The Story Continues...

Systems with Continuous Diffraction

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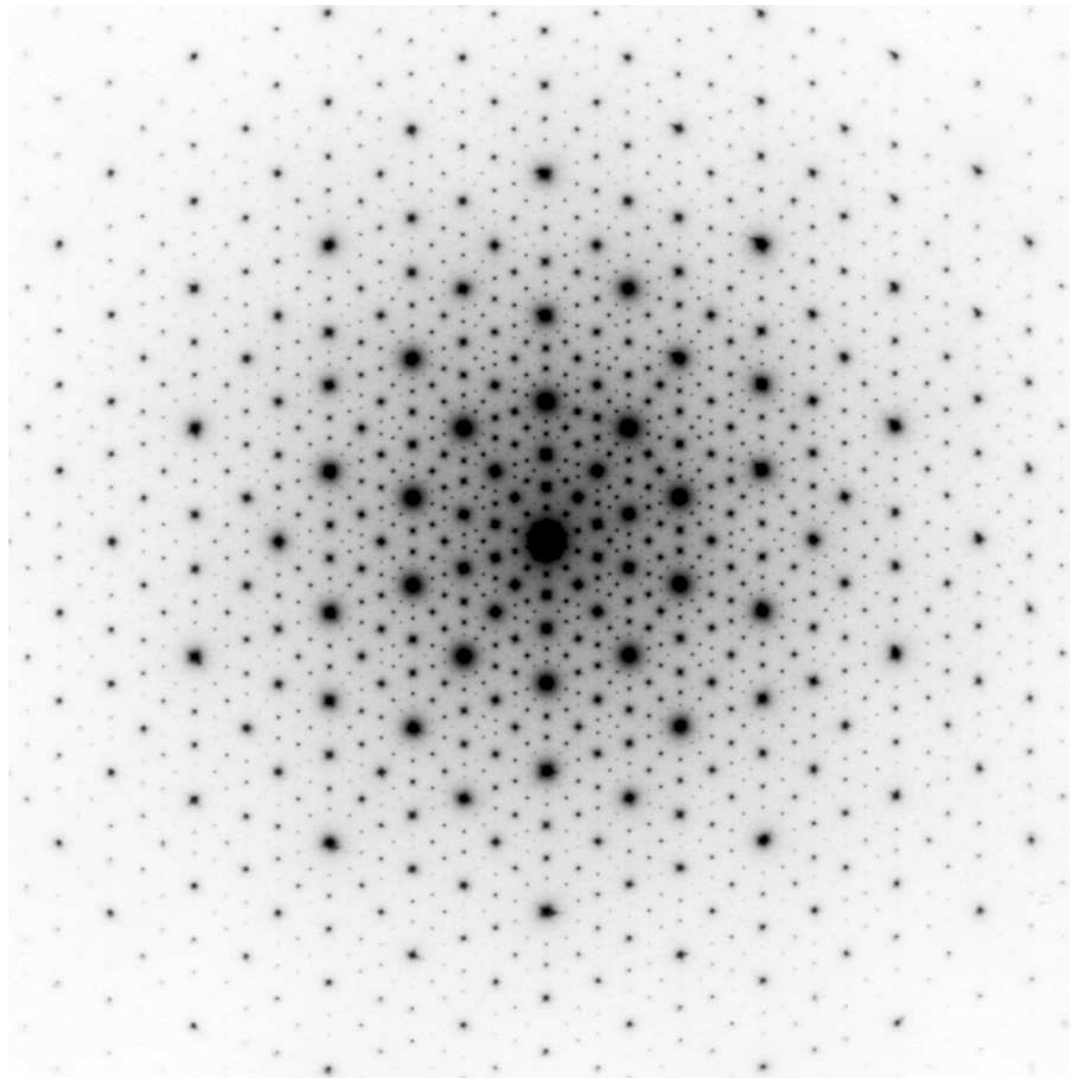
(joint work with Michael Baake)

Menue

- Diffraction theory
- Homometry
- Coin tossing
- Rudin-Shapiro
  - sequence
  - autocorrelation
  - diffraction
- Bernoullisation
- Outlook



Diffraction theory

Setting: $\omega \quad \curvearrowright \quad \gamma = \omega \circledast \widetilde{\omega} \quad \curvearrowright \quad \widehat{\gamma} \quad \not\curvearrowright \quad \omega$

Dirac comb on $\mathbb{Z}$:

$$\omega = \sum_{n\in\mathbb{Z}} w(n)\, \delta_n \quad \curvearrowright \quad \gamma = \sum_{m\in\mathbb{Z}} \eta(m)\, \delta_m$$

Autocorrelation coefficients:

$$\eta(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} w(n)\, w(n+m)$$

Homometry

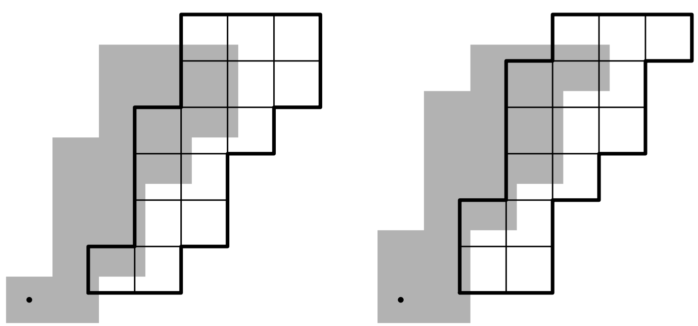
Problem: distinct structures with identical autocorrelation

Example 1: $\delta_{6\mathbb{Z}} * \sum_{j=0}^{5} c_j \, \delta_j$

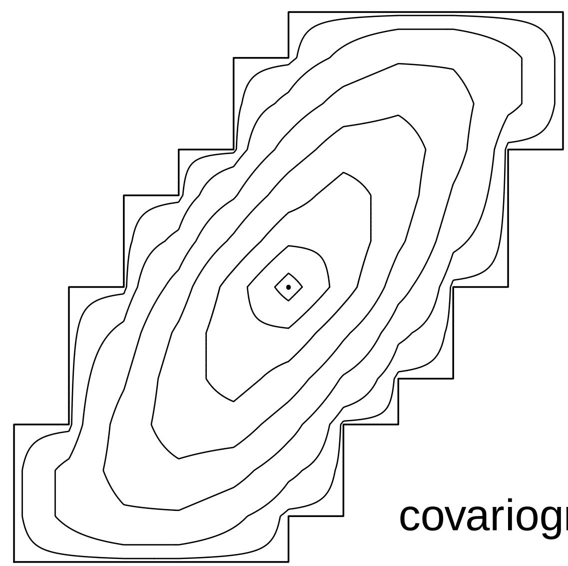
j	0	1	2	3	4	5
c_j	11	25	42	45	31	14
c_j	10	21	39	46	35	17

same correlations up to order 5 (Grünbaum & Moore)

Example 2: homometric models sets with distinct windows



windows



covariogram

Coin tossing sequence

Sequence: i.i.d. random variables $W_n \in \{\pm 1\}$
with probabilities p and $1-p$

Metric entropy: $H(p) = -p\log(p) - (1-p)\log(1-p)$

Autocorrelation: $\gamma_{\mathrm{B}} = \sum_{m\in\mathbb{Z}} \eta_{\mathrm{B}}(m)\delta_m$ with

$$\eta_{\mathrm{B}}(m) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} W_n W_{n+m} \overset{\text{(a.s.)}}{=} \begin{cases} 1, & m = 0 \\ (2p-1)^2, & m \neq 0 \end{cases}$$

(strong law of large numbers)

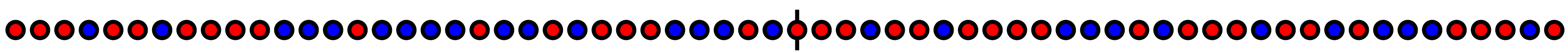
Diffraction measure: $\widehat{\gamma_{\mathrm{B}}} \overset{\text{(a.s.)}}{=} (2p-1)^2\delta_{\mathbb{Z}} + 4p(1-p)\,\lambda$

Rudin-Shapiro sequence

Substitution: $\varrho : \quad a \mapsto ac,\ b \mapsto dc,\ c \mapsto ab,\ d \mapsto db$

Fixed point: $b|a \xrightarrow{\varrho^2} dbab|acab \xrightarrow{\varrho^2} \dots \longrightarrow u = \varrho^2(u)$

Reduction: $\varphi : \quad a, c \mapsto 1, \quad b, d \mapsto -1,$ $\quad \boxed{w := \varphi(u)}$



Alternative description: $w(-1) = -1,\ w(0) = 1,$ with

$$w(4n+\ell) = \begin{cases} w(n), & \text{for } \ell \in \{0,1\} \\ (-1)^{n+\ell}\, w(n), & \text{for } \ell \in \{2,3\} \end{cases}$$

Autocorrelation: $\gamma_{\mathrm{RS}} = \sum_{m\in\mathbb{Z}} \eta(m)\delta_m$

Rudin-Shapiro autocorrelation

Define: $$\left.\begin{array}{c}\eta(m)\\ \vartheta(m)\end{array}\right\} := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} w(n)\, w(n+m) \begin{cases} 1 \\ (-1)^n \end{cases}$$

(all limits exist by Birkhoff's ergodic theorem)

Recursion: $\eta(0) = 1,\ \vartheta(0) = 0,$ and

$$\begin{aligned}
\eta(4m) &= \tfrac{1+(-1)^m}{2}\,\eta(m), \qquad \eta(4m+2) \;=\; 0,\\
\eta(4m+1) &= \tfrac{1-(-1)^m}{4}\,\eta(m) + \tfrac{(-1)^m}{4}\,\vartheta(m) - \tfrac{1}{4}\,\vartheta(m+1),\\
\eta(4m+3) &= \tfrac{1+(-1)^m}{4}\,\eta(m+1) - \tfrac{(-1)^m}{4}\,\vartheta(m) + \tfrac{1}{4}\vartheta(m+1),
\end{aligned}$$

$$\begin{aligned}
\vartheta(4m) &= 0, \qquad \vartheta(4m+2) \;=\; \tfrac{(-1)^m}{2}\,\vartheta(m) + \tfrac{1}{2}\,\vartheta(m+1),\\
\vartheta(4m+1) &= \tfrac{1-(-1)^m}{4}\,\eta(m) - \tfrac{(-1)^m}{4}\,\vartheta(m) + \tfrac{1}{4}\,\vartheta(m+1),\\
\vartheta(4m+3) &= -\tfrac{1+(-1)^m}{4}\,\eta(m+1) - \tfrac{(-1)^m}{4}\,\vartheta(m) + \tfrac{1}{4}\,\vartheta(m+1).
\end{aligned}$$

Rudin-Shapiro diffraction

Unique solution: $\vartheta(1) = 0 \quad \curvearrowright \quad \vartheta(m) = 0$ for all $m \in \mathbb{Z}$

and $\eta(m) = 0$ for all $m \neq 0$

Theorem: $\boxed{\gamma_{\mathrm{RS}} = \delta_0}$ and $\boxed{\widehat{\gamma_{\mathrm{RS}}} = \lambda}$

$\Longrightarrow$ homometric with coin tossing for $p = \frac{1}{2}$,

but zero entropy !

General weights: $h_{\pm}$ instead of ± 1:

$$\boxed{\widehat{\gamma_h} = \left|\frac{h_+ + h_-}{2}\right|^2 \delta_{\mathbb{Z}} + \left|\frac{h_+ - h_-}{2}\right|^2 \lambda}$$

Bernoullisation

Sequence: $S \in \{\pm 1\}^{\mathbb{Z}}$ (assumed ergodic)
with Dirac comb $\omega_S = \sum_{n \in \mathbb{Z}} S_n \, \delta_n$
and autocorrelation γ_S

Bernoullisation: $\omega := \sum_{n \in \mathbb{Z}} S_n W_n \, \delta_n$ $\quad \big(W_n \in \{\pm 1\}\big)$

Autocorrelation: $\gamma \overset{\text{(a.s.)}}{=} (2p-1)^2 \, \gamma_S + 4p(1-p) \, \delta_0$

(strong law of large numbers)

Application: Rudin-Shapiro, with $\gamma_S = \gamma_{\mathrm{RS}} = \delta_0$

$\curvearrowright$ $\gamma = \delta_0$ *independently* of p

$\curvearrowright$ diffraction $\boxed{\widehat{\gamma} \equiv \lambda}$

$\curvearrowright$ homometric, irrespective of entropy

Outlook

- Diffraction as useful tool
- Continuous spectra accessible
- Homometry more difficult
- Insensitivity to entropy
- Generalisation beyond lattice systems
- Extension to higher dimension
- Lower rank entropy (Ledrappier)
- Randomness with interaction

References

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