Short time asymptotics of the heat kernel on (the usual and) the harmonic Sierpinski gasket

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Fractals and Tilings

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§0 The Main Question

Given a '*Laplacian*' Δ, let $p_t(x,y)$ be the *heat kernel*

$$e^{t\Delta}f(x) = \int p_t(x,y)f(y)dy.$$

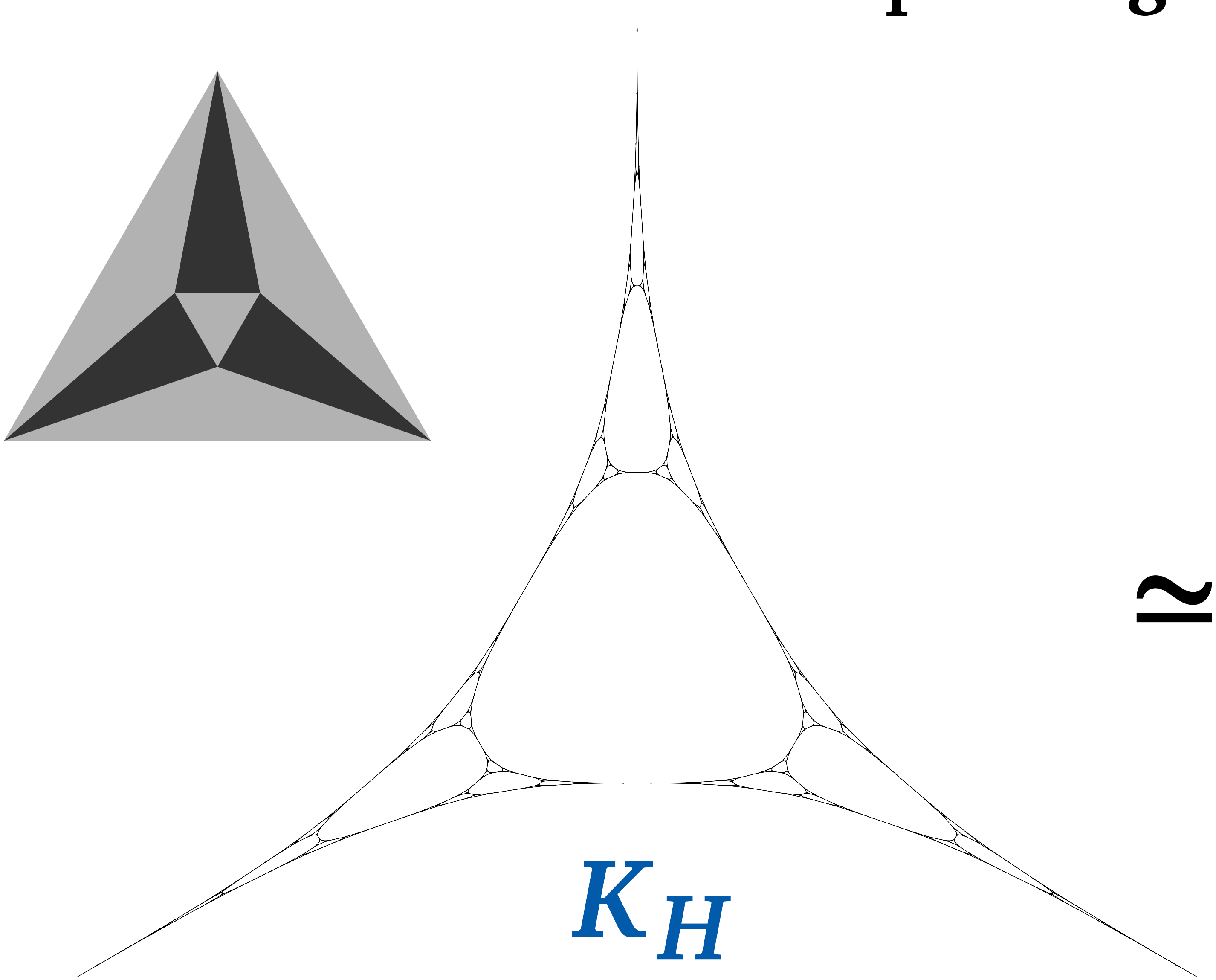
Q. How does $p_t(x,x)$ behave as $t \downarrow 0$?

cf. M^N: Riem. mfd $\Rightarrow p_t^M(x,x) \underset{t\downarrow 0}{\sim} (2\pi t)^{-N/2}(1 + \frac{1}{12}S(x)t + \dots)$

$S(x) := \mathrm{Tr}(\mathrm{Ric}_x)$ scalar curvature

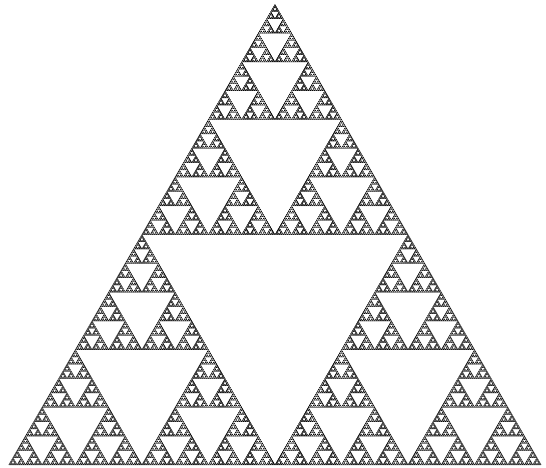
- **What happens for heat kernels on fractals?**

Usual and Harmonic Sierpinski gaskets K and K_H



K_H

$\simeq$

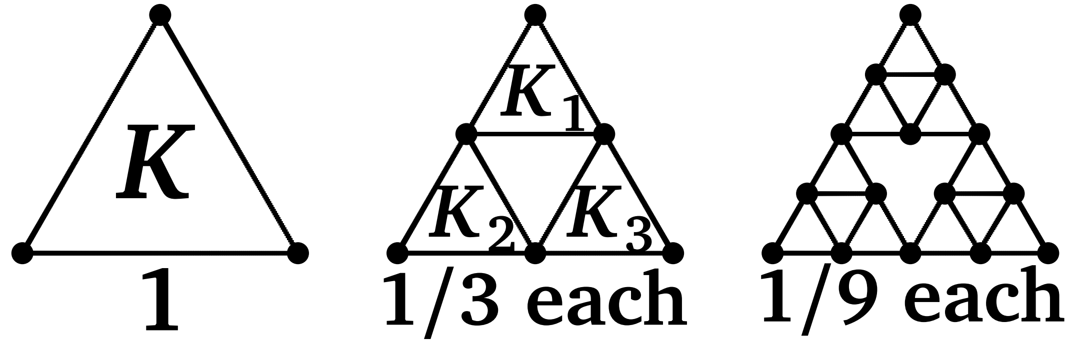


K

The topological structure of K_H is the same as that of K.

§1 Heat kernel on the usual SG

μ: The self-similar measure with weight $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$



$(\mathcal{E}, \mathcal{F})$: the standard Dirichlet form on K (note: $\mathcal{F} \subset C(K)$)

$(\mathcal{E}, \mathcal{F})$ Dirichlet form $\mathcal{E}(u^+ \wedge 1) \le \mathcal{E}(u)$	$\xrightarrow{\mathcal{E}(u,v)=\langle -Au,v\rangle_{L^2(\mu)}}$ $\xleftarrow[\mathcal{E}(\sqrt{-A}u,\sqrt{-A}v)]{}$	A selfad, ≤ 0 'Laplacian'	$\xrightarrow{T_t = e^{tA}}$ $\xleftarrow[A=\lim_{t\downarrow 0}\frac{T_t - I}{t}]{}$	$\{T_t\}_{t\in(0,\infty)}$ $0 \le f \le 1 \Rightarrow$ $0 \le T_t f \le 1$

$\Updownarrow$ $T_t f(x) = \mathrm{E}_x[f(X_t)]$

$X = (\{X_t\}_{t\ge 0}, \{\mathrm{P}_x\}_{x\in K})$: μ-symmetric conservative diffusion

(the 'Brownian motion' on the SG)

Heat kernel $p_t(x,y)$ and its sub-Gaussian bound

$$T_t f(x) = \mathrm{E}_x[f(X_t)] = \int_K p_t(x,y) f(y)\, d\mu(y)$$

Theorem 1 (Barlow-Perkins '88) For $t \in (0,1]$, $x, y \in K$,

$$\boxed{p_t(x,y) \asymp c_1 t^{-d_f/d_w} \exp\left(-c_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)}$$

- $d_f := \dim_{\mathrm{H,Euc}} K = \log_2 3$
- $d_w := \log_2 5 > 2$

In particular, $\quad c_3 \le t^{d_f/d_w} p_t(x,x) \le c_4, \quad t \in (0,1], x \in K.$

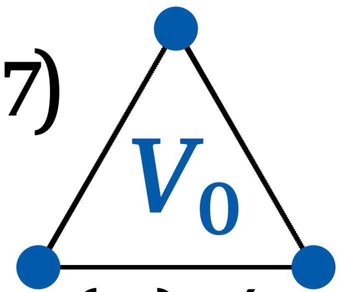
Main Theorem for Usual SG: Oscillatory behavior

Recall: $c_3 \le t^{d_f/d_w} p_t(x,x) \le c_4, \quad t \in (0,1],\ x \in K.$

Theorem 2 (K., in preparation) For **any** $x \in K$,

$$\boxed{\liminf_{t \downarrow 0} t^{d_f/d_w} p_t(x,x) < \limsup_{t \downarrow 0} t^{d_f/d_w} p_t(x,x).}$$

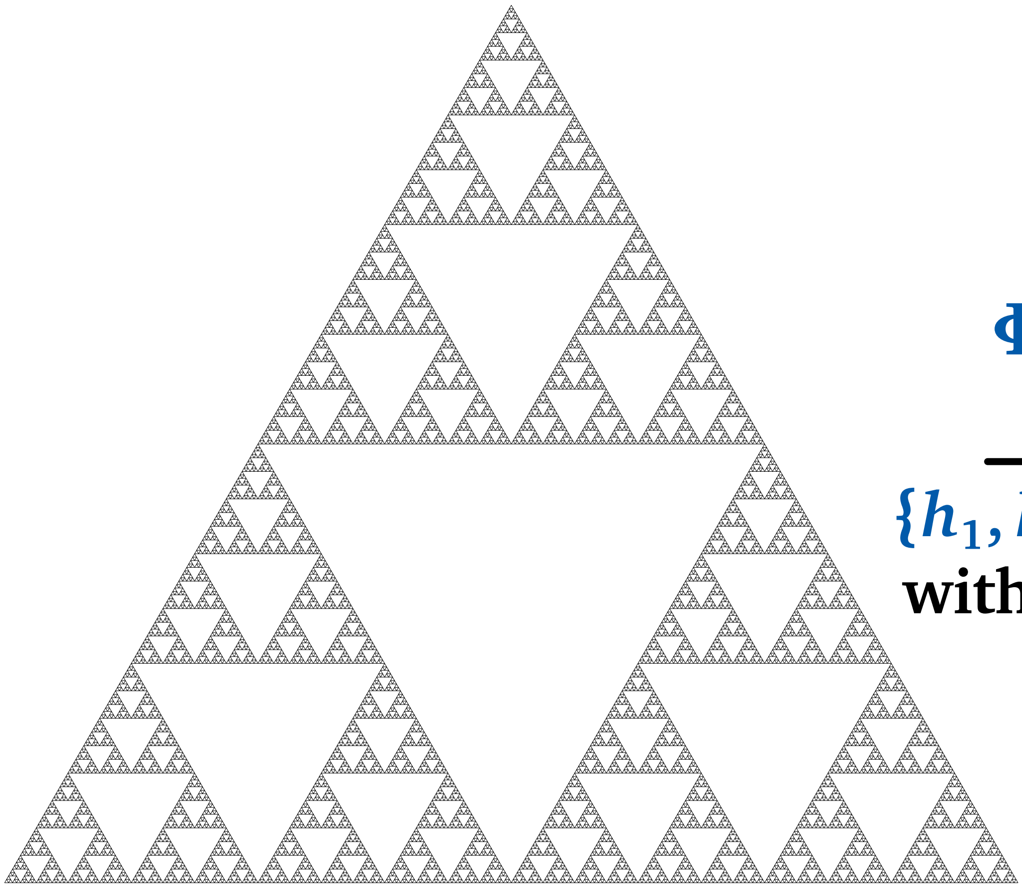
(For $x \in V_0$: Essentially implied by Grabner-Woess '97)



Proof: Construct (weakly) localized eigenfunct. φ_n, $\varphi_n(x) \neq 0$ (based on symmetry) and use $p_t(x,x) = \sum_n e^{-\lambda_n t} \varphi_n(x)^2$.

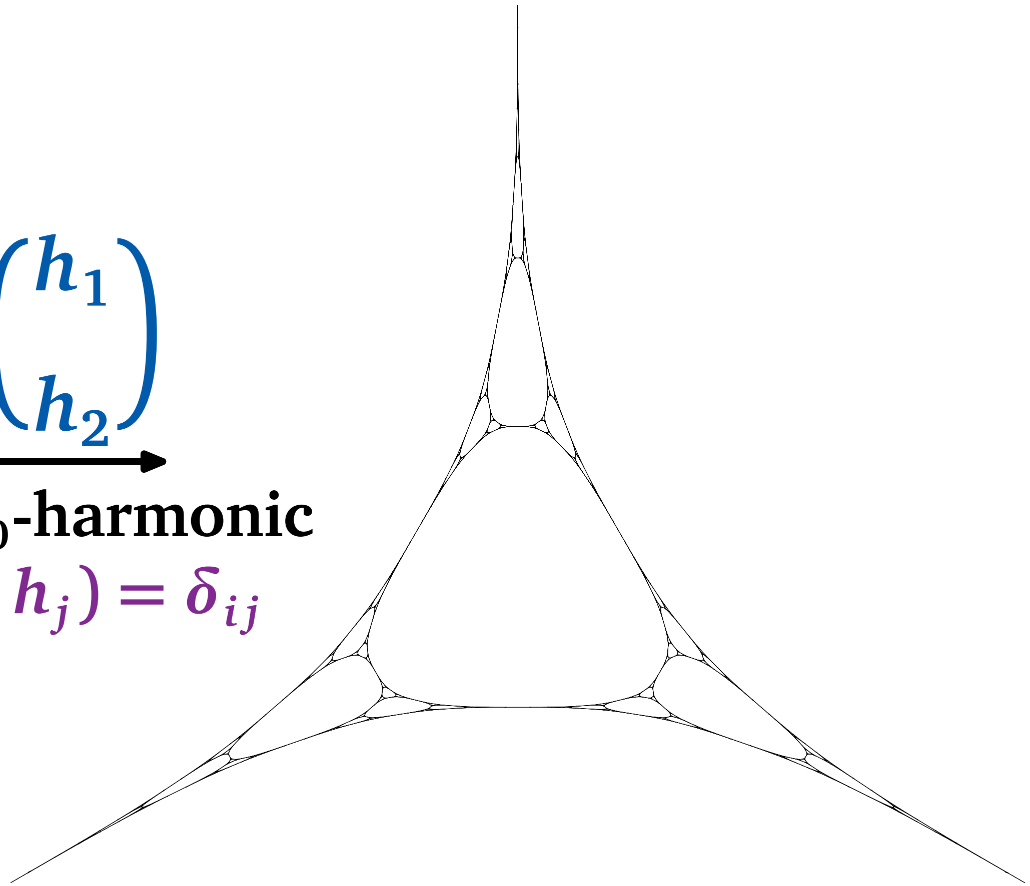
§2 Harmonic SG and its heat kernel

Homeomorphism $\Phi : K \to K_H$



$$\Phi := \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \longrightarrow$$

$\{h_1, h_2\}$: V_0-harmonic with $\mathscr{E}(h_i, h_j) = \delta_{ij}$



the Sierpinski gasket K

the harmonic Sier. gasket K_H

Energy measures ν_u, $u \in \mathcal{F}$

$$\int_{K_H} f \, d\nu_u = 2\mathcal{E}(fu, u) - \mathcal{E}(f, u^2), \quad {}^\forall f \in \mathcal{F}.$$

$$\boxed{d\nu_u = |\nabla u|^2 dx}$$

Define $\nu_1 := \nu_{h_1}$, $\nu_2 := \nu_{h_2}$ and

$\boxed{\nu := \nu_1 + \nu_2}$: **the Kusuoka measure**

(the energy of the 'harmonic map' Φ)

- $p_t^H(x, y)$: the heat kernel associated with $(K_H, \nu, \mathcal{E}, \mathcal{F})$

(Note: the measure has been changed from μ to ν)

Shortest path metric and Gaussian estimate of p_t^H

(J. Kigami, Math. Ann. 340 (2008), 781-804.)

Definition 3 For $x, y \in K_H$, define

$$\boxed{d_H(x,y)} := \inf\left\{ |c|_{\mathrm{Euc}} \;\middle|\; \begin{array}{c} c : [0,1] \to K_H,\ c \text{ is continuous}, \\ c(0) = x,\ c(1) = y \end{array} \right\}.$$

Theorem 4 (Gaussian Heat Kernel Estimate)

For $t \in (0,1]$, $x, y \in K_H$,

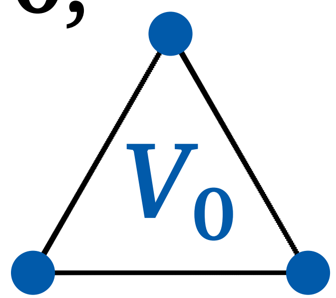
$$\boxed{p_t^H(x,y) \asymp \frac{c_1}{\nu\left(B_{d_H}(x,\sqrt{t})\right)} \exp\left(-c_2 \frac{d_H(x,y)^2}{t}\right).}$$

Main Theorem for Harmonic SG: No Oscill. at $x \in V_0$

Theorem 5 (K., in preparation)

Let $\beta := \log_{5/3} 5 - 2\,(> 1)$. For $x \in V_0$, as $t \downarrow 0$,

$$\boxed{p_t^H(x,x) = \frac{1}{\sqrt{2\pi t}} + O\left(t^{\beta/2}\right).}$$



cf. (K., in preparation) $^{\exists}\alpha$, $1 < \alpha \le \dim_{\mathrm{H}}(K_H, d_H)$,

$$\lim_{t \downarrow 0} \frac{2 \log p_t^H(x,x)}{-\log t} = \alpha \qquad \nu\text{-a.e. } x \in K_H.$$