Fourier series on fractals

D. Dutkay, joint work with D. Han, P. Jorgensen, G. Picioroaga, Q. Sun

July, 2009
Definition

A set $\Omega$ of positive finite Lebesgue measure is called spectral if there exists a set $\Lambda \subset \mathbb{R}^d$, such that $\{\exp(2\pi i \lambda \cdot x) \mid \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\Omega)$. 
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Spectral sets

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Conjecture (Fuglede)
A set $\Omega$ is spectral if and only if it tiles $\mathbb{R}^d$ by translations.
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Tao, Matolcsi et.al.: The Fuglede Conjecture fails in dimension $d \geq 3$. 

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**Definition**

Let $e_\lambda(x) := e^{2\pi i \lambda \cdot x}$. A Borel probability measure $\mu$ on $\mathbb{R}^d$ is called **spectral** if there exists a set $\Lambda \subset \mathbb{R}^d$ such that $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. Then $\Lambda$ is called a **spectrum** for the measure $\mu$. 
Example: Cantor set, using division by 4, keep the first and the third quarter. The Hausdorff measure $\mu_4$ on this Cantor set, with dimension $\ln 2 / \ln 4$, is a spectral measure with spectrum

$$\Lambda := \left\{ \sum_{k=0}^{n} 4^k a_k \mid a_k \in \{0, 1\} \right\}.$$
The Jorgensen-Pedersen example

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The Middle Third Cantor measure is far from spectral: there are no three mutually orthogonal exponential functions.
Let $R$ be a $d \times d$ expansive integer matrix, let $B$ be a finite subset of $\mathbb{Z}^d$, $0 \in B$, and let $N := \#B$. Define the affine maps

$$\tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d, b \in B)$$

Then $(\tau_b)_{b \in B}$ is called an affine iterated function system (IFS).
Affine iterated function systems

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**Theorem (Hutchinson)**

There exists a unique compact set such that

$$X_B = \bigcup_{b \in B} \tau_b(X_B)$$
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**Theorem (Hutchinson)**

*There exists a unique compact set such that*

$$X_B = \bigcup_{b \in B} \tau_b(X_B)$$

*There is a unique Borel probability measure $\mu = \mu_B$ on $\mathbb{R}^d$ such that*

$$\int f \, d\mu = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b \, d\mu, \quad (f \in C_c(\mathbb{R}^d))$$

*The measure $\mu$ is supported on $X_B$.*
Theorem (Strichartz)

For the Jorgensen-Pedersen Cantor set, the Fourier series of continuous functions converge uniformly, Fourier series of $L^p$-functions converge in $L^p$. 
The Fourier transform of $\mu$:

$$\hat{\mu}(x) = \prod_{n=1}^{\infty} \hat{\delta}_B((R^*)^{-n}x), \quad \hat{\delta}_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi ib \cdot x}$$
Connections to wavelet theory

The Fourier transform of $\mu$:

$$\hat{\mu}(x) = \prod_{n=1}^{\infty} \hat{\delta}_B \left( (R^*)^{-n} x \right), \quad \hat{\delta}_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b \cdot x}$$

Orthogonality:

$$\sum_{\lambda \in \Lambda} |\hat{\mu}(x + \lambda)|^2 = 1$$
Hadamard pairs

Let $L$ be a subset of $\mathbb{Z}^d$ of the same cardinality as $B, 0 \in L$. We say that $(B, L)$ form a Hadamard pair if one of the following equivalent conditions is satisfied

1. The matrix

$$\frac{1}{\sqrt{N}} \left( e^{2\pi i R^{-1} b \cdot l} \right)_{b \in B, l \in L}$$

is unitary.
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is unitary.

2. The following QMF condition is satisfied:

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\frac{1}{N} \sum_{l \in L} \hat{\delta}_B \left( (R^*)^{-1}(x + l) \right) = 1, \quad (x \in \mathbb{R}).
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   is unitary.

2. The following QMF condition is satisfied:
   
   \[ \frac{1}{N} \sum_{l \in L} \hat{\delta}_B \left( ((R^*)^{-1}(x + l)) \right) = 1, \quad (x \in \mathbb{R}). \]

3. The measure $\delta_B = \frac{1}{N} \sum_{b \in B} \delta_b$ is spectral with spectrum $(R^*)^{-1}L$. 

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Suppose \((B, L)\) form a Hadamard pair. Want to get the following spectrum for \(\mu\).

\[
\Lambda := \left\{ \sum_{n=0}^{\infty} R^k l_k \mid l_k \in L \right\}
\]
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**Definition**

A set \(\{x_0, \ldots, x_{p-1}\}\) is called a \(\delta\)-cycle, if there exist

\(l_0, \ldots, l_{p-1} \in L\) such that \((R^*)^{-1}(x_i + l_i) = x_{i+1}\), where \(x_p := x_0\),

and \(|\hat{\delta}_B(x_i)| = 1\), for all \(i \in \{0, \ldots, p-1\}\)
The Łaba-Wang theorem

**Theorem (Łaba-Wang)**

In dimension $d = 1$, suppose $R \in \mathbb{Z}$ and $0 \in B$, $L \subset \mathbb{Z}$ form a Hadamard pair. Let $\mu_B$ be the invariant measure for the IFS $(\tau_b)_{b \in B}$. Then the set

$$\Lambda := \left\{ \sum_{n=0}^{\infty} R^k l_k \mid l_k \in L \right\}$$

is a spectrum for the measure $\mu_B$ if and only if the only $\delta$-cycle is $\{0\}$. 

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Theorem (D, Jorgensen)

In dimension $d = 1$, suppose $0 \in B$, $L \subset \mathbb{Z}$, $(B, L)$ form a Hadamard pair, and let $\mu_B$ be the invariant measure of the IFS $(\tau_b)_{b \in B}$. Then $\mu_B$ is a spectral measure.
Theorem (D, Jorgensen)

In dimension $d = 1$, suppose $0 \in B$, $L \subset \mathbb{Z}$, $(B, L)$ form a Hadamard pair, and let $\mu_B$ be the invariant measure of the IFS $(\tau_b)_{b \in B}$. Then $\mu_B$ is a spectral measure. A spectrum for $\mu_B$ is the smallest set that contains $-C$ for all $\delta$-cycles $C$, and such that

$$R^* \Lambda + L \subset \Lambda.$$
Examples

For the Jorgensen-Pedersen example $R = 4$, $B = \{0, 2\}$. We can take $L = \{0, 3\}$. Then $\hat{\delta}_B(x) = \frac{1}{2}(1 + e^{2\pi i \cdot 2x})$. 
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For the Jorgensen-Pedersen example $R = 4$, $B = \{0, 2\}$. We can take $L = \{0, 3\}$. Then $\hat{\delta}_B(x) = \frac{1}{2}(1 + e^{2\pi i \cdot 2x})$.

Other than the trivial $\delta$-cycle $\{0\}$, there is an additional one $\{1\}$. $1 = \frac{1}{4}(1 + 3)$, and $|\hat{\delta}_B(1)| = 1$.

$$\Lambda(0) = \left\{ \sum_{k=0}^{n} 4^k l_k \mid l_k \in \{0, 3\} \right\}$$

$$\Lambda(1) = \left\{ -1 - \sum_{k=0}^{n} 4^k l_k \mid l_k \in \{0, 3\} \right\}$$

Then $\Lambda(0) \cup \Lambda(1)$ is a spectrum for $\mu_B$. 
Conjecture (D-Jorgensen)

Let $0 \in B, L \subset \mathbb{Z}^d$, and suppose $(B, L)$ form a Hadamard pair. The invariant measure $\mu_B$ for the IFS $\left(\tau_b\right)_{b \in B}$ is a spectral measure.
Higher dimensions

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Let $0 \in B, L \subset \mathbb{Z}^d$, and suppose $(B, L)$ form a Hadamard pair. The invariant measure $\mu_B$ for the IFS $(\tau_b)_{b \in B}$ is a spectral measure.

1. True for dimension $d = 1$. 

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Conjecture (D-Jorgensen)

Let $0 \in B, L \subset \mathbb{Z}^d$, and suppose $(B, L)$ form a Hadamard pair. The invariant measure $\mu_B$ for the IFS $(\tau_b)_{b \in B}$ is a spectral measure.

1. True for dimension $d = 1$.
2. True for higher dimensions under the assumption that $(B, L)$ is “reducible”.

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Examples

Figure: The Eiffel Tower. \( R = 2I_3, \ B = \{0, e_1, e_2, e_3\} \)
Let $\mu$ be a spectral measure with spectrum $\Lambda$. The Fourier transform $\mathcal{F} : L^2(\mu) \to l^2(\Lambda)$ is defined by

$$(\mathcal{F}f)(\lambda) = \langle f, e_\lambda \rangle, \quad (f \in L^2(\mu), \lambda \in \Lambda).$$
The group of local translations

Define the multiplication operator $M_{e_t}$ on $l^2(\Lambda)$

$$M_{e_t}(a_\lambda)_{\lambda} = (e^{2\pi i t \cdot \lambda} a_\lambda)_{\lambda}.$$
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Define the multiplication operator $M_{e_t}$ on $l^2(\Lambda)$

$$M_{e_t}(a_{\lambda})_{\lambda} = (e^{2\pi i t \cdot \lambda} a_{\lambda})_{\lambda}.$$  

The group of local translations $U_\lambda$ is defined by

$$U_\Lambda(t) = F^{-1} M_{e_t} F, \quad (t \in \mathbb{R}^d).$$
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$$U_\Lambda(t) = \mathcal{F}^{-1} M_{e_t} \mathcal{F}, \quad (t \in \mathbb{R}^d).$$

**Theorem**

*Suppose $O, O + t$ is contained in supp$(\mu)$. Then*

$$(U_\Lambda(t)f(x) = f(x + t), \quad (x \in O)$$
The group of local translations

Define the multiplication operator \( M_{e^t} \) on \( l^2(\Lambda) \)

\[
M_{e^t}(a_{\lambda})_{\lambda} = (e^{2\pi i t \cdot \lambda} a_{\lambda})_{\lambda}.
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The group of local translations \( U_{\lambda} \) is defined by

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U_{\lambda}(t) = F^{-1} M_{e^t} F, \quad (t \in \mathbb{R}^d).
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**Theorem**

*Suppose \( O, O + t \) is contained in \( \text{supp}(\mu) \). Then*

\[
(U_{\lambda}(t)f(x) = f(x + t), \quad (x \in O)
\]

**Corollary**

*If \( \mu \) is a spectral measure and \( O, O + t \subset \text{supp}(\mu) \) then*

\[
\mu(O) = \mu(O + t).
\]
Finite spectral sets

Theorem

Let $A$ be a finite subset of $\mathbb{R}^n$. The following affirmations are equivalent:

1. The set $A$ is spectral.

2. There exists a continuous group of unitary operators $(U(t))_{t \in \mathbb{R}^n}$ on $L^2(A)$, i.e., $U(t + s) = U(t)U(s)$, $t, s \in \mathbb{R}^n$ such that

$$U(a - a')\chi_a = \chi_a' \quad (a, a' \in A), \quad (3.1)$$

where

$$\chi_a(x) = \begin{cases} 
1, & x = a \\
0, & x \in A \setminus \{a\}.
\end{cases}$$
Frames

Definition

A family of vectors \((v_i)_{i \in I}\) in a Hilbert space \(H\) is called a frame if there exist \(A, B > 0\) such that

\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, v_i \rangle|^2 \leq B \|f\|^2, \quad (f \in H).
\]
Consider the Middle Third Cantor set with its invariant measure $\mu_3$, i.e., $R = 3, B = \{0, 2\}$. Jorgensen and Pedersen proved that there are not more than two orthogonal exponentials in $L^2(\mu_3)$. 
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**Definition**

Let $\mu$ be a finite Borel measure on $\mathbb{R}^d$. A set $\Lambda$ in $\mathbb{R}^d$ is called a frame spectrum if $\{e^\lambda \mid \lambda \in \Lambda\}$ is a frame for $L^2(\mu)$. 
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**Definition**

Let $\mu$ be a finite Borel measure on $\mathbb{R}^d$. A set $\Lambda$ in $\mathbb{R}^d$ is called a **frame spectrum** if $\{e^\lambda \mid \lambda \in \Lambda\}$ is a frame for $L^2(\mu)$.

**Question**

*Construct a frame spectrum for the Middle Third Cantor set.*
Frame spectrum and geometry

Question (Mark Kac)

*Can one hear the shape of a drum?*
Frame spectrum and geometry

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Can one hear the shape of a drum?

Question

What geometric properties of the measure \( \mu \) can be deduced if we know a spectrum/frame spectrum of \( \mu \)?
Beurling dimension

Definition

Let $Q = [0, 1]^d$ be the unit cube. Let $\Lambda$ be a discrete subset of $\mathbb{R}^d$, and let $\alpha > 0$. Then the $\alpha$-upper Beurling density is

$$D_{\alpha}(\Lambda) := \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap (x + hQ))}{h^\alpha}.$$ 

Then $D_{\alpha}(\Lambda)$ is constant $\infty$ then $0$, with discontinuity at exactly one point. This point is called the upper Beurling dimension of $\Lambda$. 

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Hausdorff meets Beurling

Theorem

Let $\mu_B$ be the invariant measure for an affine IFS, with no overlap. Suppose $\Lambda$ is a frame spectrum for $\mu_B$, and $\Lambda$ is “not too sparse”. Then the Beurling dimension of $\Lambda$ is equal to the Hausdorff dimension of the attractor $X_B (= \text{supp}(\mu))$. 

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