Connectivity via Jordan Forms

Eva Curry and Avra Laarakker

Acadia University
Wolfville, Nova Scotia, Canada

Workshop on Fractals and Tilings
Strobl, Austria
July 6, 2009

Definitions

- A *dilation matrix* on a lattice $\Gamma \subset \mathbb{R}^m$ is an $m \times m$ matrix $A : \Gamma \to \Gamma$ with all eigenvalues λ of A satisfying $|\lambda| > 1$. For us, $\Gamma = \mathbb{Z}^m$, $A \in M_m(\mathbb{Z})$.
- A *digit set* for A is a complete set of coset representatives of $\Gamma / A(\Gamma)$. We use a centered canonical digit set $D = A(F) \cap \mathbb{Z}^m$, where $F = \left(-\frac{1}{2}, \frac{1}{2}\right]^m$ is a fundamental domain for $\Gamma = \mathbb{Z}^m$ centered at the origin. Eg. $A = 4$, $D = \{-1, 0, 1, 2\}$.
- $T(A, D) = \left\{ x \in \mathbb{R}^m : \; x = \sum_{j=1}^{\infty} A^{-j} d_j, \; d_j \in D \right\}$

Basic Facts

- $T(A, D)$ is the attractor of the IFS $\{f_d(x) = A^{-1}(x + d) : \ d \in D\}$.
- $T(A, D)$ is *self-affine*: $T(A, D) = \bigcup_{d \in d} A^{-1}\left(T(A, D) + d\right)$.
- (Curry) When A yields a *radix representation* for Γ with digit set D, $T(A, D)$ is a set of “fractions”, congruent to the fundamental domain of Γ.
- (Gröchenig and Haas) $T(A, D)$ tiles $\mathbb{R}^m$ under translation by Γ or some sub-lattice of Γ.

Goal

Conjecture

For every dilation matrix $A \in M_m(\mathbb{Z})$, there exists a digit set D for which $T(A, D)$ is a connected set in $\mathbb{R}^m$.

Approximates

- Set
$$T_0 = F = \left(-\frac{1}{2}, \frac{1}{2}\right]^m, \quad T_n = \bigcup_{d \in D} A^{-1}(T_{n-1} + d).$$
Then $\lim_{n \to \infty} T_n = T(A, D)$ in the Hausdorff metric.
- Define the *level sets* of the digit set D by
$$D_1 = D, \quad D_n = \left\{ k \in \Gamma : \; k = \sum_{j=0}^{n-1} A^j d_j, \; d_j \in D \right\}$$
(as in Gröchenig and Haas).

Multiplying T_n by A^n: $\quad A^n T_n = F + D_n$.

Connectedness and Lattice Connectedness

Lemma

The set T_n is connected if and only if the level set D_n is lattice connected.

Lemma (Kirat and Lau)

Suppose that T_n is a sequence of compact, connected subsets of $\mathbb{R}^m$, and that in the Hausdorff metric $T = \lim_{n \to \infty} T_n$. Then T is connected.

That is, $T(A, D)$ is connected whenever the approximates T_n are connected for all sufficiently large n, equivalently, whenever the level sets D_n are lattice connected for all sufficiently large n.

Matrix Transformations

Matrices can:

- dilate (or contract) along an axis
- rotate in some plane
- skew along an axis

Dilation and rotation will not affect lattice connectedness of D_n; skewing might.

Example: Skew Leading to Disconnected D_n

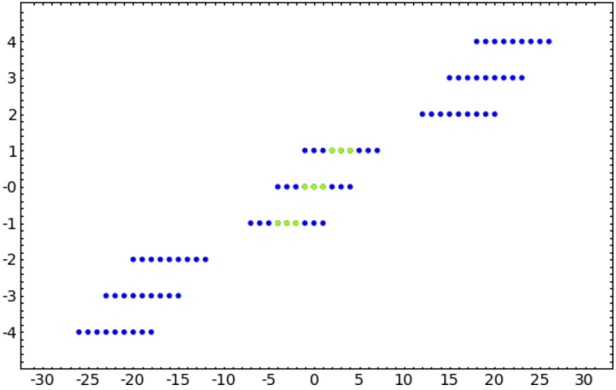


Figure: D_2 for $A = \begin{bmatrix} 3 & 10 \\ 0 & 3 \end{bmatrix}$

Similar Matrices

Definition

A matrix B is similar to the matrix A if there exists an invertible matrix P such that $A = PBP^{-1}$.

Let B be similar to A, let D_B be a digit set for B, and set $D_A = P(D_B)$. Then $T(A, D_A) = P(T(B, D_B))$ (following Lagarias and Wang).

Lemma

The set $T(A, D_A)$ is connected if and only if $T(B, D_B)$ is connected.

Jordan Form I

Every matrix $A \in M_m$ is similar to a diagonal or almost diagonal matrix $J \in M_m$,

$$J = \begin{bmatrix} J_{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_r} \end{bmatrix}$$

where

- $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of A (with multiplicity)
- $J_{\lambda_i} = \lambda_i$ if the corresponding eigenspace is 1-dimensional

Jordan Form II

- if the eigenspace corresponding to λ_i is s-dimensional, then

$$J_{\lambda_i} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Real Jordan Form

If $A \in M_m(\mathbb{R})$, then eigenvalues are roots of a polynomial with real coefficients. Thus if $\lambda_+ = a + bi$ is an eigenvalue, then $\lambda_- = a - bi$ is also an eigenvalue.

- $\dim \lambda_+ = \dim \lambda_-$
- $\begin{bmatrix} J_{\lambda_+} & 0 \\ 0 & J_{\lambda_-} \end{bmatrix} \sim \begin{bmatrix} J_{\lambda_*} & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & J_{\lambda_*} \end{bmatrix}$ with $J_{\lambda_*} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$
- number of blocks $J_{\lambda_*} = \dim \lambda_{+/-}$

Extension: Rational Jordan Form

If $A \in M_m(\mathbb{Z})$, then eigenvalues are roots of a polynomial with integer coefficients. Thus if $\lambda_+ = a + b\sqrt{c}$ is an eigenvalue, with $\sqrt{c}$ irrational, then $\lambda_- = a - b\sqrt{c}$ is also an eigenvalue.

- $\dim \lambda_+ = \dim \lambda_-$
- $\begin{bmatrix} J_{\lambda_+} & 0 \\ 0 & J_{\lambda_-} \end{bmatrix} \sim \begin{bmatrix} J_{\lambda_*} & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & J_{\lambda_*} \end{bmatrix}$ with $J_{\lambda_*} = \begin{bmatrix} a & bc \\ b & a \end{bmatrix}$
- number of blocks $J_{\lambda_*} = \dim \lambda_{+/-}$

A Better Rational Form

- J_{λ_i} not very skew in Jordan form
- J_{λ_*} a rotation in real Jordan form, and the Jordan block in this case not very skew
- J_{λ_*} may still be quite skew in rational Jordan form

Prefer: a different similar form as close to a diagonal matrix as possible while still in $M_m(\mathbb{Q})$

Eg., for $k^2 < c < (k+1)^2$,

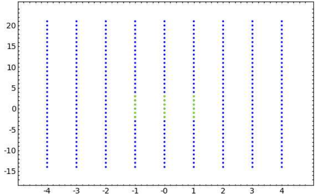
$$C_{\lambda_*} = \begin{bmatrix} a + bk & b(c - k^2) \\ b & a - bk \end{bmatrix}$$

Jordan Form in 2 Dimensions

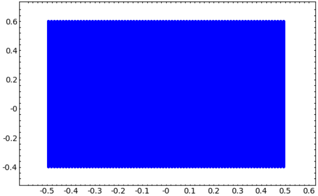
Three cases for J:

1. A has two distinct or non-distinct (but with one dimensional eigenspaces) integer (or half-integer) eigenvalues λ_1 and λ_2:
$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$
2. A has a single integer eigenvalue λ with a two dimensional eigenspace:
$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$
3. A has two distinct non-integer eigenvalues λ_+ and λ_-:
$$J = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},$$
where $\lambda_+ = a + b\sqrt{c}$, and $\lambda_- = a - b\sqrt{c}$, with $a, b \in \frac{1}{2}\mathbb{Z}$, $c \in \mathbb{Z}$.

Case 1: Distinct Integer Eigenvalues

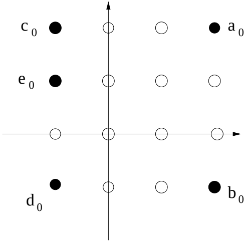


(a) D_n

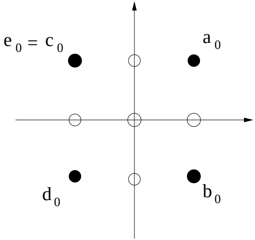


(b) T_n

Case 2: Integer Eigenvalue with 2-dimensional Eigenspace

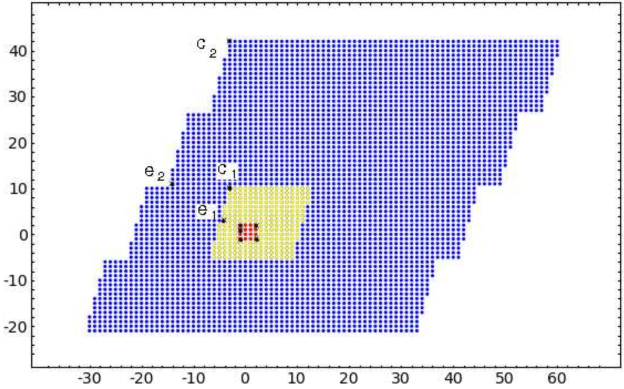


(c) $\lambda = 4$.

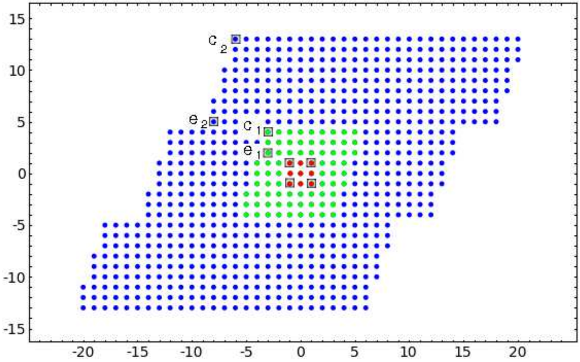


(d) $\lambda = 3$.

c_1 and e_1, and c_2 and e_2 for $\lambda = 4$



c_1 and e_1, and c_2 and e_2 for $\lambda = 3$



Values of c_0, c_1, c_2, and c_n for λ even and odd

	λ even	λ odd
c_0	$\frac{1}{2}\begin{bmatrix} -\lambda+2 \\ \lambda \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda+1 \\ \lambda-1 \end{bmatrix}$
c_1	$\frac{1}{2}\begin{bmatrix} -\lambda^2+2\lambda+2 \\ \lambda^2+\lambda \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda^2-\lambda \\ \lambda^2-1 \end{bmatrix}$
c_2	$\frac{1}{2}\begin{bmatrix} -\lambda^3+3\lambda^2+2\lambda+2 \\ \lambda^3+\lambda^2+\lambda \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda^3+2\lambda^2-\lambda \\ \lambda^3-1 \end{bmatrix}$
$\vdots$	$\vdots$	$\vdots$
c_n	$\frac{1}{2}\begin{bmatrix} -\lambda^{n+1}+(n+1)\lambda^n+n\lambda^{n-1}+\ldots+2\lambda+2 \\ \lambda^{n+1}+\lambda^n+\ldots+\lambda^2+\lambda \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda^{n+1}+n\lambda^n-\lambda^{n-1}-\lambda^{n-2}-\ldots-\lambda^2-\lambda \\ \lambda^{n+1}-1 \end{bmatrix}$

Values of e_0, e_1, e_2, and e_n for λ even and odd

	λ even	λ odd
e_0	$\frac{1}{2}\begin{bmatrix} -\lambda+2 \\ 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda+1 \\ 1 \end{bmatrix}$
e_1	$\frac{1}{2}\begin{bmatrix} -\lambda^2+\lambda+4 \\ \lambda+2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda^2+3 \\ \lambda+1 \end{bmatrix}$
e_2	$\frac{1}{2}\begin{bmatrix} -\lambda^3+\lambda^2+4\lambda+4 \\ \lambda^2+\lambda+2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda^3+3\lambda+2 \\ \lambda^2+1 \end{bmatrix}$
$\vdots$	$\vdots$	$\vdots$
e_n	$\frac{1}{2}\begin{bmatrix} -\lambda^{n+1}+\lambda^n+(n+2)\lambda^{n-1}+(n+1)\lambda^{n-2}+\ldots+4\lambda+4 \\ \lambda^n+\lambda^{n-1}+\ldots+\lambda+2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\lambda^{n+1}+(n+1)\lambda^{n-1}+\lambda^{n-2}+\ldots+\lambda+2 \\ \lambda^n+1 \end{bmatrix}$

Connectedness of $T(J, D_J)$

Lemma

For the Jordan matrices of the form $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, where $\lambda \geq 3$ is an integer, each $D_{J,n}$ is lattice connected.

Theorem

For matrices of the form $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, such that $\lambda \geq 3$ an integer, we have that $T(J, D_J)$ is connected.

Case 3: Complex Conjugate Eigenvalues

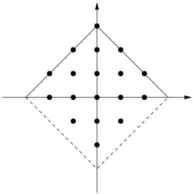


Figure: Digit set D for $J_{\lambda_*} = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$ $(\lambda_+ = 3 + 3i,\ \lambda_- = 3 - 3i)$

Note that J_{λ_*} is a real Jordan form, and can be decomposed as the product of a dilation matrix and a rotation matrix

$$J_{\lambda_*} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}.$$

Real Conjugate Eigenvalues and Higher Dimensions

- Experimental evidence suggests our alternate rational form will generate connected attractors.
- Calculations to extend results to higher dimensions will be similar to Case 2 calculations.

Acknowledgments

Special Thanks:

- NSERC (Discovery Grant Program and PGS-M)
- SAGE Development Team
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