Projections of Bedford-McMullen Carpets

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Marstrand's Projection Theorem

Let $E \subseteq \mathbb{R}^2$ with $\dim_{\mathcal{H}} E \leq 1$. We might expect orthogonal projections to preserve the dimension. In fact we have

Theorem (Marstrand-1954)

For almost all $\theta \in [0, \pi/2]$,

$$dim_{\mathcal{H}} proj_{\theta}(E) = dim_{\mathcal{H}} E.$$

However there are examples where

$$\dim_{\mathcal{H}} \{\theta : \dim_{\mathcal{H}} \mathrm{proj}_{\theta}(E) < \dim_{\mathcal{H}} E\} = \dim_{\mathcal{H}} E.$$

We look at the same question where E is the attractor for certain iterated function systems.

Conjecture of Furstenberg

Let

$$\begin{aligned} T_0(x,y) &= 1/3(x,y) \\ T_1(x,y) &= 1/3(x,y) + (1/3,0) \\ T_3(x,y) &= 1/3(x,y) + (0,1/3) \end{aligned}$$

and Λ be the attractor, (sometimes called the 1-dimensional Sierpiński gasket). The conjecture is

$$\dim_{\mathcal{H}} \mathrm{proj}_\theta(\Lambda) = 1 \text{ if } \tan\theta \notin \mathbb{Q}.$$

Some result

If $\tan\theta \in Q$ we write $\tan\theta = \frac{p}{q}$ p, q coprime.

1. $\text{leb}(\text{proj}_\theta(\Lambda)) > 0$ iff $\tan\theta \in \mathbb{Q}$ and $p + q = 0 \mod 3$.
2. If $p + q \neq 0 \mod 3$ then $\dim_{\mathcal{H}} \text{proj}_\theta(\Lambda) < 1$.
3. By the standard projection theorem

$$\text{leb}\{\theta : \dim_{\mathcal{H}} \text{proj}_\theta(\Lambda) < 1\} = 0$$

   but not much more is known about this set.

The problem can be stated about overlapping self-similar sets on the unit interval but here the results you get tend to be.

1. Results for specific parameters based on some algebraic property (for the Furstenburg conjecture these parameters are exactly the rational values).
2. An almost all result based on transversality techniques (Marstrand's Projection Theorem) already gives this.

Let C_α be the middle α-Cantor set.

Theorem (Peres-Shmerkin 2007)

If $\frac{\log \alpha}{\log \beta} \notin \mathbb{Q}$ *then*

$$dim_{\mathcal{H}} proj_\theta(C_\alpha \times C_\beta) = \min\{dim_{\mathcal{H}} C_\alpha + dim_{\mathcal{H}} C_\beta, 1\}$$

for all $\theta \in (0, \pi/2)$.

So given a natural irrationality condition we know how dimension behaves under orthogonal projection for the product of Cantor sets. Our work tries to extend this to more general self-affine sets (the work crucially relies on the fact they are self-affine but not self-similar).

Let $m, n \in \mathbb{N}$ with $m < n$ and

$$D \subseteq \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}.$$

For each $(i, j) \in D$ let

$$T_{i,j}(x, y) = (n^{-1}x, m^{-1}y) + (i, j).$$

Let Λ be the attractor of this iterated function system, i.e.

$$\Lambda = \cup_{(i,j)\in D} T_{i,j}(\Lambda).$$

Let

$$n_j = \#\{i : (i, j) \in D\}$$

and $\gamma = \frac{\log m}{\log n}$.

Theorem (McMullen 1984 Bedford 1984)

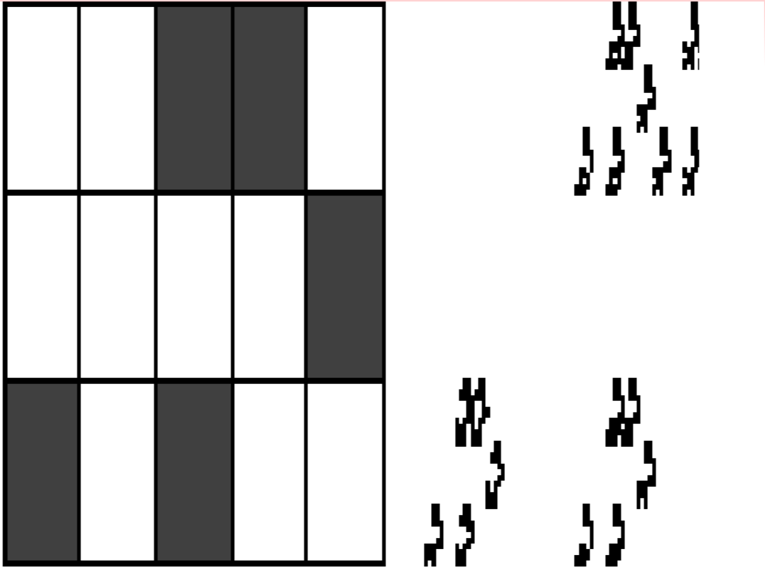
$$dim_{\mathcal{H}}\Lambda = s$$

where s is the solution to

$$m^{-s}\sum_{j=0}^{m-1} n_j^{\gamma} = 1.$$

Furthermore the self-affine measure which gives the rectangle (i,j) probability $m^{-s}n_j^{\gamma-1}$ is a measure of maximal dimension.

In the case where the n_js are either constant or zero then this Bernoulli measure has evenly distributed weights and the Hausdorff and box counting dimensions are the same. (uniform horizontal fibres)



Theorem (Ferguson-J-Shmerkin)

If Λ is a Bedford-McMullen carpet where $\gamma \notin Q$ then

$$dim_{\mathcal{H}} proj_{\theta}(\Lambda) = \min\{dim_{\mathcal{H}}\Lambda, 1\}$$

for all $\theta \in (0, \pi/2)$.

In the previous example

$$\dim_{\mathcal{H}}\Lambda = \frac{\log(1 + 2^{\frac{\log 5 + \log 3}{\log 5}})}{\log 3} = 1.308...$$

So for all $\theta \in (0, \pi/2]$

$$\dim_{\mathcal{H}} \mathrm{proj}_{\theta}(\Lambda) = 1.$$

It is easy to see that

$$\dim_{\mathcal{H}} \mathrm{proj}_0(\Lambda) = \frac{\log 4}{\log 5}.$$

1. For a given Λ show that we can find a sequence Λ_n such that $\lim_{n\to\infty} \dim_{\mathcal{H}}\Lambda_n = \dim_{\mathcal{H}}\Lambda$,
   $\Lambda_n \subseteq \Lambda$ and
   Λ_n has uniform horizontal fibres.

2. A discrete-Marstrand Theorem modified from the one used in Peres-Shmerkin.

3. Using approximate squares where the ratio of the side lengths can be modified by an irrational rotation.

The same results hold for the self-affine sets studied by Gatzouras-Lalley.

Gatzouras-Lalley construction. Let $m, n_1, ldots, n_m \in \mathbb{N}$ and

$$D = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n_i\}.$$

For each $(i,j) \in D$ let $0 < a_{ij} < b_i < 1$ and

$$S_{ij}(x,y) = (a_{ij}x + c_{ij}, b_i y + d_i).$$

The self similar systems

$$\{b_i x + d_i\}_{i=1}^{m}, \{a_{ij}x + c_{ij}\}_{j=1}^{n_i}$$

all need to satisfy the open set condition. The attractor Λ satisfies

$$\Lambda = \cup_{(i,j)\in D} S_{(i,j)\in D} S_{ij}(\Lambda).$$

An appropriate irrational condition under which our results hold is that for some $(i,j) \in D$ $\frac{\log a_{ij}}{\log b_i} \notin \mathbb{Q}$.

Let

$$\{a_i x + c_i\}_{i=1}^n \text{ and } \{b_i x + d_i\}_{i=1}^m$$

be self-similar iterated function systems where $\sum_{i=1}^n a_i = 1$ and $\sum_{j=1}^m b_j = 1$, $c_1 = d_1 = 0$ and for $i, j > 1$

$$c_i = \sum_{l=1}^{i-1} a_l \text{ and } d_j = \sum_{k=1}^{j-1} c_k.$$

The digit set is of the same type as for Bedford McMullen

$$D \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}.$$

The self affine system consists of maps of the form

$$T_{ij}(x, y) = (a_i x + c_i, b_j y + d_j)$$

where $(i, j) \in D$. There is no condition on a_i or b_j.

If there exists $(i,j) \in D$ with $\frac{\log a_i}{\log b_j} \notin Q$ then our results hold for these sets.
That is the Hausdorff dimension is preserved (or drops to 1) by all orthogonal projections with angles between $\left(0, \frac{\pi}{2}\right)$.

Gatzouras-Lalley and Barański constructions