Discrete Minimum Energy Problems

D.P. Hardin

Department of Mathematics
Vanderbilt University

TU Graz, July 3, 2009

Reisz s energy

The Riesz s-energy of $\omega_N = \{x_1, x_2, ..., x_N\} \subset \mathbf{R}^p$ is, for $s > 0$,

$$E_s(\omega_N) := \sum_{i=1}^{N} \sum_{j \neq i} k_s(x_i, x_j)$$

where

$$k_s(x,y) := \begin{cases} |x-y|^{-s}, & s > 0 \\ -\log\left(|x-y|\right), & s = 0 \end{cases}$$

Note: $\dfrac{|x-y|^{-s} - 1}{s} \to -\log|x-y|$ as $s \to 0$.

Constrained optimization problem:

Given a compact set $A \subset \mathbf{R}^p$.

Minimize the objective function

$$E_s(\omega_N) := \sum_{i=1}^{N} \sum_{j \neq i} k_s(x_i, x_j)$$

subject to the constraint $\omega_N = \{x_1, \ldots, x_N\} \subset A$.

Constrained optimization problem:

Given a compact set $A \subset \mathbf{R}^p$.

Minimize the objective function

$$E_s(\omega_N) := \sum_{i=1}^{N} \sum_{j \neq i} k_s(x_i, x_j)$$

subject to the constraint $\omega_N = \{x_1, \ldots, x_N\} \subset A$.

Let $\omega_N^* := \{x_{1,N}, \ldots, x_{N,N}\}$ denote an optimal configuration and let $\mathcal{E}_s(A, N) := E(\omega_N^*)$.

Cases $s = p - 1$ and $s \to \infty$

The function $k_{p-1}(\cdot, y)$ is harmonic on $\mathbf{R}^p \setminus \{y\}$.

Cases $s = p - 1$ and $s \to \infty$

As $s \to \infty$ and fixed N,

$$\left(\sum_{i \neq j} \frac{1}{|x_i - x_j|^s} \right)^{1/s} \to \frac{1}{\min\{|x_i - x_j|,\ i \neq j\}}.$$

Thus, minimal energy configurations become best-packing configurations, i.e., they maximize the minimum pairwise distance between N points on A as $s \to \infty$.

Example: $A = S^2$

Describe optimal 2 point configurations.

Example: $A = S^2$

Describe optimal 3 point configurations.

Example: $A = S^2$

Describe optimal 4 point configurations.

Example: $A = S^2$

Describe optimal 5 point configurations.

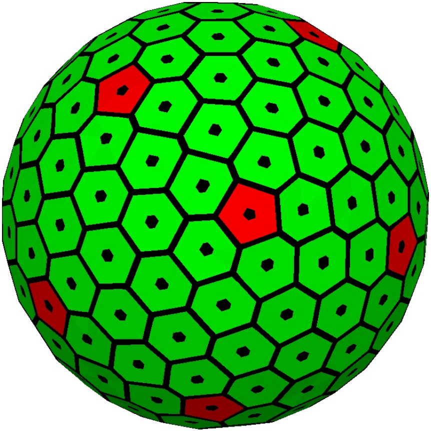
Example: $A = S^2$

For a configuration of points $\omega_N = \{x_1, \ldots, x_N\} \subset S^2$, let ν_i denote the number of nearest neighbors in ω_N to x_i (i.e., the number of edges for the Voronoi cell for x_i). A simple application of Euler's characteristic formula gives: $\sum_{i=1}^{N}(6 - \nu_i) = 12.$

Example: $A = S^2$; $N = 174$; $s = 0, 1, 2$

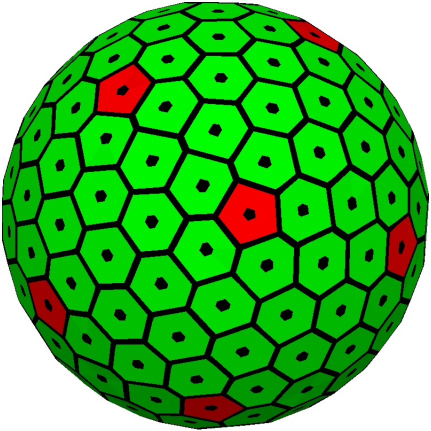
Matthew Calef: Vanderbilt PhD student

Example: $A = S^2$; $N = 174$; $s = 0, 1, 2$



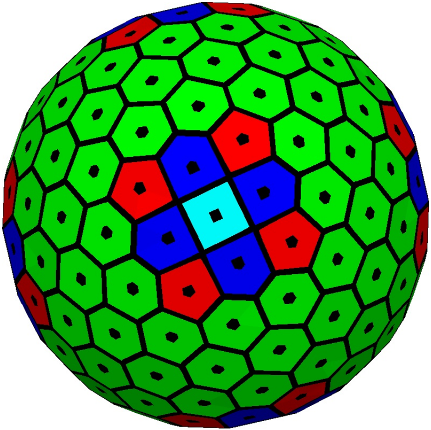
$N = 174$; $s = 2$

Example: $A = S^2$; $N = 174$; $s = 0, 1, 2$



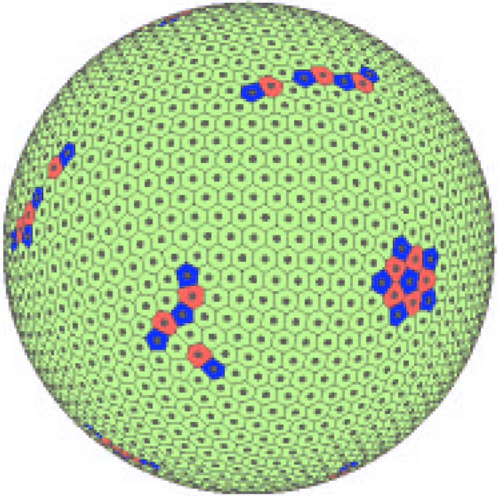
$N = 174$; $s = 1$

Example: $A = S^2$; $N = 174$; $s = 0, 1, 2$



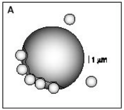
$N = 174$; $s = 0$

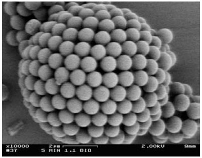
Red = heptagon, Green = hexagon, Blue = pentagon



$$N = 1600, \;\; s = 4$$

Spherical crystallography of "colloidosomes"



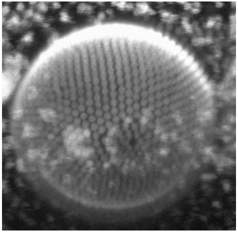


"Colloidosome" = colloids of radius ***a*** coating water droplet (radius ***R***) -- Weitz Laboratory

- ✱ Adsorb, say, latex spheres onto lipid bilayer vesicles or water droplets
- ✱ Useful for encapsulation of flavors and fragrances, drug delivery

  [H. Aranda-Espinoza e.t al. Science **285**, 394 (1999)]
- ✱ Strength of colloidal 'armor plating' influenced by defects in shell….
- ✱ For water droplets, surface tension prevents buckling….

Ordering on a sphere ➔ a minimum of 12 5-fold disclinations, as in soccer balls and fullerenes -- what happens for $R/a >> 1$?



Confocal image: P. Lipowsky, & A. Bausch

Questions from physics

- How does long range order (crystalline structure) arise out of simple pairwise interactions?
- How does the structure depend on the geometry of the world A (dimension, curvature, ...) in which the particles live. How does the structure depend on the interaction?
- How does the order break down as we move away from the ground state?

Asymptotics of configurations as $N \to \infty$

Problem: What is the asymptotic behavior of $\mathcal{E}_s(A, N)$ and of ω_N^* as $N \to \infty$?

Q1: How are minimal s-energy configurations for A distributed for large N?

Q2: How does the asymptotic behavior of $\mathcal{E}_s(A, N)$ depend on A and s?

Connections to Potential Theory

Let $A \subset \mathbf{R}^p$ be compact with Hausdorff dimension $d = \dim_{\mathcal{H}}(A)$.

$$\mathfrak{M}_A := \{\text{all Borel probability measures } \mu \text{ on } A\}.$$

- For $\mu \in \mathfrak{M}_A$, let

$$I_s(\mu) := \int\int \frac{1}{|x-y|^s} d\mu(y) d\mu(x).$$

- Frostman (1935): For $s < d$, there exists a unique equilibrium measure μ_s in $\mathfrak{M}_A$ such that

$$I_s(\mu_s) \leq I_s(\nu) \quad \text{for all} \quad \nu \in \mathfrak{M}_A$$

  and $I_s(\nu) = \infty$ for $s \geq d$ and all $\nu \in \mathfrak{M}_A$.
- The s-capacity of A is $\mathrm{cap}_s(A) = I_s(\mu_s)^{-1}$.
- Points in an optimal configuration are also called Fekete points.

Connection Between Continuous & Discrete Problems

Theorem (Fekete, 1923; Pólya and Szegő, 1931)

Let $A \subset \mathbf{R}^p$ be compact, $s < d := \dim_{\mathcal{H}}(A)$, and μ_s denote the Riesz s-equilibrium measure on A. Then

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N(N-1)} = I_s(\mu_s)$$

and minimal s-energy configurations $\omega_N^* = \omega_N^*(A,s)$ satisfy in the weak-star topology

$$\nu_N := \frac{1}{N}\sum_{x\in\omega_N^*} \delta_x \xrightarrow{*} \mu_s \quad \text{as} \quad N \to \infty.$$

Remark: Weak-star convergence of ν_N to μ_s means

$$\frac{1}{N}\sum_{x\in\omega_N^*}^{N} f(x) \to \int f\, d\mu_s$$

for any $f \in C(A)$.

Sketch of Proof

Step 1: First observe that

$$\mathcal{E}_s(A, N) = E_s(\omega_N^*) = \frac{1}{N-2} \sum_{k=1}^{N} E_s\left(\omega_N^* \setminus \{x_{k,N}\}\right) \geq \frac{N}{N-2} \mathcal{E}_s(A, N-1).$$

Then

$$\tau_N := \frac{\mathcal{E}_s(A, N)}{N(N-1)} \geq \frac{\mathcal{E}_s(A, N-1)}{N(N-1)} \frac{N}{N-2} = \tau_{N-1}$$

showing that τ_N is **increasing** with N.

Let

$$\tau := \lim_{N \to \infty} \frac{\mathcal{E}_s(A, N)}{N(N-1)}$$

Sketch of Proof

Step 2: Show $\tau \leq I_s(\mu_s)$.

$$\mathcal{E}_s(A,N) \leq \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^s}, \quad \forall x_1, \ldots, x_N \in A.$$

Then

$$\mathcal{E}_s(A,N) \leq \int_A \cdots \int_A \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^s} d\mu_s(x_1) \cdots d\mu_s(x_N)$$

Sketch of Proof

Step 2: Show $\tau \leq I_s(\mu_s)$.

$$\mathcal{E}_s(A,N) \leq \sum_{i\neq j}^{N} \frac{1}{|x_i - x_j|^s}, \quad \forall x_1, \ldots, x_N \in A.$$

Then

$$\mathcal{E}_s(A,N) \leq \sum_{i\neq j}^{N} \int_A \cdots \int_A \frac{1}{|x_i - x_j|^s} d\mu_s(x_1) \cdots d\mu_s(x_N)$$

Sketch of Proof

Step 2: Show $\tau \leq I_s(\mu_s)$.

$$\mathcal{E}_s(A, N) \leq \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^s}, \quad \forall x_1, \dots, x_N \in A.$$

Then

$$\mathcal{E}_s(A, N) \leq \sum_{i \neq j}^{N} \int_A \int_A \frac{1}{|x_i - x_j|^s} d\mu_s(x_i) d\mu_s(x_j)$$

Sketch of Proof

Step 2: Show $\tau \leq I_s(\mu_s)$.

$$\mathcal{E}_s(A,N) \leq \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^s}, \quad \forall x_1, \ldots, x_N \in A.$$

Then

$$\mathcal{E}_s(A,N) \leq \sum_{i \neq j}^{N} \int_A \int_A \frac{1}{|x_i - x_j|^s} d\mu_s(x_i) d\mu_s(x_j)$$
$$\leq \sum_{i \neq j}^{N} I_s(\mu_s)$$

Sketch of Proof

Step 2: Show $\tau \leq I_s(\mu_s)$.

$$\mathcal{E}_s(A,N) \leq \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^s}, \quad \forall x_1, \ldots, x_N \in A.$$

Then

$$\begin{aligned} \mathcal{E}_s(A,N) &\leq \sum_{i \neq j}^{N} \int_A \int_A \frac{1}{|x_i - x_j|^s} d\mu_s(x_i) d\mu_s(x_j) \\ &\leq \sum_{i \neq j}^{N} I_s(\mu_s) = N(N-1) I_s(\mu_s) \end{aligned}$$

Sketch of Proof

Step 2: Show $\tau \leq I_s(\mu_s)$.

$$\mathcal{E}_s(A,N) \leq \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^s}, \quad \forall x_1, \ldots, x_N \in A.$$

Then

$$\begin{aligned}\mathcal{E}_s(A,N) &\leq \sum_{i \neq j}^{N} \int_A \int_A \frac{1}{|x_i - x_j|^s} d\mu_s(x_i) d\mu_s(x_j) \\ &\leq \sum_{i \neq j}^{N} I_s(\mu_s) = N(N-1) I_s(\mu_s)\end{aligned}$$

and so:

$$\tau_N \leq I_s(\mu_s) \quad \Rightarrow \quad \tau \leq I_s(\mu_s).$$

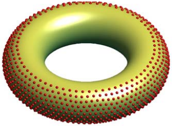
Sketch of Proof

Step 3: By Banach-Alaoglu Thm, ν_N has a weak-star cluster point μ. Consider

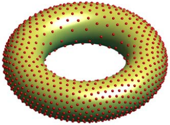
$$
\begin{aligned}
I_s(\mu) &= \int\int \frac{1}{|x-y|^s} d\mu(x) d\mu(y) \\
&= \lim_{M\to\infty} \int\int \min\left\{\frac{1}{|x-y|^s}, M\right\} d\mu(x) d\mu(y) \\
&= \lim_{M\to\infty} \lim_{N\to\infty} \int\int \min\left\{\frac{1}{|x-y|^s}, M\right\} d\nu_N(x) d\nu_N(y) \\
&\leq \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \left\{\mathcal{E}_s(A,N) + NM\right\} \\
&= \tau \leq I_s(\mu_s).
\end{aligned}
$$

So $\mu = \mu_s$ and hence $\tau = I_s(\mu_s)$ and $\nu_N \xrightarrow{*} \mu_s$.

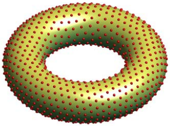
$\boxed{\textbf{N} = 1000 \textbf{ points}}$



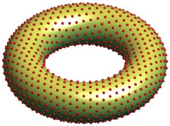
$\boxed{s = 0.2}$



$\boxed{s = 1.0}$

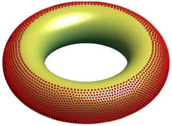


$\boxed{s = 2.0}$

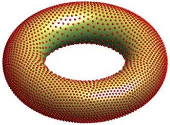


$\boxed{s = 4.0}$

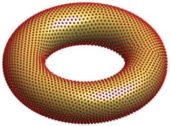
$\boxed{\mathbf{N} = 4000\ \textbf{points}}$



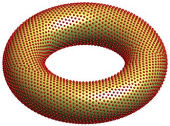
$s = 0.2$



$s = 1.0$



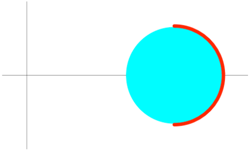
$s = 2.0$



$s = 4.0$

Surfaces of revolution–the case $s = 0$

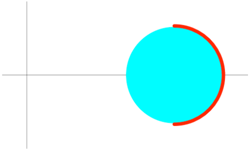
For A in the right-half xy-plane, let $\Gamma(A)$ denote the set in $\mathbb{R}^3$ obtained by rotating A about the y-axis. Let $\mu_{0,Gamma(A)}$ denote the log energy equilibrium on $\Gamma(A)$.



Let A_+ denote the 'right-most' portion of A.

Surfaces of revolution–the case $s = 0$

For A in the right-half xy-plane, let $\Gamma(A)$ denote the set in $\mathbb{R}^3$ obtained by rotating A about the y-axis. Let $\mu_{0,Gamma(A)}$ denote the log energy equilibrium on $\Gamma(A)$.



Let A_+ denote the 'right-most' portion of A.

Theorem (H., Saff, and Stahl, 2006)

Suppose A is a compact set in the right-half plane $\mathbb{R}_+ \times \mathbb{R}$. Then the support of the equilibrium measure $\mu_{0,\Gamma(A)}$ is contained in $\Gamma(A_+)$.

J. Brauchart, H., and Saff (2008) also provide related results for $0 < s < 1$.

Ed Saff, Vanderbilt

Herbert Stahl, Technische Fachhochschule Berlin

Johann Brauchart, (TU Graz PhD) Vanderbilt

What about $s \geq d$?

For $s > d = \dim A$, $I_s(\mu) = \infty$ for *any* $\mu \in \mathfrak{M}_A$. Also

$$\tau = \lim_{N \to \infty} \frac{E_s(A)}{N^2} = \infty.$$

What about $s \geq d$?

For $s > d = \dim A$, $I_s(\mu) = \infty$ for *any* $\mu \in \mathfrak{M}_A$. Also

$$\tau = \lim_{N \to \infty} \frac{E_s(A)}{N^2} = \infty.$$

So new methods are required for $s \geq d$.

The case $A = S^d$ and $s = d$.

The case $A = S^d$ and $s = d$.

Theorem (Kuijlaars & Saff, 1998)

$$\lim_{N\to\infty} \frac{\mathcal{E}_d(S^d, N)}{N^2 \log N} = \frac{1}{d}\frac{\Gamma((d+1)/2)}{\sqrt{\pi}\,\Gamma(d/2)} = \frac{\mathrm{Vol}(\mathcal{B}_d)}{\mathrm{Area}(S^d)} = \frac{\mathcal{H}_d(\mathcal{B}_d)}{\mathcal{H}_d(S^d)},$$

and (Götz & Saff, 2001) d-energy optimal configurations are asymptotically uniformly distributed on S^d as $N \to \infty$.

Here $\mathcal{H}_d$ denotes d dimensional Hausdorff measure on $\mathbf{R}^p$ appropriately normalized.

Grabner and Damelin, 2003, give discrepancy bounds for d-energy optimal configurations on S^d.

Compact sets in $\mathbf{R}^d$

Theorem (H. & Saff, 2005)

Suppose A is a compact set in $\mathbf{R}^d$. Then

(a) $$\lim_{N\to\infty} \frac{\mathcal{E}_d(A,N)}{N^2 \log N} = \frac{\mathcal{H}_d(\mathcal{B}_d)}{\mathcal{H}_d(A)},$$

(b) $$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} = \frac{C_{s,d}}{[\mathcal{H}_d(A)]^{s/d}}, \quad s > d.$$

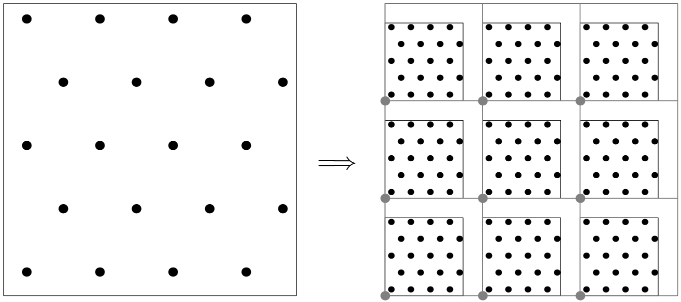
If $\mathcal{H}_d(A) > 0$, then

$$\nu_N := \frac{1}{N} \sum_{x \in \omega_N^*} \delta_x \xrightarrow{*} \mathcal{H}_d^A \quad \text{as} \quad N \to \infty.$$

where $\mathcal{H}_d^A = \frac{\mathcal{H}_d(\cdot \cap A)}{\mathcal{H}_d(A)}$.

About the proof.

An important step in the proof is to show the existence of the limit (2) for $A = U^d := [0,1]^d$. The unit cube is self-similar with scaling $1/m$ for any $m = 2, 3, ...$ We use this to find bounds relating $E_s(A, N)$ and $E_s(A, m^d N)$:



d-rectifiable sets

$A \subset \mathbf{R}^p$ is a **d-rectifiable set** if A is the image of a bounded set in $\mathbf{R}^d$ under a Lipschitz mapping. If A is a d-rectifiable set, then A is almost the finite disjoint union of almost isometric images of compact sets in $\mathbf{R}^d$:

d-rectifiable sets

$A \subset \mathbf{R}^p$ is a **d-rectifiable set** if A is the image of a bounded set in $\mathbf{R}^d$ under a Lipschitz mapping. If A is a d-rectifiable set, then A is almost the finite disjoint union of almost isometric images of compact sets in $\mathbf{R}^d$:

Lemma (Federer, 1969)

If A is a d-rectifiable set then for every $\epsilon > 0$ there exist compact sets K_1, K_2, $K_3, \ldots \subset \mathbf{R}^d$ and bi-Lipschitz mappings $\psi_i : K_i \to \mathbf{R}^p$ with constant $1+\epsilon$, $i = 1, 2, 3, \ldots$, such that $\psi_1(K_1)$, $\psi_2(K_2)$, $\psi_3(K_3), \ldots$ are disjoint subsets of A with

$$\mathcal{H}_d(A \setminus \bigcup_i \psi_i(K_i)) = 0.$$

Any compact subset of a smooth d dimensional manifold is a d-rectifiable set.

Minimum energy on d-rectifiable sets

Theorem (H. & Saff, 2005; Borodachov, H., & Saff, 2007)

Suppose $s \geq d$ and $A \subset \mathbf{R}^p$ is a d-rectifiable set. When $s = d$ we further assume A is a subset of a d-dimensional C^1 manifold. Then

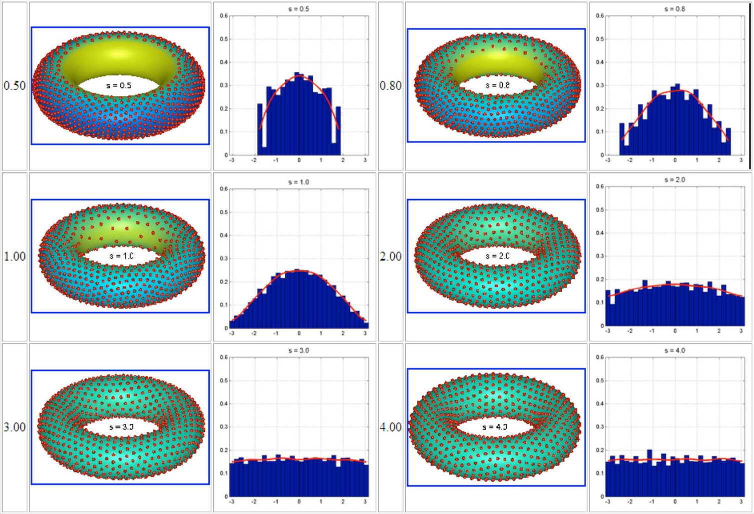
(a) $$\lim_{N\to\infty} \frac{\mathcal{E}_d(A,N)}{N^2 \log N} = \frac{\mathcal{H}_d(\mathcal{B}_d)}{\mathcal{H}_d(A)},$$

(b) $$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} = \frac{C_{s,d}}{[\mathcal{H}_d(A)]^{s/d}}, \quad s > d.$$

If $\mathcal{H}_d(A) > 0$, then

$$\nu_N := \frac{1}{N} \sum_{x \in \omega_N^*} \delta_x \xrightarrow{*} \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)} \quad \text{as} \quad N \to \infty.$$

Sergiy Borodachov (Vanderbilt PhD), Towson University


0.50
s = 0.5
s = 0.5
0.80
s = 0.8
s = 0.8
1.00
s = 1.0
s = 1.0
2.00
s = 2.0
s = 2.0
3.00
s = 3.0
s = 3.0
4.00
s = 4.0
s = 4.0

The constant $C_{s,d}$

- $d = 1$: Since N-th roots of unity are optimal on unit circle, we obtain

$$C_{s,1} = 2\zeta(s) \quad \text{for} \quad s > 1,$$

where $\zeta(s)$ is the classical Riemann zeta function.

The constant $C_{s,d}$

- $d = 1$: Since N-th roots of unity are optimal on unit circle, we obtain

$$C_{s,1} = 2\zeta(s) \quad \text{for} \quad s > 1,$$

  where $\zeta(s)$ is the classical Riemann zeta function.

- For $s > d$, the Epstein zeta function for a d dimensional lattice Λ is

$$\zeta_\Lambda(s) := \sum_{0 \neq v \in \Lambda} |v|^{-s}.$$

  Summing over lattice configurations gives:

$$C_{s,d} \leq \zeta_d^{\min}(s) := \min_{\Lambda} |\Lambda|^{s/d} \zeta_\Lambda(s), \quad s > d. \tag{1}$$

The constant $C_{s,d}$

- $d = 1$: Since N-th roots of unity are optimal on unit circle, we obtain

$$C_{s,1} = 2\zeta(s) \quad \text{for} \quad s > 1,$$

where $\zeta(s)$ is the classical Riemann zeta function.

- For $s > d$, the Epstein zeta function for a d dimensional lattice Λ is

$$\zeta_\Lambda(s) := \sum_{0 \neq v \in \Lambda} |v|^{-s}.$$

Summing over lattice configurations gives:

$$C_{s,d} \leq \zeta_d^{\min}(s) := \min_\Lambda |\Lambda|^{s/d} \zeta_\Lambda(s), \quad s > d. \tag{1}$$

- It is almost surely true (although not proved) that $C_{s,2} = |\Lambda|^{s/d}\zeta_\Lambda(s)$ where Λ is the equilateral hexagonal lattice. It is conjectured (Cohn and Kumar, 2007) that the E_8 and Leech lattices play the same role in dimensions $d = 8$ and $d = 24$.

Sphere packing density in $\mathbf{R}^d$.

Let $\triangle_d$ denote the largest sphere packing density in $\mathbf{R}^d$. Connection between $C_{s,d}$ and Δ_d (Borodachov, H., & Saff, 2007)

$$(C_{s,d})^{1/s} \to (1/2)\left(\frac{\mathrm{Vol}(\mathcal{B}_d)}{\Delta_d}\right)^{1/d} \text{ as } s \to \infty,.$$

- $\Delta_1 = 1$,
- $\Delta_2 = \pi/\sqrt{12} \approx .9069$ (Fejes Toth, 1940's),
- $\Delta_3 = \pi/\sqrt{18} \approx .7405$ (Hales, 2007).

For $d > 3$, Δ_d is unknown, although extremely precise bounds are available for $d = 2$, 8 and 24 (Cohn and Elkies, 2003). Ratio of upper bound to lower bound is

Sphere packing density in $\mathbf{R}^d$.

Let $\triangle_d$ denote the largest sphere packing density in $\mathbf{R}^d$. Connection between $C_{s,d}$ and Δ_d (Borodachov, H., & Saff, 2007)

$$(C_{s,d})^{1/s} \to (1/2)\left(\frac{\mathrm{Vol}(\mathcal{B}_d)}{\Delta_d}\right)^{1/d} \text{ as } s \to \infty,.$$

- $\Delta_1 = 1$,
- $\Delta_2 = \pi/\sqrt{12} \approx .9069$ (Fejes Toth, 1940's),
- $\Delta_3 = \pi/\sqrt{18} \approx .7405$ (Hales, 2007).

For $d > 3$, Δ_d is unknown, although extremely precise bounds are available for $d = 2$, 8 and 24 (Cohn and Elkies, 2003). Ratio of upper bound to lower bound is

$$1.00\ldots001.$$

In each of these dimensions the densest packing appears to be a lattice packing (the hexagonal lattice, E_8, Leech lattice, respectively).

Best packing in dimensions $d = 2, 8$, and 24

Cohn & Elkies' bounds for Δ_d for $d = 2, 8, 24$ are based on the following:

Theorem (Cohn & Elkies, 2003)

Suppose $f : \mathbf{R}^d \to \mathbf{R}$ is an admissible function satisfying the following three conditions :

(1) $f(0) = \hat{f}(0) > 0$,

(2) $f(x) \leq 0$ *for* $|x| \geq r$, *and*

(3) $\hat{f}(t) \geq 0$ *for all* t.

Then $\Delta_d \leq Vol(B_d)(r/2)^d$.

Fork in the road

- Fejes Toth's proof of best packing in $\mathbf{R}^2$.
- Movie (Rob Womersley)

Densest packing in $\mathbf{R}^2$.

The fact that the largest sphere packing $\Delta_2 = \pi/\sqrt{12}$ follows directly from the following:

Theorem (Fejes Toth)

Suppose Ω is a convex polygon in $\mathbf{R}^2$ with six or fewer sides. Suppose Ω contains N pairwise disjoint open discs of radius $r > 0$. Then

$$N \leq \frac{A(\Omega)}{r^2 a(6)}$$

where $A(\Omega)$ denotes the area of Ω and $a(n) = n \tan(\pi/n)$ denotes the area of the regular n-gon with inradius 1.

Then

$$\frac{N\pi r^2}{A(\Omega)} \leq \frac{\pi}{a(6)} = \pi/\sqrt{12}$$

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x\tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x\tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.
- $A(\Omega) = \sum_{i=1}^N A(V_i)$

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x \tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.
- $A(\Omega) = \sum_{i=1}^N A(V_i) \geq r^2 \sum_{i=1}^N a(\nu_i)$

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x\tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.
- $A(\Omega) = \sum_{i=1}^N A(V_i) \geq r^2 \sum_{i=1}^N a(\nu_i) \geq Nr^2 \sum_{i=1}^N a(\nu_i)\frac{1}{N}$

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x \tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.
- $A(\Omega) = \sum_{i=1}^N A(V_i) \geq r^2 \sum_{i=1}^N a(\nu_i) \geq Nr^2 \sum_{i=1}^N a(\nu_i) \frac{1}{N}$
  $\geq Nr^2 a\left(\frac{1}{N} \sum_{i=1}^N \nu_i\right)$

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x \tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.
- $A(\Omega) = \sum_{i=1}^N A(V_i) \geq r^2 \sum_{i=1}^N a(\nu_i) \geq Nr^2 \sum_{i=1}^N a(\nu_i) \frac{1}{N}$
  $\geq Nr^2 a\left(\frac{1}{N} \sum_{i=1}^N \nu_i\right) \geq Nr^2 a(6)$

$\square$

Sketch of proof.

Proof.

Let $\{B(x_i, r)\}$ be a collection of N non-overlapping discs in Ω and let $\{V_i\}$ denote the Voronoi decomposition of Ω associated with the centers $\{x_i\}_{i=1}^N$. Then each V_i is a ν_i-gon containing a disc of radius r.

- $A(V_i) \geq r^2 a(\nu_i)$.
- Euler's formula for planar graphs: $\sum_{i=1}^N \nu_i \geq 6N$.
- The function $a(x) = x \tan(\pi/x)$ is decreasing and strictly convex on the interval $[3, \infty)$.
- $A(\Omega) = \sum_{i=1}^N A(V_i) \geq r^2 \sum_{i=1}^N a(\nu_i) \geq Nr^2 \sum_{i=1}^N a(\nu_i) \frac{1}{N}$
  $\geq Nr^2 a\left(\frac{1}{N} \sum_{i=1}^N \nu_i\right) \geq Nr^2 a(6)$

$\square$

References

- S. Borodachov, D. Hardin, and E. Saff. Asymptotics for discrete weighted minimal energy problems on rectifiable sets. Trans. Amer. Math. Soc., 360(3):1559–1580, 2008.
- M. Calef and D. Hardin. Riesz s-equilibrium measures on d-rectifiable sets as s approaches d. Potential Analysis, 30(4):385-401, 2009.
- H. Cohn and N. Elkies. New upper bounds on sphere packings I. Ann. Math., 157:689–714, 2003.
- H. Cohn and A. Kumar. Universally optimal distribution of points on spheres. J. Am. Math. Soc., 157:99–148, 2006.
- P. Grabner and S. Damelin. Energy functionals, numerical integration and asymptotic equidistribution on the sphere. Journal of Complexity 19(3): 231–246, 2003.
- T. Hales. A proof of the Kepler conjecture. Ann. Math., 162:1065–1185, 2005.
- D. Hardin and E. Saff. Minimal Riesz energy point configurations for rectifiable d- dimensional manifolds. Adv. Math, 193:174–204, 2005.