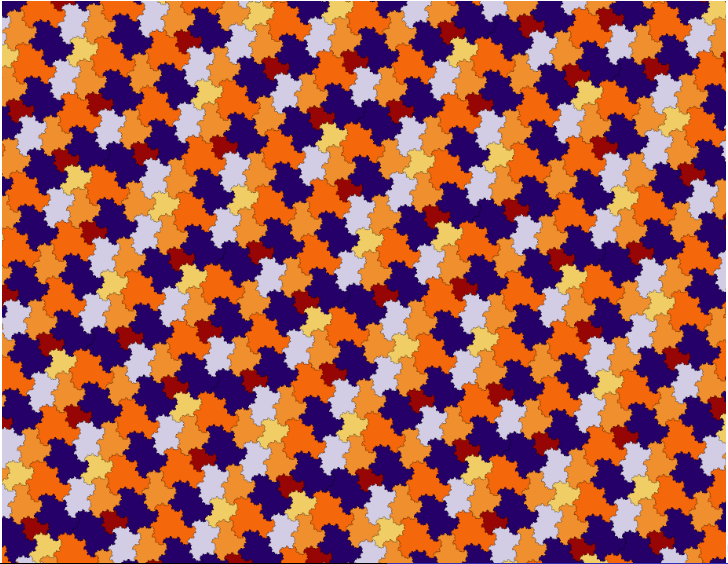
Duality

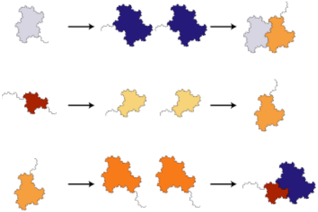
We can do exactly the same for the contracting plane:
Define the dual map $E^2(\sigma)$.
It is also positive.
Get dual substitution tiling and a dual self-similar tiling.





Conch_fract_patch_03.gif 926x734 pixels 5/09/06 1:41





Conclusions

Formalism extendable in all dimensions
Geometric models for a family of automorphisms
Problems with cancellations

Cut-and-project tiling

- The fractal tiles of the expanding tiling are solution of a GIFS.
- The vertices of the contracting tiling are solution of a GIFS.
- After projection on the expanding space, we can observe a very much curious phenomenon:
- The second IFS is the opposite of the first!
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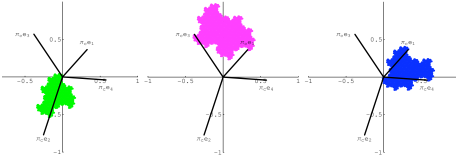
Generalized Rauzy fractals

The window for the tiling of the expanding plane is the contracting Rauzy fractal $X^c = \cup X^c_{i \wedge j}$.
It can be obtained by projecting on the contracting plane the vertices of the discrete approximation to the expanding plane.
It can also be obtained by renormalization of the projection of the image of a patch of faces by the action of the dual map:

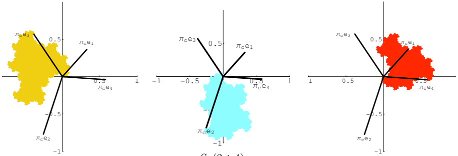
$$X^c = \lim M^{-n}(\pi_c(E_2^*(\sigma)^n(\mathcal{U})))$$

the same property is true for the expanding Rauzy fractal.

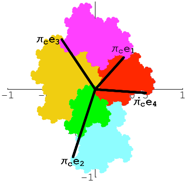
The window



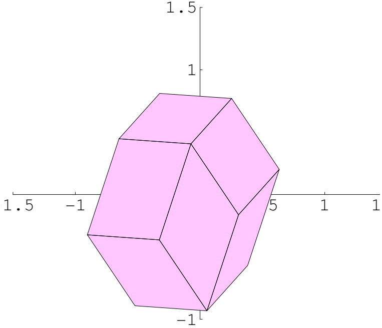
$S_c(2 \wedge 1)$ $S_c(1 \wedge 3)$ $S_c(4 \wedge 1)$



$S_c(3 \wedge 2)$ $S_c(2 \wedge 4)$ $S_c(4 \wedge 3)$



renormalization and projection



$\mathcal{U}_c^{(1)}$

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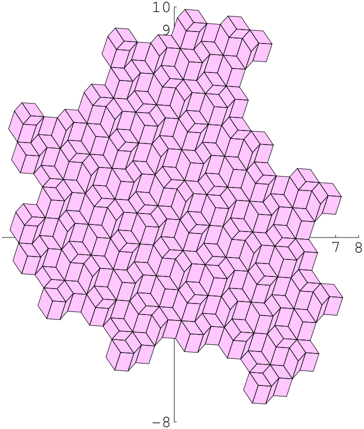
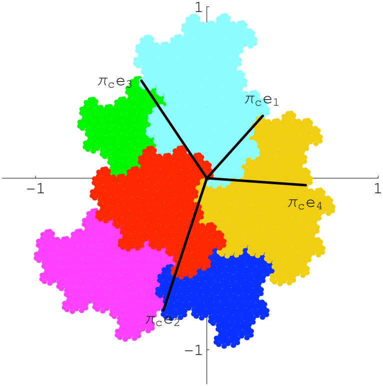


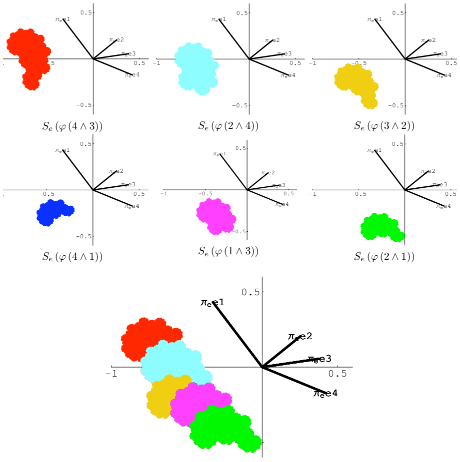
$E_2{}^{13}(\theta)\,\mathcal{U}_c^{(5)}$

renormalization and projection



$$\bigcup_{i\wedge j\in V_c} X_c\,(i\wedge 1)$$

The other window



$S_e(\varphi(4\wedge 3))$ $S_e(\varphi(2\wedge 4))$ $S_e(\varphi(3\wedge 2))$

$S_e(\varphi(4\wedge 1))$ $S_e(\varphi(1\wedge 3))$ $S_e(\varphi(2\wedge 1))$

$\bigcup_{i\wedge j\in V_e} S_e(\varphi(i\wedge j))$

Symbolic dynamics

By taking the product of the corresponding Rauzy fractals:

$$X^{c}_{i\wedge j} \times X^{e}_{k\wedge l}$$

one obtains a partition of the torus $\mathbb{T}^4$.
This partition gives a symbolic dynamics for the action of the matrix A which is a subshift of finite type.
This is the first known explicit Markov partition for a non-Pisot irreducible automorphism of the torus.
It is the natural extension of the β-expansion.

Formalism IV: geometric extensions of substitutions

- Easy to generalize to dimension k.
- Define $E_k(\sigma) : \mathcal{G}_k \to \mathcal{G}_k$.
- Boundary map $\delta_k : \mathcal{G}_k \to \mathcal{G}_{k-1}$, commutative diagram:

$$\begin{array}{ccccc} & \mathcal{G}_d & \xrightarrow{E_d(\sigma)} & \mathcal{G}_d & \\ \delta_d & \downarrow & & \downarrow & \delta_d \\ & \mathcal{G}_{d-1} & \xrightarrow{E_{d-1}(\sigma)} & \mathcal{G}_{d-1} & \\ \delta_{d-1} & \downarrow & & \downarrow & \delta_{d-1} \\ & \vdots & & \vdots & \\ & \mathcal{G}_1 & \xrightarrow{E_1(\sigma)} & \mathcal{G}_1 & \\ \delta_1 & \downarrow & & \downarrow & \delta_1 \\ & \mathcal{G}_0 & \xrightarrow{E_0(\sigma)} & \mathcal{G}_0. & \end{array}$$

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Formalism V: dual extensions of substitutions

- $E_k(\sigma)$ linear map on $\mathcal{G}_k$.
- one can define the dual map $E_k^*(\sigma)$.
- If M_σ is invertible, it is easy to compute this dual map.
- $$E_1^*(\sigma)(\mathbf{x}, \mathbf{i}^*) = \sum_{W_n^{(\mathbf{j})}=\mathbf{i}} \left(A^{-1}\left(\mathbf{x} - f(P_n^{(\mathbf{j})})\right), \mathbf{j}^*\right).$$
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Geometric models for duality

- Define $\phi_k : \mathcal{G}_k^* \to \mathcal{G}_{d-k}$:

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Hiromi Ei's theorem

σ free group automorphism; we have :

$$E^{d-k}(\sigma) = E_k(\tilde{\sigma}^{-1})$$

where $\tilde{\sigma}$ is the mirror image of σ.
$E^k(\sigma)$ pseudo-inverse for σ.

Topology of tiles and their boundary

The boundary can be studied by a suitable extension.
Alternative and more efficient methods have been developed:
Prefix-suffix graphs and their generalizations
Due to Siegel-Thuswaldner

Transversal dynamics

- Study the transversal flow of these tilings.
- Find a good symbolic dynamics for this $\mathbb{R}^2$-action
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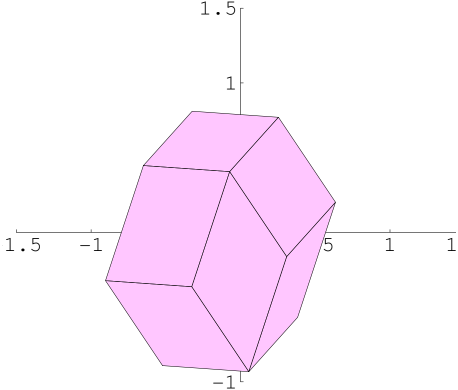

$$\mathcal{U}_c^{(1)}$$

Classification problems

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