Diffraction Theory

of Tiling Dynamical Systems

Michael Baake

Faculty of Mathematics, Bielefeld

(joint work with Uwe Grimm)

Menue

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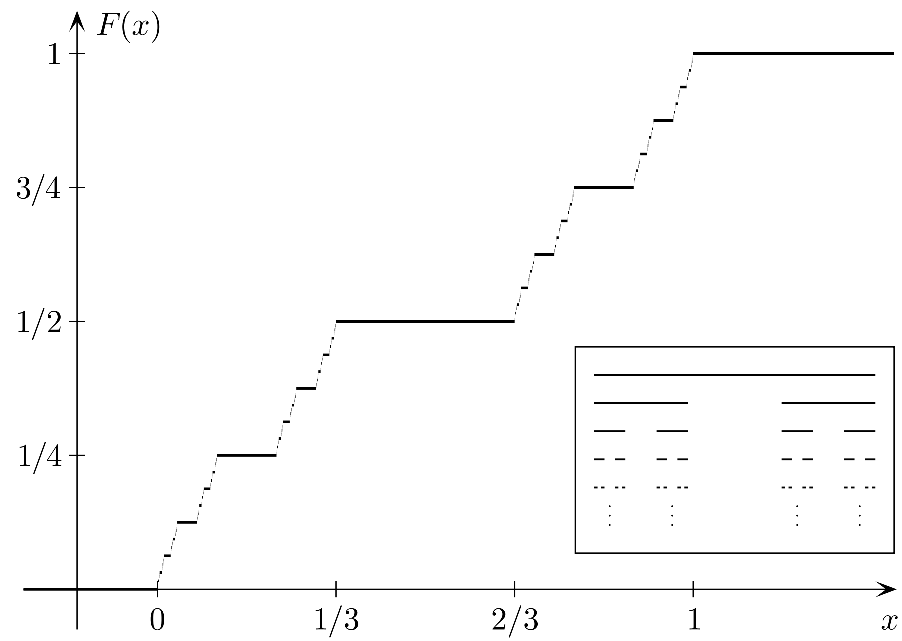


Diffraction theory

Structure: translation bounded measure ω
assumed 'amenable'

Autocorrelation: $$\gamma = \gamma_\omega = \omega \circledast \widetilde{\omega} := \lim_{R\to\infty} \frac{\omega|_R * \widetilde{\omega|_R}}{\mathrm{vol}(B_R)}$$

Diffraction: $$\boxed{\widehat{\gamma} = \widehat{\gamma}_{\mathsf{pp}} + \widehat{\gamma}_{\mathsf{sc}} + \widehat{\gamma}_{\mathsf{ac}}}$$ (relative to λ)

- pp: Bragg peaks only (lattices, model sets)
- ac: diffuse scattering with density
- sc: whatever remains ...

Thue-Morse chain

Substitution: $\varrho : \begin{array}{l} 1 \mapsto 1\bar{1} \\ \bar{1} \mapsto \bar{1}1 \end{array}$ $\qquad (\,\bar{1} \mathrel{\hat{=}} -1\,)$

Iteration and fixed point:

$$1 \mapsto 1\bar{1} \mapsto 1\bar{1}\bar{1}1 \mapsto 1\bar{1}\bar{1}1\bar{1}11\bar{1} \mapsto \ldots \longrightarrow v = \varrho(v) = v_0 v_1 v_2 v_3 \ldots$$

- $\boxed{v_{2i} = v_i}$ and $\boxed{v_{2i+1} = \bar{v}_i}$
- v is (strongly) cube-free
- hull of v is aperiodic and strictly ergodic
- $v_i = (-1)^{\text{sum of the binary digits of } i}$

Two-sided version: $w_i = \begin{cases} v_i, & \text{for } i \geq 0 \\ v_{-i-1}, & \text{for } i < 0 \end{cases}$

Autocorrelation

Autocorrelation: $$\gamma = \lim_{n\to\infty} \frac{1}{2n+1} \left(\omega|_n * \widetilde{\omega|_n}\right)$$

with $$\omega|_n = \sum_{i=-n}^{n} w_i \, \delta_i$$

Structure: $$\gamma = \textstyle\sum_{m\in\mathbb{Z}} \eta(m)\, \delta_m$$

with $$\eta(m) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i \, v_{i+m}$$

and $\eta(-m) = \eta(m)$ for $m \geq 0$

Recursion: $\eta(0) = 1$, $\eta(1) = -\frac{1}{3}$ and, for all $m \geq 0$,

$\boxed{\eta(2m) = \eta(m)}$ and $\boxed{\eta(2m+1) = -\frac{1}{2}\big(\eta(m) + \eta(m+1)\big)}$

Diffraction: Absence of pp part

$\boxed{\widehat{\gamma} = \mu * \delta_{\mathbb{Z}}}$ with $\mu = \widehat{\gamma}\big|_{[0,1)}$ and $\eta(m) = \int_0^1 \mathrm{e}^{2\pi \mathrm{i} m y} \, \mathrm{d}\mu(y)$

(Herglotz-Bochner)

Wiener's criterion: $\mu_{\mathsf{pp}} = 0 \iff \Sigma(N) = o(N)$

where $\Sigma(N) = \sum_{m=-N}^{N} \big(\eta(m)\big)^2$

Argument: $\Sigma(4N) \leq \frac{3}{2} \Sigma(2N)$ (by recursion for η)

$\Longrightarrow \mu = \mu_{\mathsf{cont}} = \mu_{\mathsf{sc}} + \mu_{\mathsf{ac}}$

Define: $F(x) = \mu\big([0,x]\big)$ for $x \in [0,1]$, where $F = F_{\mathsf{ac}} + F_{\mathsf{sc}}$

Diffraction: Absence of ac part

Functional relation:

$$\boxed{\begin{aligned} \mathrm{d}F\left(\tfrac{x}{2}\right) + \mathrm{d}F\left(\tfrac{x+1}{2}\right) &= \mathrm{d}F(x) \\ \mathrm{d}F\left(\tfrac{x}{2}\right) - \mathrm{d}F\left(\tfrac{x+1}{2}\right) &= -\cos(\pi x)\,\mathrm{d}F(x) \end{aligned}}$$

valid for F_{ac} and F_{sc} separately ($\mu_{\mathsf{ac}} \perp \mu_{\mathsf{sc}}$)

Define: $\eta_{\mathsf{ac}}(m) = \int_0^1 \mathrm{e}^{2\pi \mathrm{i} m x}\,\mathrm{d}F_{\mathsf{ac}}(x)$

$\curvearrowright$ same recursion as for $\eta(m)$, but $\eta_{\mathsf{ac}}(0)$ free

Riemann-Lebesgue lemma: $\lim_{m\to\pm\infty} \eta_{\mathsf{ac}}(m) = 0$

$$\Longrightarrow \quad \eta_{\mathsf{ac}}(0) = 0 \quad \Longrightarrow \quad \eta_{\mathsf{ac}}(m) \equiv 0 \quad \Longrightarrow \quad F_{\mathsf{ac}} = 0$$

(Fourier uniqueness thm)

Theorem: $\boxed{\mu = \mu_{\mathsf{sc}}}$ and $\widehat{\gamma}$ is purely sc.

Fourier series

Functional equation: $F(1-x)+F(x)=1$ on $[0,1]$ and

$$F(x)=\frac{1}{2}\int_0^{2x}\big(1-\cos(\pi y)\big)\,\mathrm{d}F(y) \quad \text{for } x\in[0,\tfrac{1}{2}]$$

$$\Longrightarrow \qquad \boxed{F(x)=x+\sum_{m\geq 1}\frac{\eta(m)}{m\pi}\,\sin(2\pi m x)}$$

- $F(x)-x$ continuous and of bounded variation
- Uniformly convergent Fourier series
- Unique solution (contraction argument)
- F strictly increasing $\Longrightarrow$ $\boxed{\mathrm{supp}(\mu)=[0,1]}$

Volterra iteration

Define: $F_0(x) = x$ and

$$F_{n+1}(x) = \frac{1}{2}\int_0^{2x} \big(1 - \cos(\pi y)\big)\, F_n'(y)\, \mathrm{d}(y)$$

for $n \geq 0$ and $x \in [0, \frac{1}{2}]$,

extension to $[0, 1]$ by symmetry

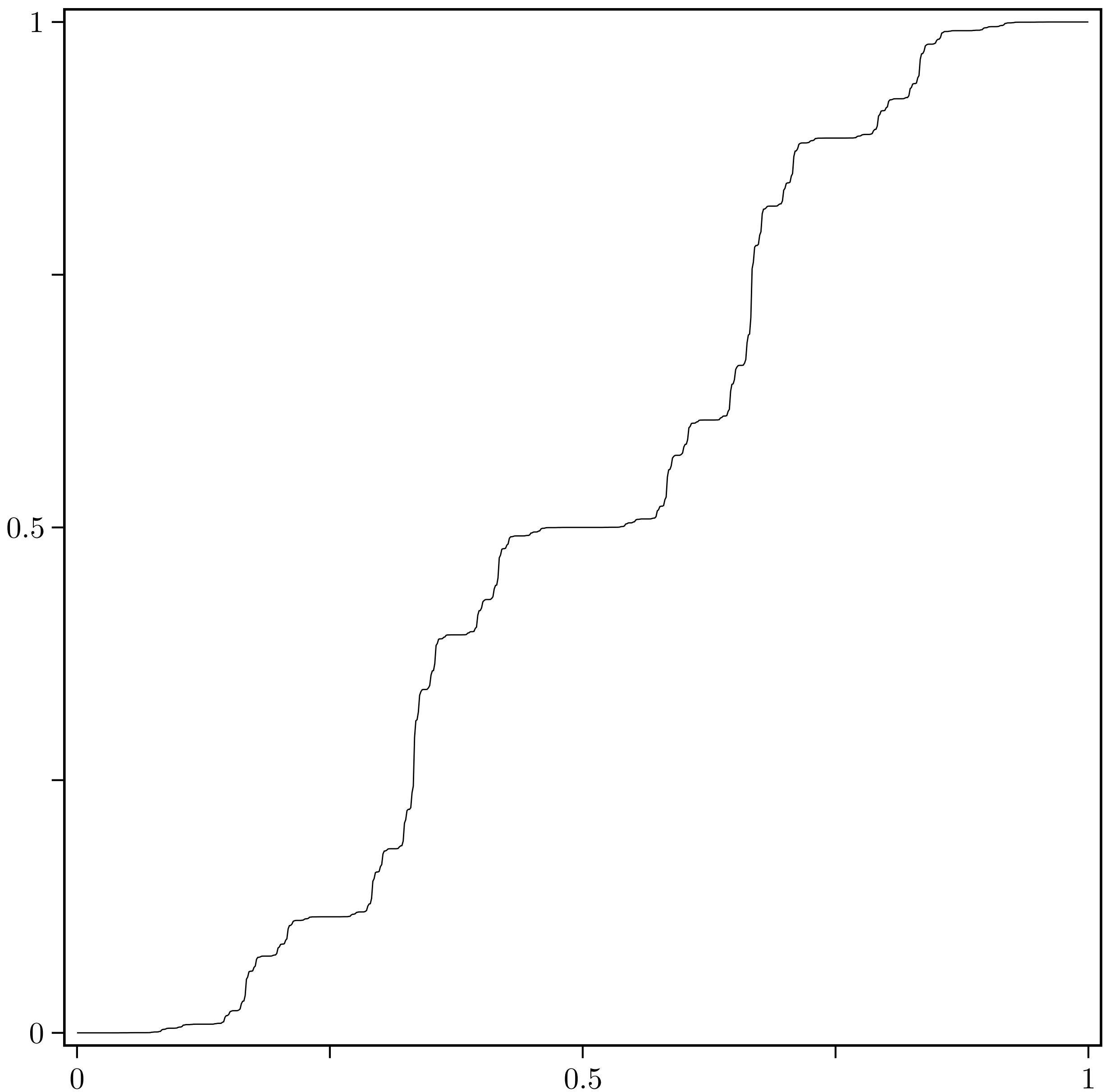
$$\Longrightarrow \; \mathrm{d}F_n(x) = g_n(x)\, \mathrm{d}x$$

$$\Longrightarrow \; g_n(x) = \prod_{k=0}^{n-1} \big(1 - \cos(2^{k+1}\pi x)\big)$$

Riesz product: sequence of ac measures,

vague convergence to $\mu = \mu_{\mathsf{sc}}$

TM measure



Generalised Morse sequences

Substitution: $\varrho: \begin{array}{l} 1 \mapsto 1^k\,\bar{1}^\ell \\ \bar{1} \mapsto \bar{1}^k\,1^\ell \end{array}$ (with $k, \ell \in \mathbb{N}$)

Fixed point: $v_0 = 1, \quad v_{m(k+\ell)+r} = \begin{cases} v_m, & \text{if } 0 \le r < k \\ \bar{v}_m, & \text{if } k \le r < k+\ell \end{cases}$

Coefficients: $\eta(0) = 1, \ \eta(1) = \frac{k+\ell-3}{k+\ell+1}$, and

$$\boxed{\eta\big((k+\ell)m + r\big) = \frac{1}{k+\ell}\big(\alpha_{k,\ell,r}\,\eta(m) + \alpha_{k,\ell,k+\ell-r}\,\eta(m+1)\big)}$$

with $m \in \mathbb{N}_0, \ 0 \le r \le k+\ell-1$, and

$\alpha_{k,\ell,r} = k + \ell - r - 2\min(k, \ell, r, k+\ell-r)$

Generalised Morse sequences, ctd

Fourier series: $$F(x) = \widehat{\gamma}\big([0,x]\big) = x + \sum_{m\geq 1} \frac{\eta(m)}{m\,\pi}\,\sin(2\pi m x)$$

(uniform convergence)

Riesz product: $$\prod_{n\geq 0} \vartheta\big((k+\ell)^n x\big)$$ with

$$\vartheta(x) = 1 + \frac{2}{k+\ell} \sum_{r=1}^{k+\ell-1} \alpha_{k,\ell,r}\,\cos(2\pi r x)$$

(vague convergence)

Period doubling sequences

Block map: $\psi : \quad 1\bar{1}, \bar{1}1 \mapsto a, \quad 11, \bar{1}\bar{1} \mapsto b$

$\curvearrowright$ gen. period doubling: $\varrho' : \begin{array}{l} a \mapsto b^{k-1} a b^{\ell-1} b \\ b \mapsto b^{k-1} a b^{\ell-1} a \end{array}$

$\uparrow$

coincidence

$\Longrightarrow$ model set

(Dekking)

$$\begin{tikzcd} \mathbb{X}_{\mathrm{TM}} \arrow[r, "\varrho"] \arrow[d, "\psi"'] & \mathbb{X}_{\mathrm{TM}} \arrow[d, "\psi"] & (2{:}1) \\ \mathbb{X}_{\mathrm{pd}} \arrow[r, "\varrho'"] \arrow[d, "\pi"'] & \mathbb{X}_{\mathrm{pd}} \arrow[d, "\pi"] & (\text{a.e. } 1{:}1) \\ \mathsf{Sol} \arrow[r, "\times 2", "\times (k+\ell)"'] & \mathsf{Sol} & \end{tikzcd}$$

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