3-dimensional Fractals & Tilings

Christoph Bandt

birs, Strobl, 7 July 2009

with Mathias Mesing & Mai The Duy

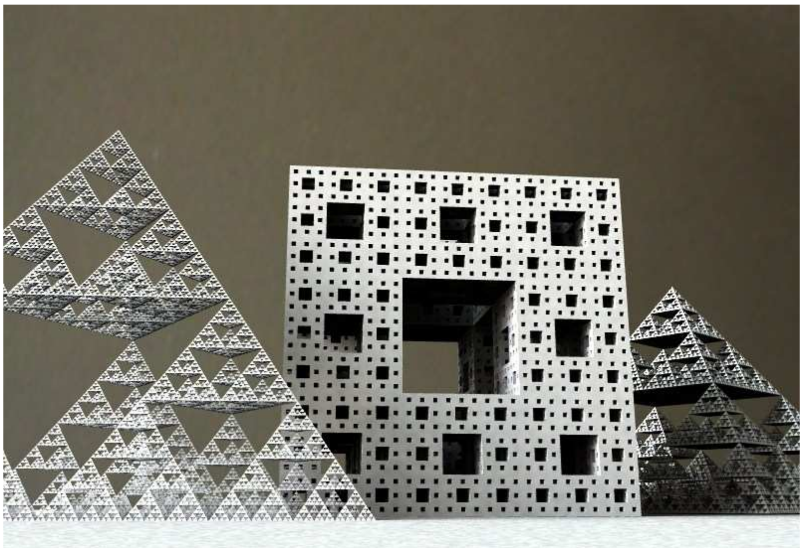
Basic object: Self-similar set

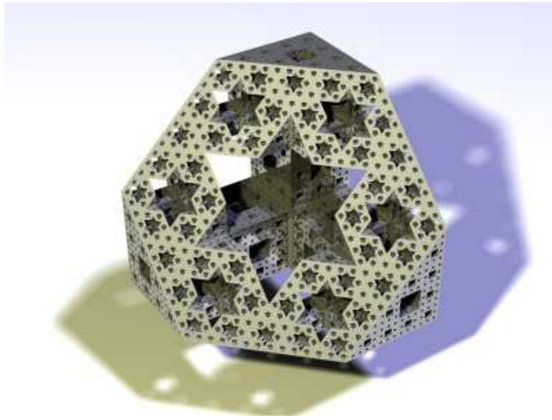
$$A = f_1(A) \cup \ldots \cup f_m(A)$$

f_i similarity maps or affine maps in $\mathbb{R}^3$

1. Difficulties and Examples in $\mathbb{R}^3$
2. The neighbor graph as a universal tool
3. The topological structure of the boundary of self-similar tiles
4. Are there 3D "twindragons" or "terdragons"? (Homeomorphic to a ball)

1. Why are 3D fractals rarely studied?





Known examples from Platonic solids

1.1 Mappings on $\mathbb{R}^3$ are not commutative

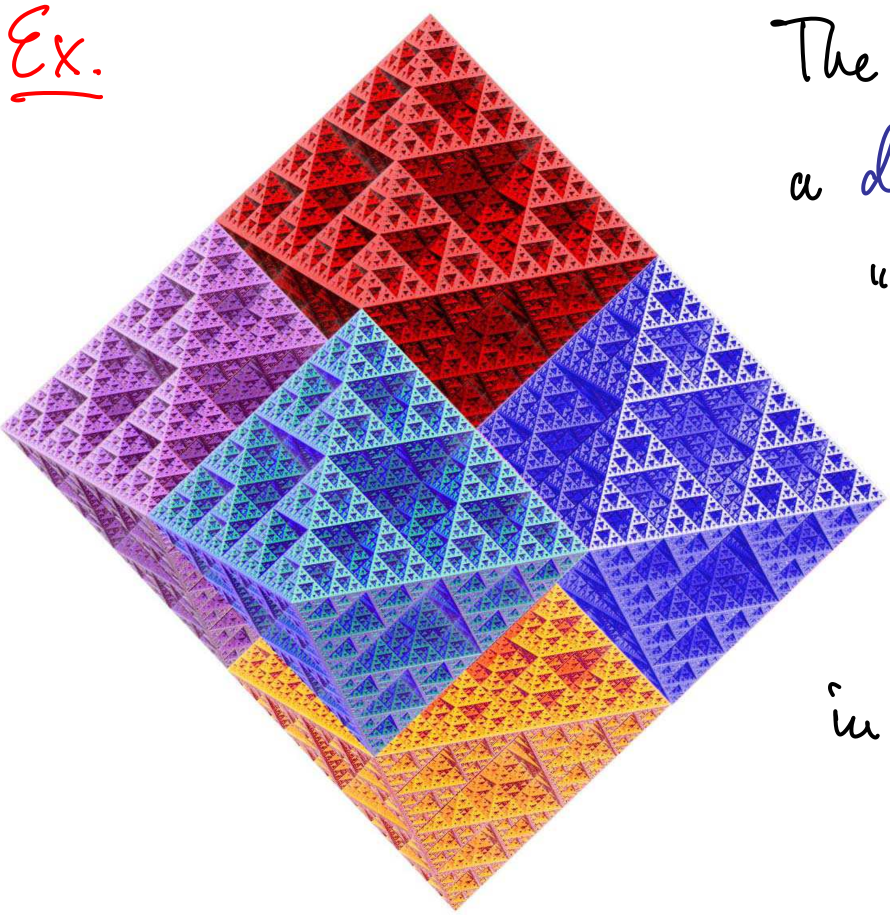
1.2 The topological structure of fractals in $\mathbb{R}^3$ can be complicate

Hui Rao + B '05: If $A \subseteq \mathbb{R}^2$ is connected and all $A_i \cap A_j$ are finite, OSC holds.

(Jordan curve argument)

Conjecture: this is not true in $\mathbb{R}^3$, even with one-point intersection and $m = 3$

Ex.

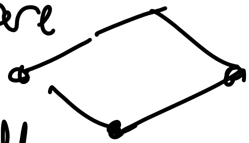


The fractal octahedron

a deflated balloon!

"Faces" are 3D Koch curves with many self-intersections

The square



in the middle is a subset of A.

1.3 It is not so easy to construct self-similar tiles in $\mathbb{R}^3$

Prop.: Let M be an integer 3×3-matrix with $\det M = m$, and let the three eigenvalues have equal modulus. Then either $m = k^3$ for some k, or M is conjugate to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix}.$$

1.4 Visualization of 3D fractals is a problem.

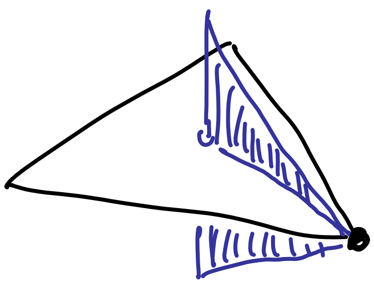
Free software <u>Chaoscope</u>

Nicolas Desprez

recommended, and used here

(needs 5 minutes to render a good picture)

1.5 Sierpiński gasket turned into $\mathbb{R}^3$



Fixed point •
90° rotation
Factor $2/3$

The three pieces intersect in a Cantor set.

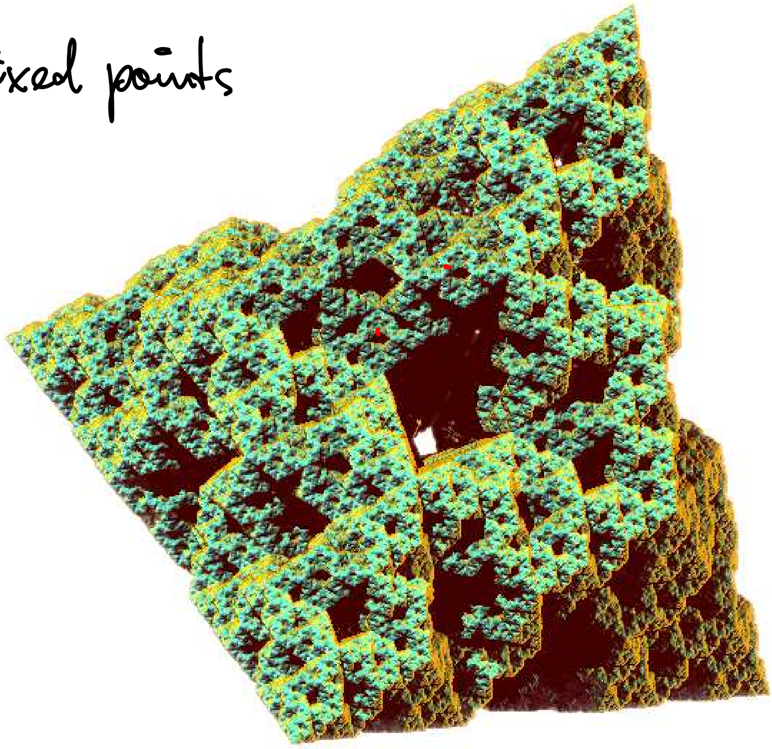
1.6 A modified fractal tetrahedron

4 maps

vertices are fixed points

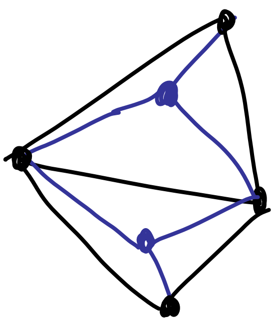
180° rotation

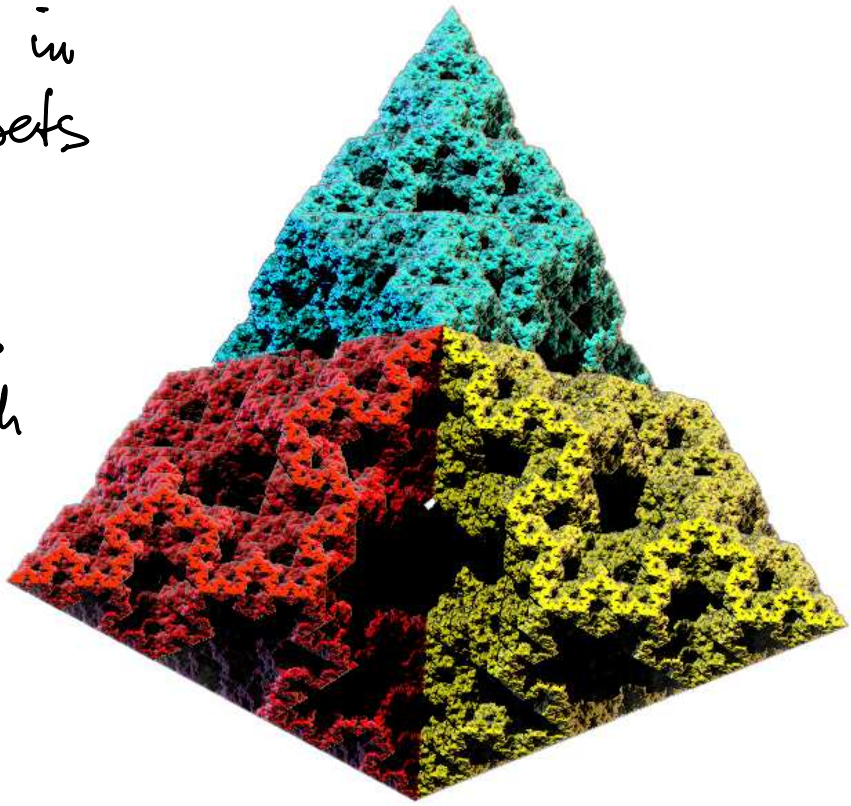
factor 3/5



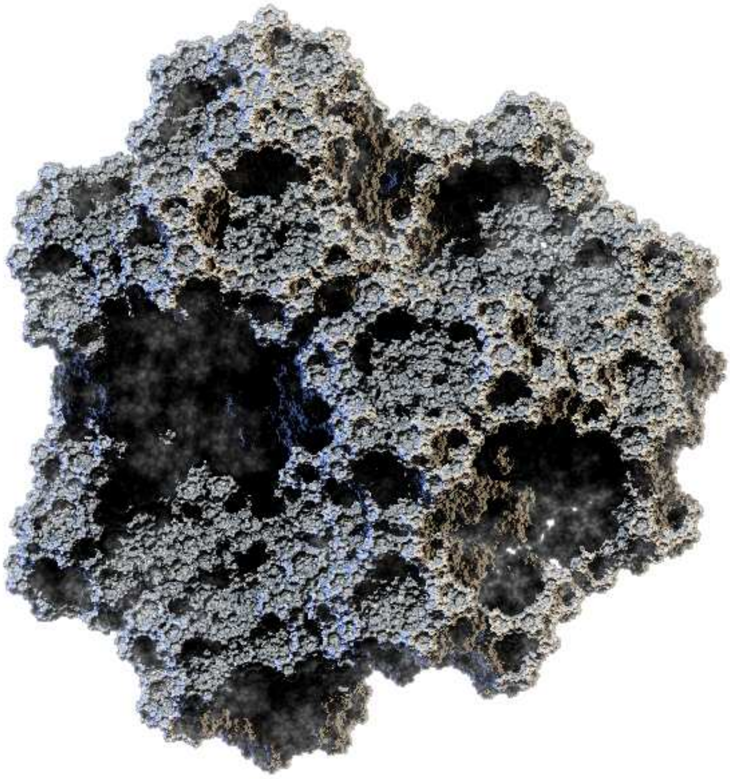
pieces intersect in Cantor sets

Open set for OSC
tetrahedron plus
pyramids on each
face





1.7 Reverse construction



If the modified tetrahedron was generated by f_1, f_2, f_3, f_4, this fractal is generated by $-f_1, -f_2, -f_3, -f_4$. OSC?

2. The neighbor graph as a universal tool

Many authors must be quoted. Among others:

Gilbert (boundary of twindragon), B+Graf,
Lau, Ngai, Rao, Deng: finite type fractals
Gröchenig and Haas, Keesling and Vince,
Kenyon, Strichartz and Wang,
Thuswaldner and Scheicher (boundaries,
Akiyama, Steiner and Rao (topology of tiles)
Solomyak: Spectrum of chair tiling

Try to unify different approaches.

2.1 Def. Neighbor graph of a self-similar fractal

$$A = f_1(A) \cup \dots \cup f_m(A)$$

(assume equal factors $r_1 = \dots = r_m = r$)

$$G = (V, E)$$

Vertices are

- neighboring positions of pieces A_u, A_w $\quad u = u_1 \dots u_n,\ w = w_1 \dots w_n$

or

- isometries $f_u^{-1} f_w$

or

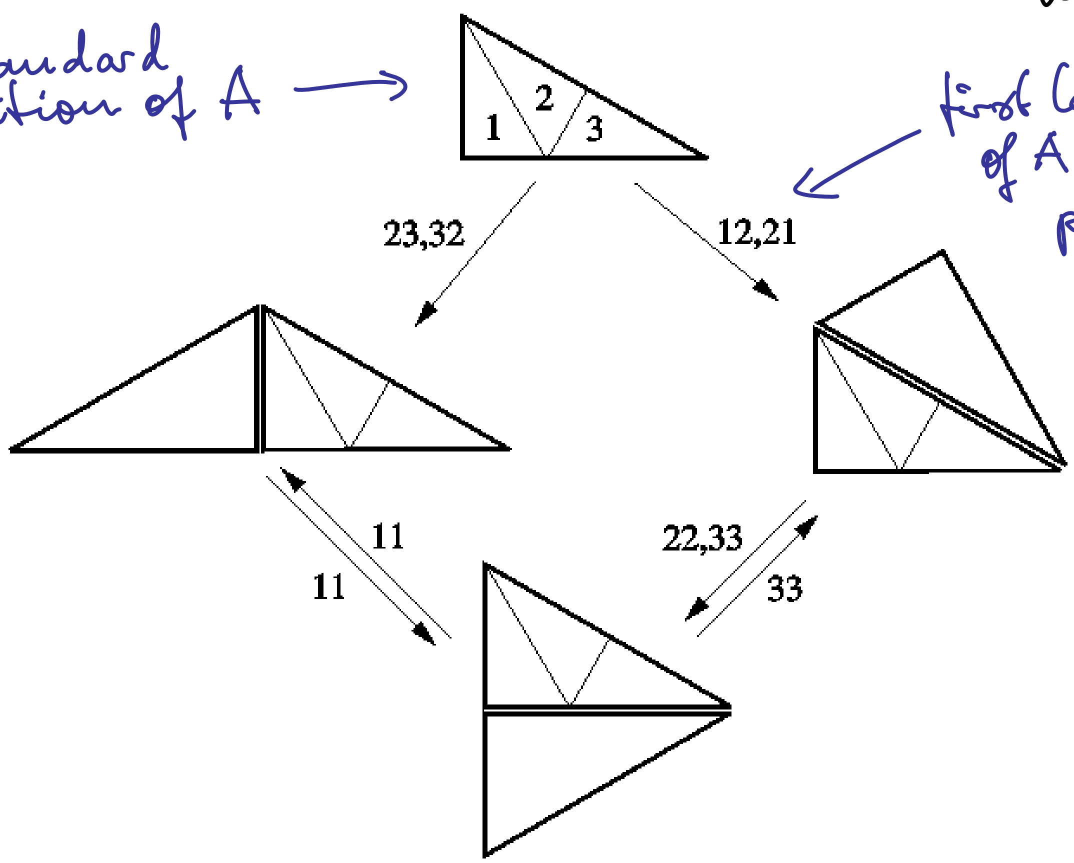
- boundary faces of A.

Edges run from pieces to subpieces with labels of the subpieces

Construction: consider subpieces of intersection sets $A_i \cap A_j$

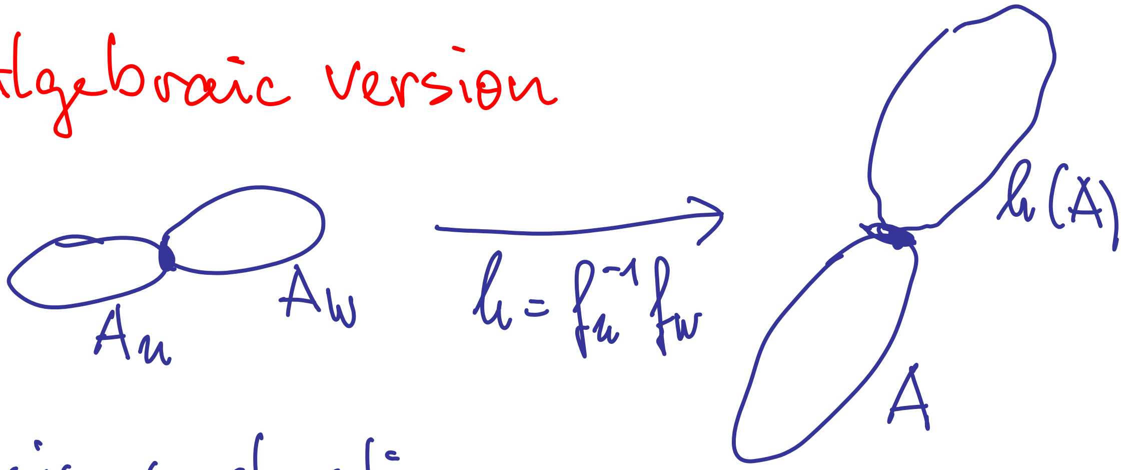
2.2 Example (pieces intersecting in a point were neglected)

standard position of A →



first label: piece of A, second: piece of neighbor

2.3 Algebraic version



Recursive construction of neighbor maps:

Start with $f_i^{-1} f_j$, $i,j \in \{1 \ldots m\}$, $i \neq j$

Given nb map h, construct $h_{ij} = f_i^{-1} h f_j$ and draw edges $h \xrightarrow[ij]{} h_{ij}$

2.4 Proper neighbor maps

We need only neighbors with $A_u \cap A_w \neq \emptyset$.

Let us assume $0 \in A$ (this is true if f_1 is linear map)

Then we cancel all maps $h(x) = Mx + b$ with $\|b\| > \operatorname{diam} A$ since A and $h(A)$ are disjoint. In the remaining graph we take only those vertices which lie on infinite directed paths. In other words, we delete dead ends in the graph.

Prop. The resulting graph describes the proper neighbors.

2.5 Finite type

If the resulting graph is finite, the fractal A is called finite type.

Almost all known examples belong to this class.

Infinite paths in G are obtained by cycles which provide equations for the mappings.

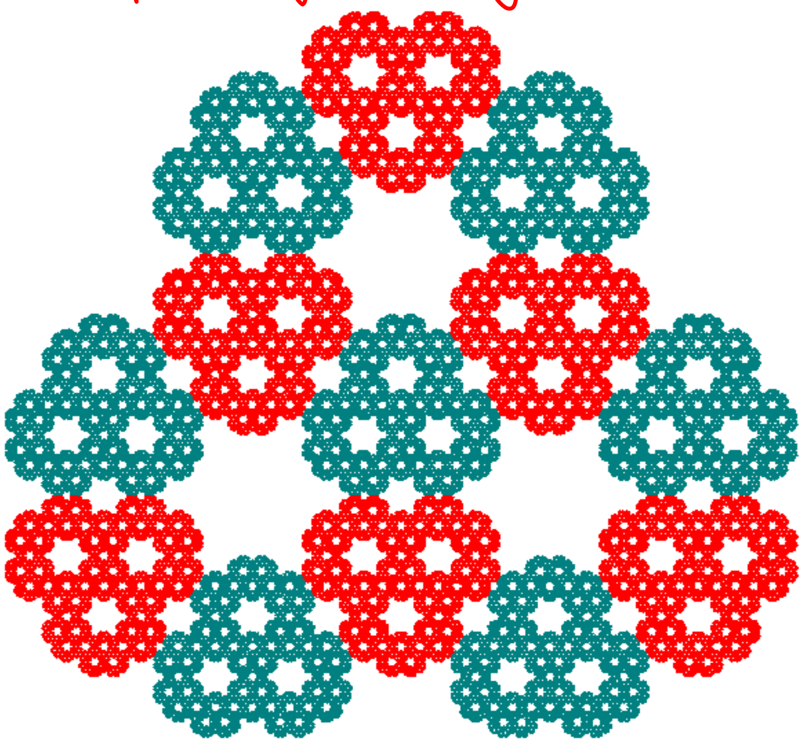
Ex: $f_i(x) = Mx + b_i \quad i = 1, \dots, m$

The nb. maps are translations $f_u^{-1} f_w(x) = x + c$

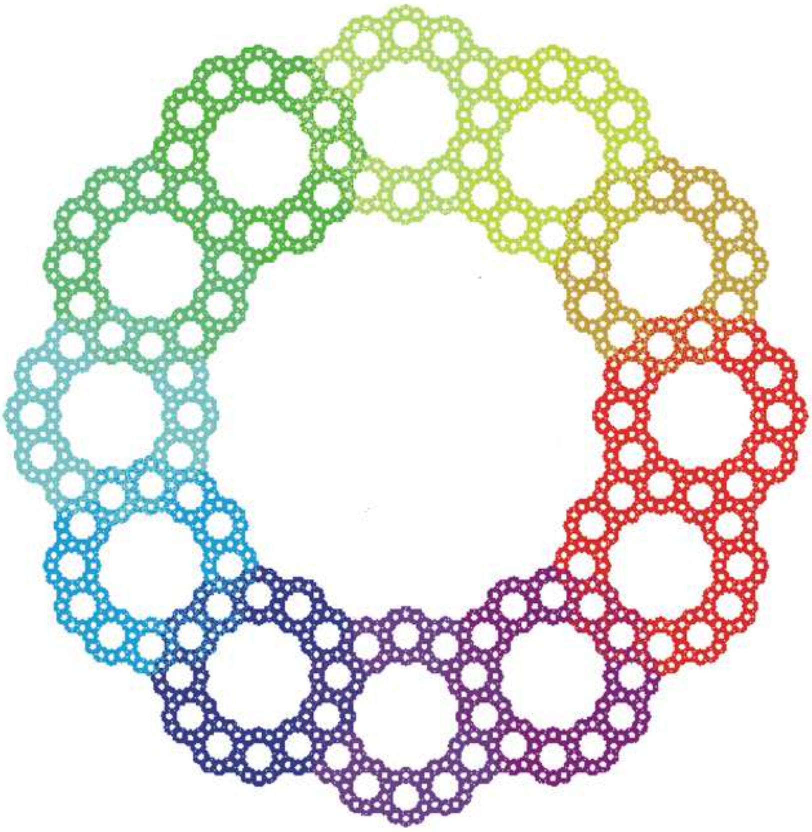
where $c = M^{n-1}(b_{w_1} - b_{u_1}) + \dots + M(b_{w_{n-1}} - b_{u_{n-1}}) + b_{w_n} - b_{u_n}$

For a cycle, two such vectors must agree.

Ex. Type 1, up to symmetry



Ex. Type 1, up to symmetry, no OSC (exact overlap)



2.6 OSC

holds for a finite type fractal iff there is no incoming edge to the root vertex (id).

Prop. (Reverse fractal)

The fractals generated by $f_1, \dots, f_m$ and by $-f_1, -f_2, \dots -f_m$ have exactly the same neighbor maps. Thus if one fulfils OSC, so does the other.

2.7 Topological structure

Consider the language $\tilde{L}$ of sequences of double labels $(s_1t_1, s_2t_2, \ldots)$ on infinite paths of directed edges in G starting in id.

This language defines the topology of a finite type fractal A by the address map

$$\pi : \{1, \ldots m\}^{\mathbb{N}} \longrightarrow A$$

Namely $\pi(s) = \pi(t)$ iff $st \in \tilde{L}$.

Cor.: Isomorphic nb. graphs $\Rightarrow$ homeomorphic fractals

2.8 Graph-directed boundary sets

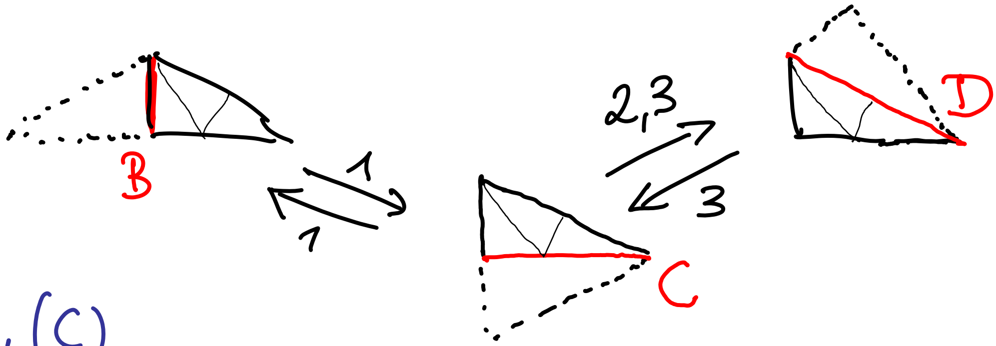
In the sequel, we consider only the first label on each edge.

We assume finite type and OSC, and we cancel id and its edges from G, obtaining G_1.

Prop. Each vertex of G_1 represents a boundary face B_i. These boundary faces form a graph-directed construction (Mauldin & Williams 1986) directed by the graph G_1.

This implies Hausdorff dimension, positive and finite / infinite Hausdorff measure.

Ex.



$B = f_1(C)$

$C = f_1(B) \cup f_3(D)$

$D = f_2(C) \cup f_3(C)$

Cor. The addresses of a boundary set B are given by the infinite paths of G_1 which start in the vertex of B.

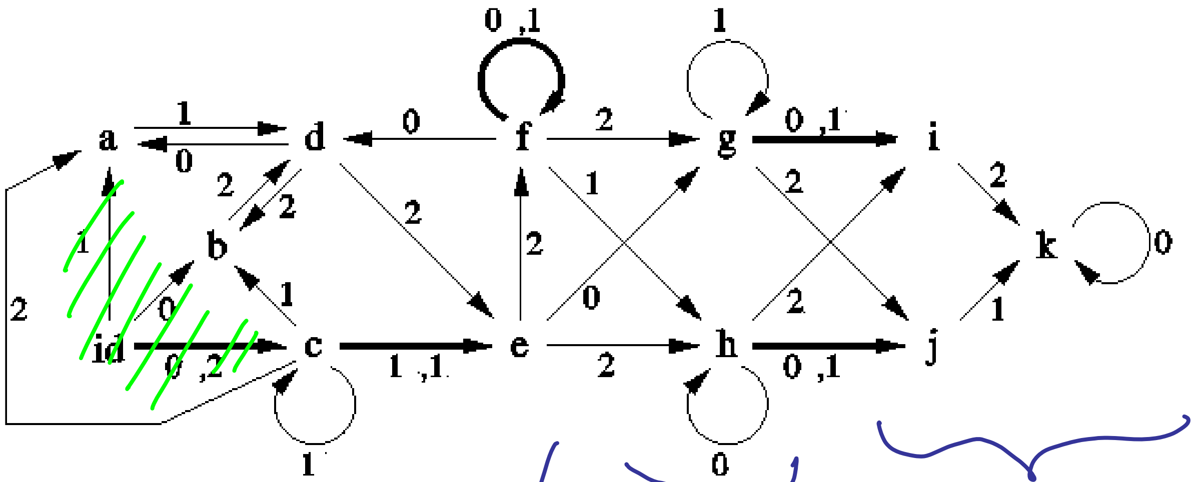
$\rightarrow$ Language L_B.

3.1 Topological structure of boundary sets

Rem. G_1 allows to decide whether a boundary set is really a "face", or an "edge", or a "vertex", or a more complicated set.

(For a tile, the cycles of G_1, run in reverse direction, also determine all exceptional tilings.)

Ex.



uncountable boundary sets

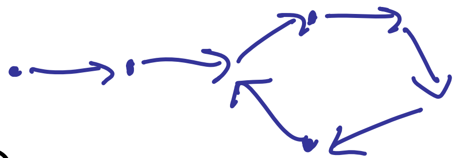
countably infinite boundary sets

singleton boundary sets

Existence of countable infinite boundary set shows that the tile is <u>not</u> disk-like.

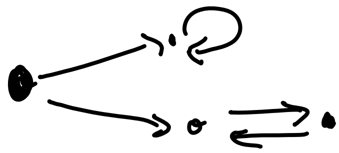
3.2 Cycle structure

a) terminal cycle

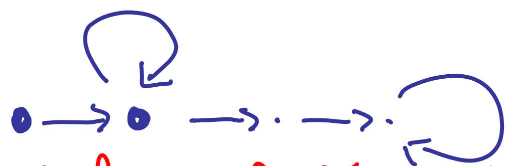


singleton

2 points



b) intermediate cycle



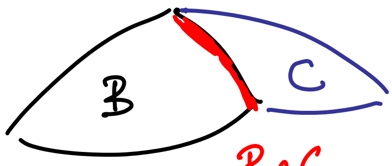
countable infinite set

c) two or more connected cycles

uncountable

3.3 Intersections of boundary sets

Like for polyhedra:



Check whether $L_{B\cap C} = L_B \cap L_C \neq \emptyset$

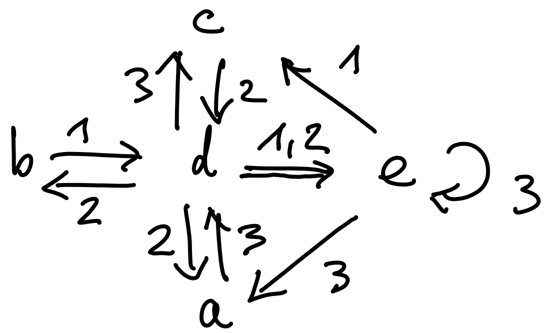
Define graph G_2 of boundary intersections.

$V_2 = \{\{v,v'\} \mid v, v' \in V_1\}$

Edge $(\{v,v'\},\{w,w'\})$ labelled i if $v \xrightarrow{i} w$ and $v' \xrightarrow{i} w'$ in G_1.

Similarly, you can define G_3, G_4 etc.

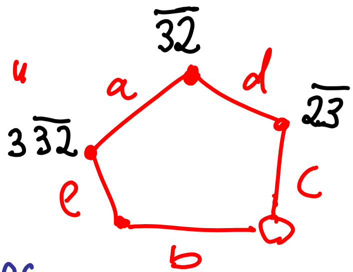
Ex. G_1



G_2 $\quad \{a,e\} \xrightarrow{3} \{a,d\} \underset{2}{\overset{3}{\rightleftarrows}} \{c,d\} \xleftarrow{1} \{b,e\}$

plus isolated vertices

disk-like tile in $\mathbb{R}^2$, "pentagon"



∘ last point from identification of addresses

4.1 A self-affine tile in $\mathbb{R}^3$

$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

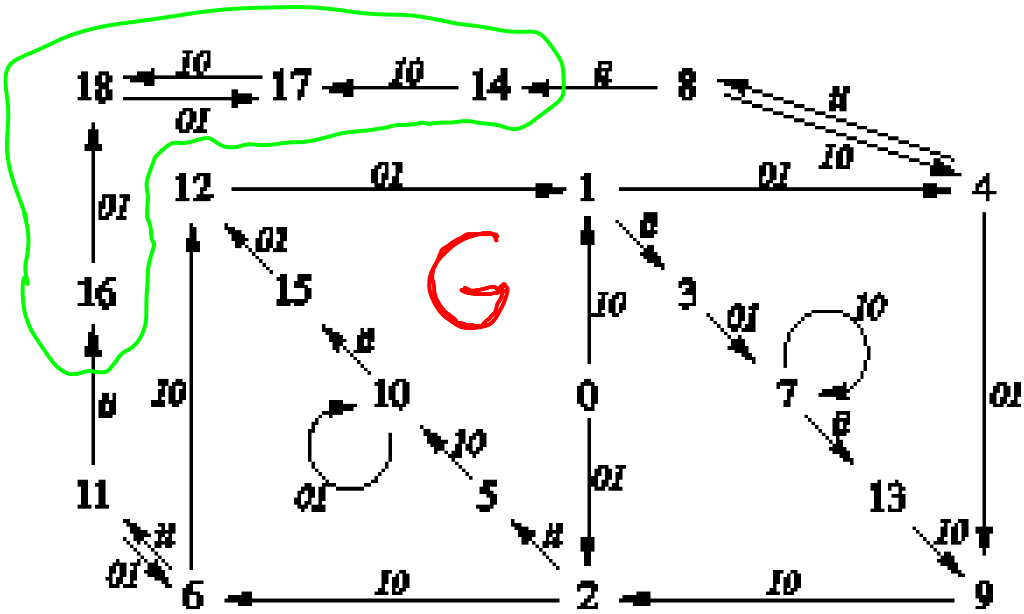
$$f_0(x) = M^{-1}x$$

$$f_1(x) = M^{-1}(x+v)$$

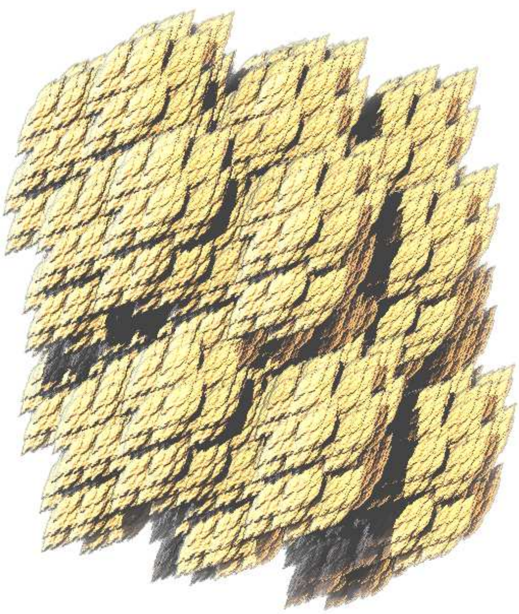
18 boundary sets

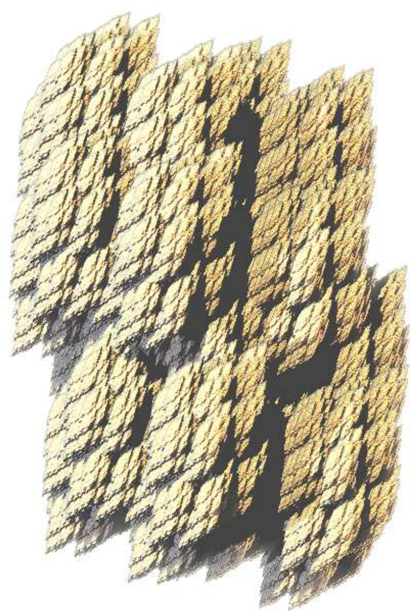
4 of them are points

the others are "faces"



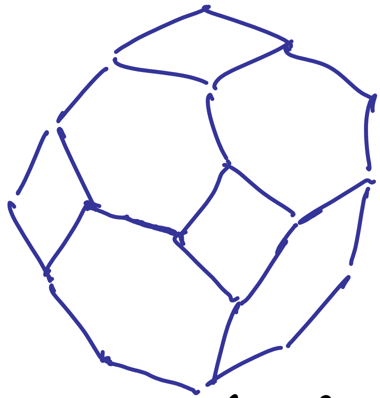
Two views





G_2 shows:

6 of the 14 faces
are 4-gons,
8 are hexagons.



Homeomorphic to truncated octahedron?

Unfortunately, no.

Two boundary faces meet in a countable set!

4.2 A self-similar tile in $\mathbb{R}^3$

With 2 tiles – no fractal example, only cube

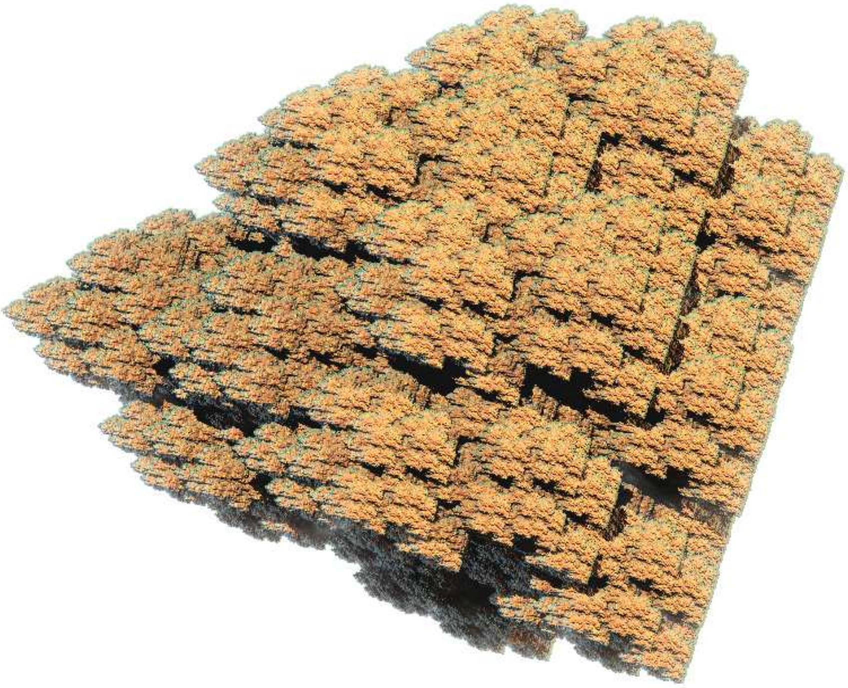
$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

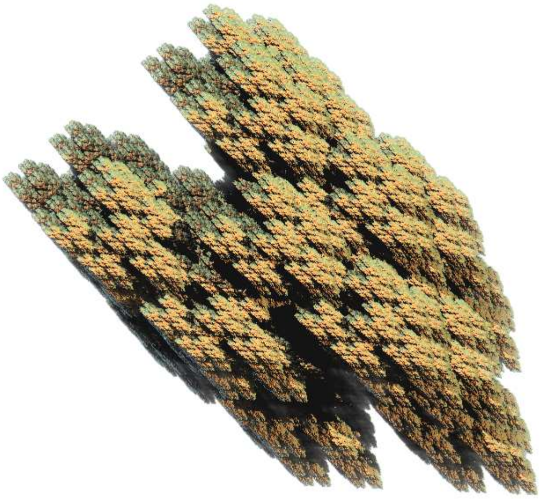
$$f_i(x) = M^{-1}(x + v_i)$$

G_1: 16 boundary sets, 2 of them are points

G_2: interesting polyhedral structure
could be homeomorphic to a ball

?

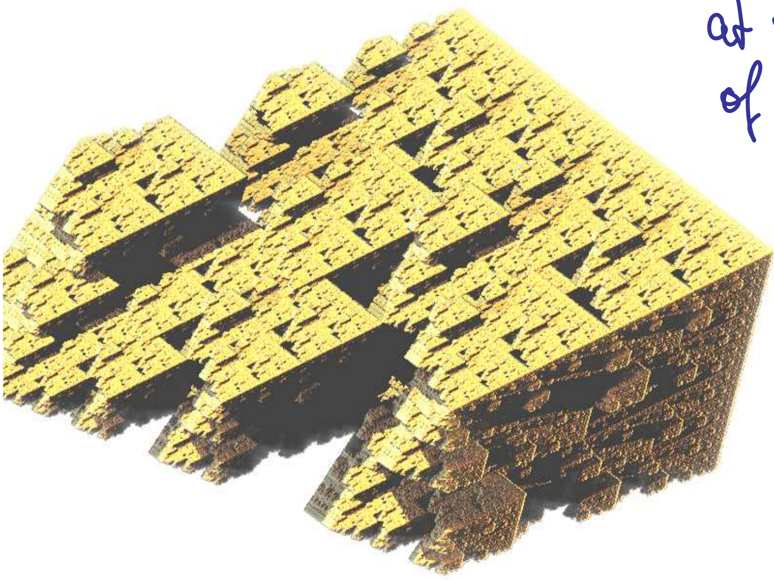




4.3 A non-standard fractal "tile" with 2 pieces

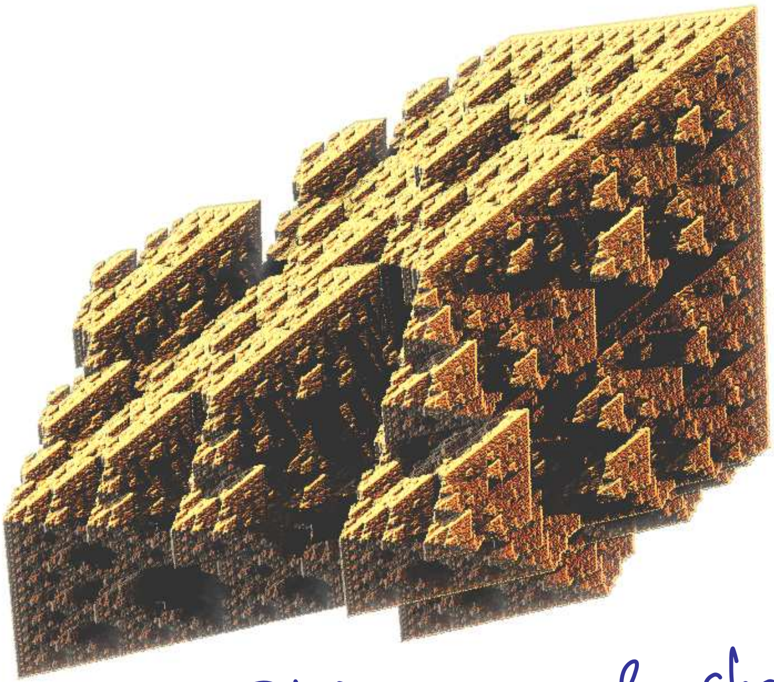
f_1 as above

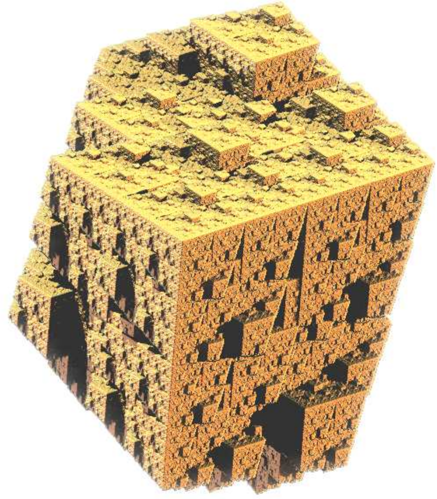
$f_2(x) = M s(x)$, where s is the reflection at the invariant plane of M.



Certainly not a ball !

$>$ 100 000 neighbor maps $\rightarrow$ probably not OSC !





This example shows the limitations of visualization.

References: The pictures of Sierpinski 3D carpet can be found at V. Alekseev's page www.mathpaint.blogspot.com among other pictures from 2008. The hexagonal cut was designed by Seb Przd.

Chaoscope is a French software project run by Nicolas Desprez: www.chaoscope.org

C. Bandt + M. Mesing, Self-affine fractals of finite type, Banach Center Publications 84, 131–148 (2009)

C. Bandt, M.T. Duy and M. Mesing, 3D fractals, submitted to Math. Intelligencer

C. Bandt and N.V. Hung, Fractal n-gons and their Mandelbrot sets, Nonlinearity 21, 2653–2670 (2008)