Fundamental groups of wild spaces

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Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.
–Henri Poincaré

Homotopy

A key notion in this talk will be the notion of *homotopy*. We say that two continuous functions $f, g : X \to Y$ are *homotopic* if there is a *homotopy* between them, which is a continuous map $H : X \times [0, 1] \to Y$ with the property that $H(*, 0) = f$ and $H(*, 1) = g$. We see that the property of being homotopic maps with the same domain and codomain is an equivalence relation.

Nulhomotopy and homotopy equivalence

A continuous map $f : X \to Y$ is *nulhomotopic* if it is homotopic to a constant map. Two spaces X, Y are *homotopy equivalent* if there exist continuous maps $f : X \to Y$, $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to the identity map on Y and X respectively. A space is said to be *contractible* if it is equivalent to a single point.

Definition

The *fundamental group* of the space X based at x_0, denoted $\pi_1(X, x_0)$, is the set of homotopy classes of continuous maps from the unit interval $I = [0, 1]$ into X with the provisions

- All maps send both 0 and 1 to x_0
- Two maps are considered equivalent if they are homotopic by a homotopy that always sends $\{0, 1\}$ to x_0.

Clearly the constant map (x_0) is an identity for this group and if $f : I \to X$ then $[f(t)]^{-1} = [f(1 - t)]$.

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We say a space is *simply connected* if it is path connected and has a trivial fundamental group.

A few standard (locally) simple examples: A point, a circle, an n-sphere $(n > 1)$, a torus, a bouquet of two circles.

- The fundamental group of a graph is an example of a *free group*.

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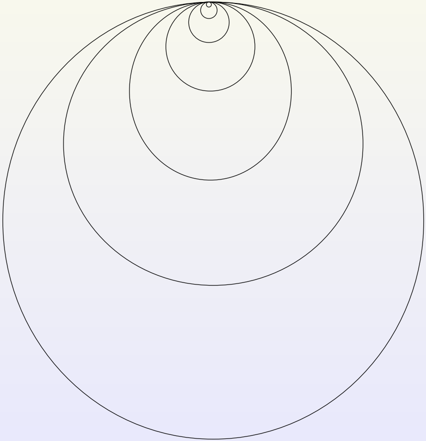
- One way to represent a free group is by *words* in a free basis. In this setting we think of the free basis as giving an *alphabet* in which to write words. A *word* is just a string consisting of elements of the alphabet and their formal inverses. A free group is isomorphic the group of equivalence classes of words on a free basis with operation being *string concatenation* and the the equivalence relation being generated by deleting or inserting occurrences of aa^{-1} or $a^{-1}a$ for a in the alphabet.

The art of doing mathematics consists in finding that special case which contains all the germs of generality.
–David Hilbert



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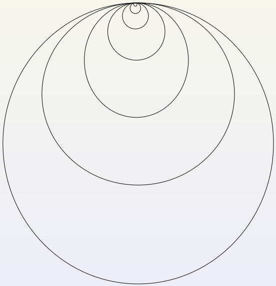

There have been numerous papers discussing this group since then. It has several interesting properties. It's uncountable, it's not a free group, it can be represented as a group of *transfinite words* similar to the words in a free group.

The combinatorial structure of the Hawaiian earring

Definition

The fundamental group of the Hawaiian earring, called the *Hawaiian earring group*, has, by a theorem of Jim Cannon and myself, an algebraic structure as a group of words, similar to that of a free group. The difference is that the word structure Hawaiian earring group is built on the notion of *countable words*.

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- with the provision that each symbol can appear only finitely many times in any word.

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We call this word a *Cantor word* since the path it represents goes through the basepoint on the Cantor set.

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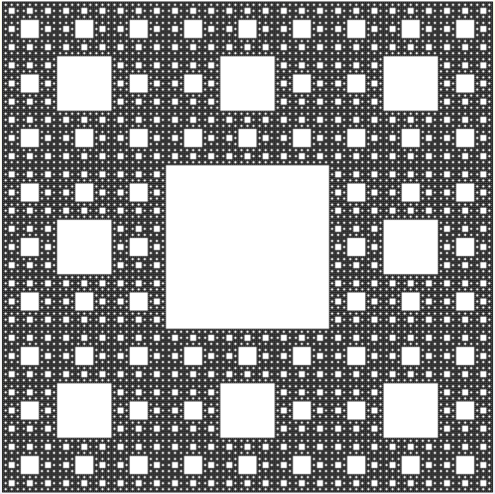
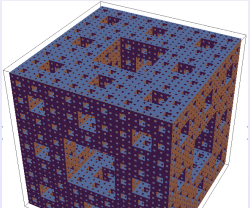
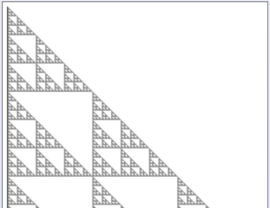
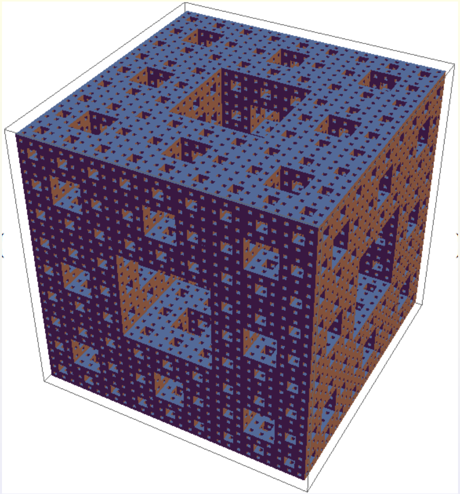
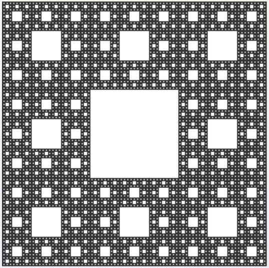
- The homomorphism is not induced by a continuous map.
- Gives a subgroup of the Hawaiian earring group of index two for which there is no covering space.

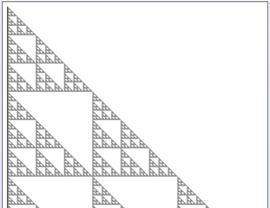


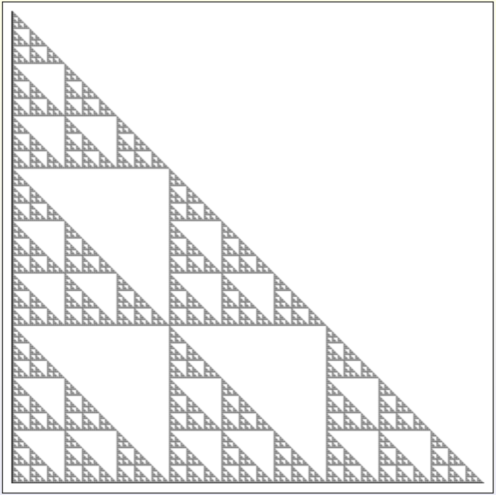


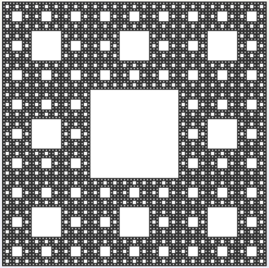


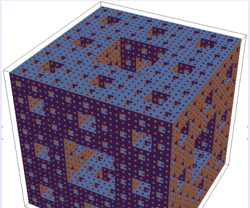


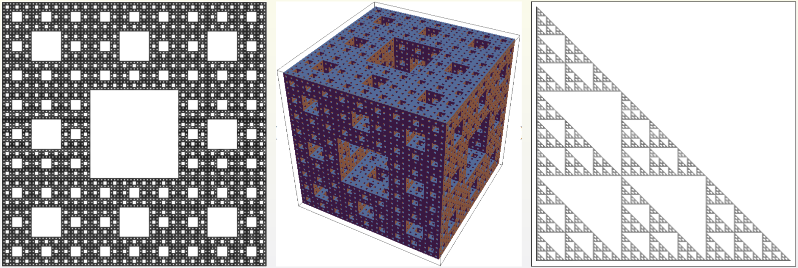




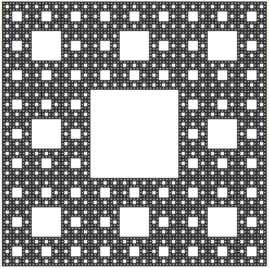
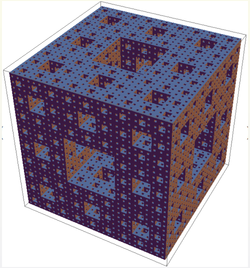
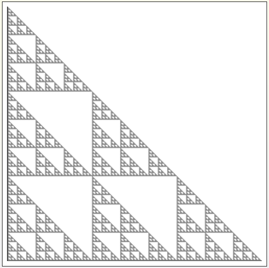








How could one tell the fundamental groups of these spaces apart?

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A theorem that I proved with Katsuya Eda shows that you can reconstruct these spaces as equivalence classes of Hawaiian earring subgroups of their fundamental groups.

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Suppose X is a one-dimensional locally connected compact metric space.

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- Give $\mathcal{H}/\sim$ the topology induced by $H_i \to H$ if $\forall i\; H_i \cap H \neq \{e\}$.

Bad set is an algebraic invariant in one-dimension

Result from 2005:

Theorem (C-Eda)

If X is a one-dimensional locally connected compact metric space then $\mathcal{H}(X)/\sim$, is isomorphic to the subspace $B(X)$ of X consisting of points which don't have simply connected neighborhoods, and thus $B(X)$ is an invariant of the fundamental group of X.

Homomorphisms into planar sets are continuous

New Result

Theorem (Curt Kent's Master's Thesis 2008)

Every homomorphism from the fundamental group of the Hawaiian earring group into the fundamental group of a connected compact subset of the Euclidean plane is induced by a continuous map.

Open Question: Is every homomorphism between fundamental groups of one-dimensional or planar Peano continua conjugate to one which is induced by a continuous function?

Bad set is an algebraic invariant of planar sets

Using Curt Kent's 2008 masters thesis, a construction from a 2007 paper of Cannon-C , and a new construction of Eda we obtain

New Result

Theorem (C-Kent)

If X is a connected, locally connected compact subset of the Euclidean plane then $\mathcal{H}(X)/\sim$, is isomorphic to the subspace $B(X)$ of X consisting of points which don't have simply connected neighborhoods, and thus $B(X)$ is an invariant of the fundamental group of X.

A beautiful theorem of Shelah, and a new tool

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Recently, Sam Corson and I have proven the following generalization of Shelah's theorem:

Theorem (New Result: C-Corson)

The n-th homotopy group, π_n, of an $(n-1)$-connected, compact, metric space is either **finitely presented** *or uncountable.*

A local version of Shelah's theorem

Definition

If X is a topological space, $x_0 \in X$, and $U \supseteq V$ then $\pi_1^U(V, x_0)$ denotes the natural image of $\pi_1(V, x_0)$ in $\pi(U, x_0)$. Also, $\pi_1(X, x_0)$ is *locally trivial* (also known as *semilocally simply-connected*) if for every U there exists a V such that $\pi_1^U(V, x_0)$ is trivial and is *locally countable* if for every U there exists a V such that $\pi_1^U(V, x_0)$ is countable.

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The following is a restatement of Shelah's theorem in this new notation

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Theorem (New Result: C-Eda '09)

If the fundamental group, π_1, of a first countable topological space is locally countable then it is locally trivial.

Terminology

Definition

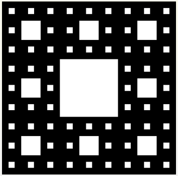
If x is an element of the path connected space X, we say that x is *bad* if every self-homotopy of X fixes x. $B(X)$ will denote the collection of bad points in X. Clearly $B(X)$ is a closed *homotopy invariant* subset of X. We say that X is *wild* if $B(X) = X$.

If X is a one-dimensional or planar Peano continuum, we will use the following equivalent formulations:

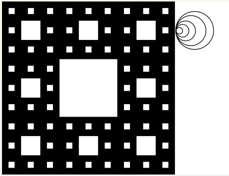
- $x \in B(X)$ if every neighborhood of x contains a curve which cannot be freely homotoped out of that neighborhood.
- $x \in B(X)$ if every neighborhood of x contains a curve which is essential (non-nullhomotopic).

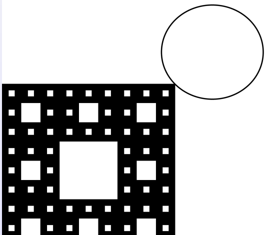
Exercise

Show that no two are homotopy equivalent:

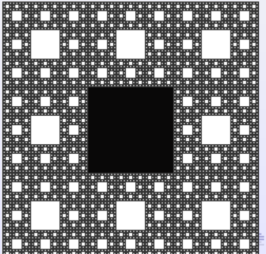


The Sierpinski curve





Zastrow's space



An Artinian property of fundamental groups

Theorem (Cannon-C)

Let X be a topological space, let $f : \pi(X, x_0) \longrightarrow L$ be a homomorphism to the group L, $U_1 \supseteq U_2 \supseteq \cdots$ be a countable local basis for X at x_0, and G_i be the image of the natural map of $\pi(U_i, x_0)$ into $\pi(X, x_0)$. Then

- *If L is countable then the sequence $f(G_1) \supseteq f(G_2) \supseteq \cdots$ is eventually constant.*
- *If L is Abelian with no infinitely divisible elements then $\bigcap_{i \in \mathbb{N}} f(G_i) = \{0_L\}$.*
- *If X is a one-dimensional Peano continuum and L is a "nice" group then $\mathrm{im}(f)$ is finitely generated and $f(G_i) = \{0_L\}$ for some $i \in \mathbb{N}$.*

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What does "nice" mean?

A Useful Invariant

Definition

If X is Peano continuum which is either planar or one-dimensional, we define $Q(X)$ to be the set of points of X for which every neighborhood contains a simple closed curve that cannnot be freely homotoped into $B(X)$. Clearly $Q(X) \subseteq \partial B(X)$. If X is wild, $Q(X)$ must be empty.

It is not difficult to show that $Q(X)$ is a homotopy invariant of X. You should now be able to solve some of the previous exercise.

Exercise

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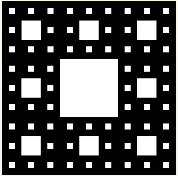

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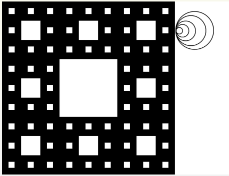

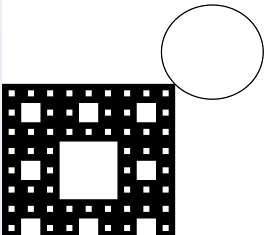

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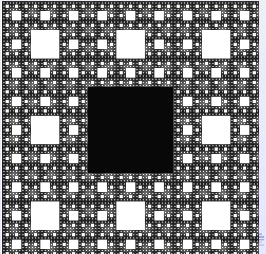
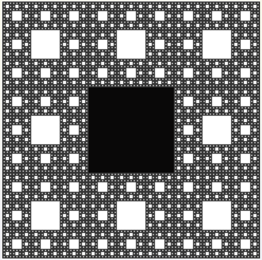


Another Exercise



Exercise: Show that Zastrow's example above cannot be homotopy equivalent to any one-dimensional Peano continuum. Hint: What would happen to the boundary of the disk under a homotopy equivalence to a one dimensional space?

Open Question

If the fundamental group of a planar set is isomorpic to the fundamental group of a one-dimensional space, must the planar set be homotopic to a one-dimensional subset of the plane?

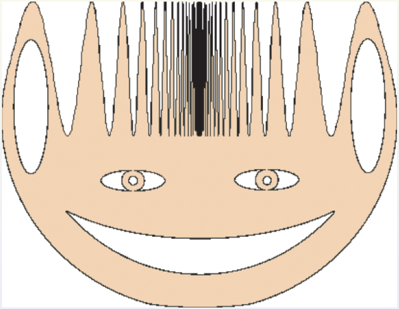
Homotopy dimension

Theorem (Cannon-C)

A planar Peano continuum,M,is homotopy equivalent to a one-dimensional Peano continuum if and only if the following two conditions are satisfied:

- *No component of $M - B(M)$ is equal to component of $\mathbb{R}^2 - B(M)$. In other words, every component of $M - B(M)$ is "missing a point of $\mathbb{R}^2$".*
- *For every closed disk D in the plane, the set of components of $D \cap (M - B(M))$ which are components of $D - B(M)$ form a null set. In other words, the collection of components of of $M - B(M)$ which do not "miss a point from D" forms a null sequence.*

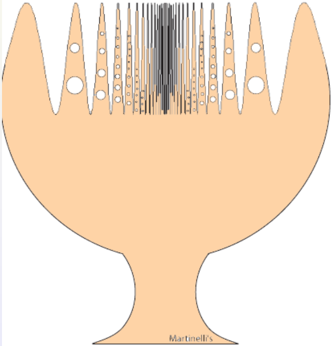
Example



Schematic: Can't be homotoped to be 1-dim.

A nicer example

This schematic represents a space which is homotopically one–dimensional.



Related topics

Theorem (Cannon-C)

The fundamental group of any planar Peano continuum embeds in that of a one-dimensional planar Peano continuum, and thus into that of the Sierpinski curve.

Conjecture A planar Peano continuum has a fundamental group which embeds in the Hawaiian earring group if and only if its bad set contains no simple closed curves.