Tree-adjoined spaces and the Hawaiian earring

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Workshop on Fractals and Tilings 2009
July 6 - 10, 2009, Strobl (Austria)

Wild topology

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Definition

Let $A = \{a_1, a_2, \ldots\}$, and W be the set of all the maps $g : \lambda \to A \cup A^{-1}$, where λ is a countable linear order type and the preimage of each a_i is finite. Let W' be the quotient set, where two elements are identified, whenever there are 'cancellations' that reduce them to the same word.

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Theorem

Together with concatenation of orders, W' forms a group and is isomorphic to the fundamental group $\pi_1(\mathbb{H}, x_0)$ of the Hawaiian earring.

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Consider $\mathcal{C}$ as a subset of the unit interval I. The adjunction space $X_f := Y \sqcup_f I$ is a compact metric space, that is wild at every point in Y.

Alternatively, instead of I, an infinite binary tree attached to the cantor set can be used. Both construction yield homotopy equivalent spaces.

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Some properties

(i) ι is injective,

(ii) $\operatorname{im} \iota \subseteq \ker \psi$,

(iii) If Y is connected and locally simply connected, then $(\operatorname{im} \iota)^{\pi_1 X} = \ker \psi$.

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Let $\gamma : I \to X_f$ be a path that is in the kernel of ψ.

- Find a cancellation on $\gamma^{-1}(X_f \setminus Y)$. That induces a relation on a partition on the rest of I.
- Only finitely many induced sub-paths in Y are not nullhomotopic.
- Remove convex nullhomotopic sub-intervals by a common homotopy.

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Thus, $\ker\psi = (\text{im}\,\iota)^{\pi_1 X_f}$.

Corollary

If Y is simply connected, the fundamental group of X_f injects into that of the Hawaiian earring.

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Definition

Define a relation $\sim$ on the Cantor set w.r.t. S:

$$x \sim y \;\; :\Leftrightarrow \;\; \begin{array}{l} \exists w \in S \; \exists k \subseteq \lambda_w : \\ k \text{ order isomorphic to } \omega \text{ or } -\omega, \\ x, y \in \mathrm{cl}_{A \cup \mathcal{C}}(\operatorname{im} w \restriction k) \; . \end{array}$$

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$S = \mathrm{im}\, \psi \ \Rightarrow \ \mathcal{C}/\!\sim \ \cong Y$.

Remark

$$f_1, f_2 : \mathcal{C} \twoheadrightarrow Y \ \Rightarrow \ \begin{array}{l} X_{f_1} \simeq_{h.e.} X_{f_2}, \\ \exists \sigma \text{ permutation on the alphabet, } (\mathrm{im}\, \psi_1)^\sigma = \mathrm{im}\, \psi_2. \end{array}$$