On the Self-affine Tiles with Polytope Convex Hulls and Their Lebesgue Measure

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Abstract

Let F be a self-affine tile generated by a finite number of affine maps. The problem of determining the convex hull of F is of geometrical interest. In regard to this problem, we give necessary and sufficient conditions for the convex hull of F to be a polytope. Additionally, we determine the vertices of such polytopes. Our constructive proofs lead us to upper bounds for the Lebesgue measure of F. We also test our technique on well-known examples.

Some Problems

Let $M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries. A matrix $T \in M_n(\mathbb{R})$ is called *expanding* (or *expansive*) if all its eigenvalues have moduli > 1. We call a finite set $D = \{d_1, \cdots, d_q\} \subset \mathbb{R}^n$ a *digit set*. Let T^* denote the adjoint of T.

Some Problems

We first recall a couple of known theorems.

THEOREM

[1] Let $F = F(T, D)$ be the compact set satisfying $TF = F + D$ for an expanding matrix T and a digit set D. Let $\{n_j\}$ be the outward unit normal vectors of $(n-1)$-dimensional faces of the convex hull P of D. Then the convex hull of F, $\mathcal{C}(F)$, is a polytope if and only if every n_j is an eigenvector of $(T^*)^k$ for some k.

[1] R. Strichartz, and Y. Wang, Geometry of self-affine tiles I, *Indiana Univ. Math. J* 48 (1999), 1-23.

(*) It is stated in [1] that even in the case when A^N is a multiple of the identity, hence $\mathcal{C}(F)$ is a polytope, it is not clear how to estimate the number of faces of $\mathcal{C}(F)$.

Some Problems

Let $\mu(F)$ denote the Lebesgue measure of T.

THEOREM

[2] Let $T \in M_n(\mathbb{R})$ be an expanding integer matrix and $D \subset \mathbb{Z}^n$ be a digit set with $\#D = |\det(T)|$. Then $\mu(F(T, D))$ is an integer.

[2] J. C. Lagarias and Y. Wang, Integral self-affine tiles in $\mathbb{R}^n$ I. Standard and non-standard digit sets, *J. London Math. Soc.* 54 (1996), 161-179.

(**) It is also mentioned by the authors that the techniques in [1] or [2] don't seem to work for affine iterated function systems (AIFSs) involving mappings with different linear parts.

Some Problems

Here we consider the attractors of AIFSs with polytope convex hulls. Regarding the problems in (*) and (**), we are interested in the following questions:

(1) How can we get the vertices of $\mathcal{C}(F)$?

(2) How can we compute or approximate $\mu(F(T, D))$?

The norm of a matrix $T \in M_n(\mathbb{R})$ induced by $\|\cdot\|$ is given by

$$\|T\| = \sup_{\|x\|=1} \{\|Tx\| : x \in \mathbb{R}^n\}.$$

For $\mathcal{T} = \{T_1, \cdots, T_q\} \subset M_n(\mathbb{R})$, we define

$$\|\mathcal{T}\| = \max\{\|T_j\| : \ T_j \in \mathcal{T}\}.$$

For $j_1 \cdots j_k \in \{1, \cdots q\}$, let $J = (j_1, \cdots, j_k)$ denote a multi-index and let $|J| = k$ be the length of J. By T_J, we mean the product $T_{j_1} \cdots T_{j_k}$ and we let $\mathcal{T}^k = \{T_J : \ |J| = k\}$.

Note on Attractors

DEFINITION

[3] *The number*

$$\lambda(\mathcal{T}) = \lim_k \|\mathcal{T}^k\|^{1/k} = \limsup \|\mathcal{T}^k\|^{1/k} = \inf_k \|\mathcal{T}^k\|^{1/k}$$

is called the joint spectral radius of $\mathcal{T}$.

[3] G.-C. Rota and W. G. Strang, A note on the joint spectral radius, *Indag. Math.* 22 (1960), 379-381.

Note on Attractors

PROPOSITION

[4] Suppose that $\mathcal{T} = \{T_1, T_2, ..., T_q\} \subset M_n(\mathbb{R})$ *satisfies* $\lambda(\mathcal{T}) < 1$. *Then for any* $D = \{d_1, \cdots, d_q\} \subset \mathbb{R}^n$, *there exists a unique nonempty compact set* $F = F(\mathcal{T}, D)$ *satisfying*

$$F = \bigcup_{j=1}^{q} T_j(F + d_j).$$

[4] I. Kirat and I. Kocyigit, Remarks on Self-affine Fractals with Polytope Convex Hulls, manuscript (2008).

REMARK

[4] The expanding assumption on T_j^{-1} cannot guarantee $\lambda(\mathcal{T}) < 1$. To see this, we give the following example of a set of expanding matrices with $\lambda(\mathcal{T}) = 1$:
Let $\mathcal{T} = \{T_1, T_2\}$ with

$$T_1^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \qquad T_2^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

Results

Assume that $0 \in D$. For $d_{j_k} \in D$, the sequence $d_{j_1} d_{j_2} \dots d_{j_k} \dots$ corresponds to the point $x = \sum_{k=1}^{\infty} T_{j_1} T_{j_2} \dots T_{j_k} d_{j_k} \in F$ and hence x will be represented by $j_1 j_2 \dots j_k \dots$. As usual, $\overline{j_1 j_2 \dots j_p}$ (i.e., the block $j_1 j_2 \dots j_p$ is repeated indefinitely) denotes a *periodic sequence* and $j_1 j_2 \cdots j_m \overline{j_{m+1} j_{m+2} \dots j_p}$ an *eventually periodic (e.p.) sequence*.

We set

$$A_k = \{ j_1 j_2 \dots j_k = j_1 j_2 \dots j_k \overline{0} = \sum_{i=1}^{k} T_{j_1} T_{j_2} \dots T_{j_i} d_{j_i} \ : \ d_{j_1}, \dots, d_{j_k} \in D \}.$$

Let $\phi_j(x) = T_j(x + d_j)$, $1 \leq j \leq q$. In the Hausdorff metric, we know that

$$A_k = \bigcup_{|J|=k} \phi_J(0), \qquad k = 1, 2, \dots,$$

converges to F as $k \to \infty$.

Results: The single-matrix case

Let $S \in \mathbb{R}^n$ be a set. We will denote the set of vertices of $\mathcal{C}(S)$ by $\mathcal{V}(S)$. Now let $\mathcal{V}_k = \mathcal{V}(A_k)$.

THEOREM

[4] Assume that $T_1 = T_2 = \cdots = T_q$ *and* $\lambda(\mathcal{T}) < 1$. *Then*

(i) $\mathcal{C}(F)$ *is a polytope if and only if* $\#\mathcal{V}_i = \#\mathcal{V}_{i+1} = t$ *for some* k. *In such a case, for any* $j > i$, $\mathcal{V}_j$ *has* t *elements too.*

(ii) If $\#\mathcal{V}_{i+1} = \#\mathcal{V}_i = t$ *for some* i, *then* $\#\mathcal{V}(F) = t$ *and for any* $j > i$, $\mathcal{V}_j$ *consists of consecutive terms of points of* $\mathcal{V}(F)$ *which are periodic points in* $\partial(F)$.

[4] I. Kirat and I. Kocyigit, Remarks on Self-affine Fractals with Polytope Convex Hulls, manuscript (2008).

Results: The multiple-matrix case

The previous theorem may not hold true if we drop the assumption that $T_1 = T_2 = \cdots = T_q$. For the general case, we will give another characterization of polytope convex hulls of similar nature. We let $(j_1 j_2 \ldots j_p)_r = \underbrace{j_1 j_2 \ldots j_p j_1 j_2 \ldots j_p \ldots j_1 j_2 \ldots j_p}_{r \quad \text{times}}$,
i.e., $(j_1 j_2 \ldots j_p)_r$ denotes a finite sequence consisting of r blocks of digits $j_1 j_2 \ldots j_p$. For bounded s and all sufficiently large $r \geq 2$, a point of the form

$$x = i_1 i_2 \ldots i_m (j_1 j_2 \ldots j_p)_r j_{p+1} \ldots j_{p+s}$$

will also be called an e.p. point.

Let $\mathcal{W}_{k,\infty}$ consist of e.p. points of the form $i_1 i_2 \dots i_m \overline{j_1 j_2 \dots j_p}$ associated to the e.p. points of $\mathcal{W}_k \subseteq \mathcal{V}_k$. Set

$$\mathcal{W}^C_{k,\infty} = \{x = i'_1 i_1 i_2 \dots i_m \overline{j_1 j_2 \dots j_p} \mid i_1 \dots i_m \overline{j_1 j_2 \dots j_p} \in \mathcal{W}_{k,\infty}, \;\; x \notin \mathcal{W}_{k,\infty}$$

THEOREM

[4] Assume that F is a self-affine set defined by $\mathcal{T} = \{T_1, T_2, ..., T_q\} \subset M_n(\mathbb{R})$ with $\lambda(\mathcal{T}) < 1$ and a digit set D. Then $\mathcal{C}(F)$ is a polytope if and only if there exists a $k_0 \in \mathbb{N}$ and a subset $\mathcal{W}_{k_0} \subseteq \mathcal{V}_{k_0}$ with e.p. points such that

$$\mathcal{W}^C_{k_0,\infty} \cup \mathcal{V}_{k_0} \subseteq \mathcal{C}(\mathcal{W}_{k_0,\infty}).$$

In such a case, $\mathcal{C}(F) = \mathcal{C}(\mathcal{W}_{k_0,\infty})$ and all periodic vertices of $\mathcal{C}(F)$ appear in $\mathcal{V}_k$ for all sufficiently large k.

Algorithm 1: The single-matrix case

1. Choose a number $k_0 \in \mathbb{N}$,

2. Start finding $\mathcal{V}_1, \mathcal{V}_2, \cdots \mathcal{V}_{k_0}$, and stop finding more of them if $\#\mathcal{V}_i = \#\mathcal{V}_{i+1}$ for some $i \in \{1, \cdots k_0\}$,

3. Extract the points in $\mathcal{V}(F)$ from $\mathcal{V}_{i+1}$ in the following way:

If $k > i+1$ and

$$\begin{aligned}\mathcal{V}_{i+1} &= \{j_1 \ldots j_{i+1},\ j_2 \ldots j_{i+1} j_{i+2},\ j_3 \ldots j_{i+2} j_{i+3}, \ldots, j_{k-i} \ldots j_k = j_1 \ldots j_{i+1}, \\ &\quad \text{(similarly other sequences here)} \ldots\},\end{aligned}$$

then

$$\mathcal{V}(F) = \{\overline{j_1 \ldots j_{i+1} \ldots j_{k-i-1}}, \overline{j_2 \ldots j_{i+1} \ldots j_{k-i-1} j_1}, \ldots, \overline{j_{k-i-1} j_1 j_2 \ldots j_{k-i-2}}, \ldots\}$$

When $k \leq i+1$ and

$$\begin{aligned}\mathcal{V}_{i+1} &= \{j_1 \ldots j_k \ldots j_{i+1},\ j_2 \ldots j_{i+1} j_{i+2},\ j_3 \ldots j_{i+2} j_{i+3},\ \ldots, j_{k+1} \ldots j_{k+i+1} = j_1 \ldots \\ &\quad \text{(similarly other sequences here)} \ldots\},\end{aligned}$$

we have $\mathcal{V}(F) = \{\overline{j_1 \ldots j_k}, \overline{j_2 \ldots j_k j_1}, \ldots, \overline{j_k j_1 \ldots j_{k-1}}, \ldots\}$.

Set $\mathcal{C}_k = \bigcup_{|J|=k} \phi_J(\mathcal{C}(F)), \qquad k = 1, 2, \ldots.$

We can use the vertex-finding algorithms together with the following simple proposition to give upper bounds for the Lebesgue measure of F.

PROPOSITION

$\{\mu(\mathcal{C}_k)\}$ *converges to* $\mu(F)$ *decreasingly.*

Example

The twin dragon. For $T^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \left\{ d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, $F(T, D)$ is called the twin dragon tile (see the Figure). For this example, we find $\#\mathcal{V}_4 = \#\mathcal{V}_5 = 8$ and

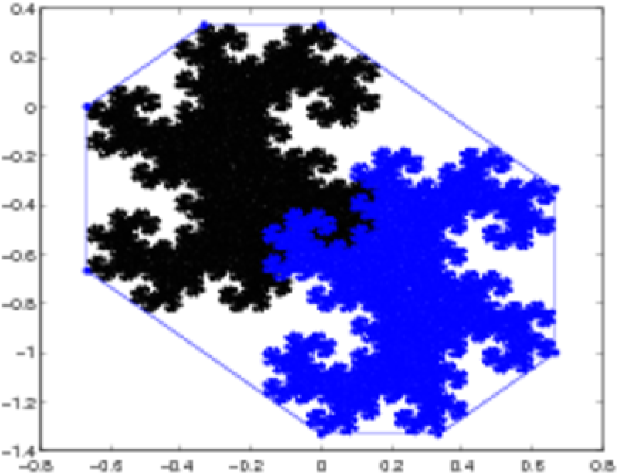
$$\mathcal{V}_5 = \{22111, 21111, 11112, 11122, 11222, 12222, 22221, 22211\}.$$

Since $\#\mathcal{V}_4 = \#V_5 = 8$, the convex hull is an octagon. Using Algorithm 1, we can extract the vertex set $V(F)$ from V_5. Expressed as coordinate pairs,

$$\mathcal{V}(F) = \{(2/3, -1), (2/3, -1/3), (0, 1/3), (-1/3, 1/3), \\ (-2/3, 0), (-2/3, -2/3), (0, -4/3), (1/3, -4/3)\}$$

as marked in the next Figure . □

Examples



Examples

A Haar tile For the expansive matrices

$$T_1^{-1} = T_2^{-1} = \begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix},$$

and $D = \left\{ d_1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right], d_2 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right], d_3 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right], d_4 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right], \right\}$, the associated self-affine set $F = F(\mathcal{T}, D)$ is a tile (see the Figure). We obtain $\#\mathcal{V}_2 = \#\mathcal{V}_3 = 6$ and $\mathcal{V}_3 = \{434, 343, 323, 232, 121, 212\}$. Since V_3 has 6 strings, the convex hull is a hexagon. Again by the vertex finding algorithm, from $\mathcal{V}_3$, we get

$$\begin{aligned} \mathcal{V}(F) &= \{\overline{43}, \overline{34}, \overline{32}, \overline{23}, \overline{12}, \overline{21}\} \\ &= \{(1, 1/3), (1, 4/3), (2/3, 4/3), (1/3, -2/3), (0, 1/3), (0, -2/3) \end{aligned}$$

For the Lebesgue measure, we get from the second column of the next Table that $0 < \mu(F) \leq 1.0104$. Since D is a complete residue system for T_1^{-1}, $\mu(F)$ is a non-zero integer. Therefore, $\mu(F) = 1$, i.e., F is a Haar tile.

Examples

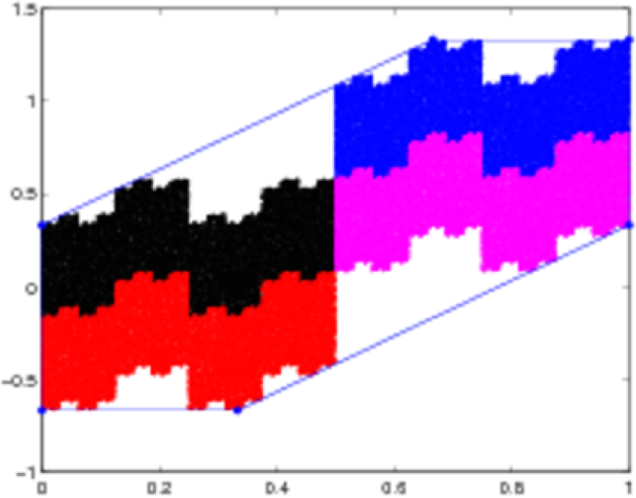


Figure: The Haar tile

Examples

Table: First few successive approximations to Lebesgue measures

	The Haar tile	Levy's dragon	Heighway's dragon	The 3-digit tile
$\mu(\mathcal{C}_1)$	1.1666	1.9375	0.52546	0.51192
$\mu(\mathcal{C}_2)$	1.0833	1.7031	0.47222	0.45152
$\mu(\mathcal{C}_3)$	1.0417	1.5156	0.4349	0.41183
$\mu(\mathcal{C}_4)$	1.0208	1.3867	0.40538	0.38557
$\mu(\mathcal{C}_5)$	1.0104	1.2969	0.38028	0.36813
$\mu(\mathcal{C}_8)$		1.0806	0.32872	0.34363

Notice that the table suggests that the convergence of $\mu(\mathcal{C}_k)$ to the actual value of $\mu(F)$ in the single-matrix case is faster than the other case.

Algorithm 2: The multiple-matrix case

1. Choose a number $k_0 \in \mathbb{N}$,
2. Start finding $\mathcal{V}_1, \mathcal{V}_2, \cdots \mathcal{V}_{k_0}$, and stop finding more of them if a there is a subset $\mathcal{W}_i \subseteq \mathcal{V}_i$ with e.p. points such that

$$\mathcal{W}_{i,\infty}^{C} \cup \mathcal{V}_i \subseteq \mathcal{C}(\mathcal{W}_{i,\infty}).$$

for some $i \in \{1, \cdots k_0\}$, In such a case, $\mathcal{C}(F) = \mathcal{C}(\mathcal{W}_{i,\infty})$.

For the following examples, it is easy to show that $\lambda(\mathcal{T}) < 1$.

Example

A disconnected attractor.
Let

$$T_1^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad T_2^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and $D = \left\{ d_1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right], d_2 = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix} \right] \right\}$. Then $F = \cup_{j=1}^{2} F_j$ together with some $\mathcal{C}(\mathcal{V}_k) = \mathcal{C}(A_k)$ are depicted in the next Figure. By Algorithm 2, $\mathcal{C}(F)$ is a heptagon with vertex set

$$\mathcal{V}_{19,\infty} = \mathcal{V}(F) \quad = \quad \{\overline{2}, 1\overline{2}, 11\overline{2}, 111\overline{2}, 1111\overline{2}, 11111\overline{2}, 111111\overline{2}\}.$$

□

Examples

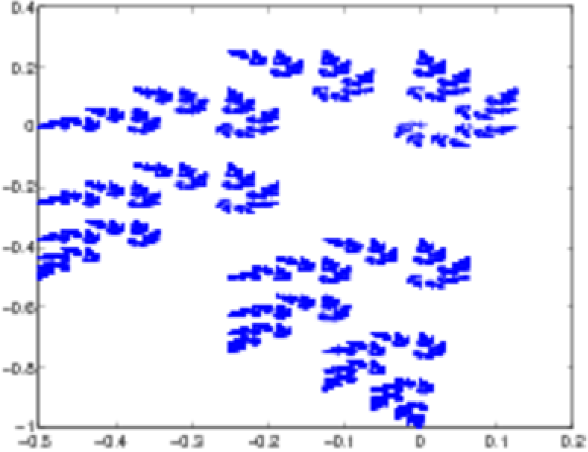


Figure: The disconnected attractor

Examples

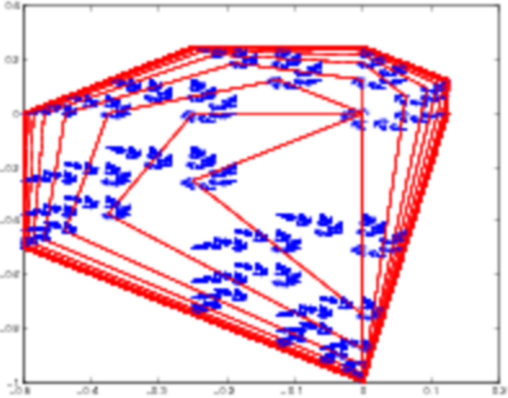


Figure: The disconnected attractor

Example

Levy's Dragon.

For the expansive matrices

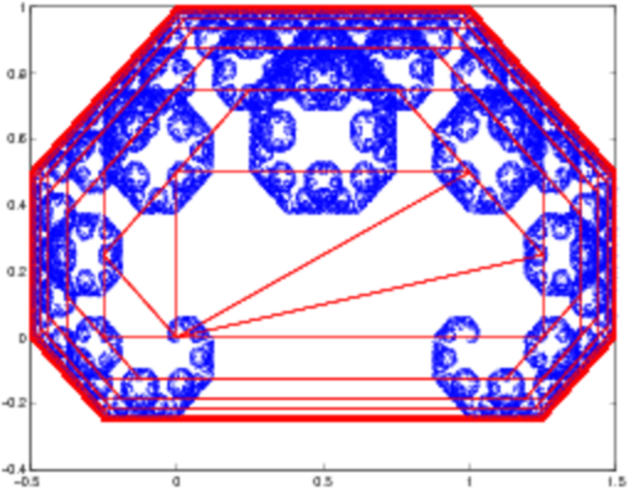
$$T_1^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad T_2^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and $D = \left\{ d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, the associated self-affine set $F = F(\mathcal{T}, D)$ is a tile, called the Levy dragon (see the Figure for F together with some $\mathcal{C}(\mathcal{V}_k)$). By Algorithm 2,

$$\mathcal{V}_{19,\infty} = \mathcal{V}(F) = \{222\overline{21}, 22\overline{21}, 2\overline{21}, \overline{21}, 1\overline{21}, 11\overline{21}, 111\overline{21}, 1111\overline{21}\}.$$

Then the convex hull of Levy's dragon is an octagon. From the third column of the Table, we see that $0 < \mu(F) \leq 1.0806$. □

Examples



Examples

Example

Heighway's Dragon.

For the expansive matrices

$$T_1^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad T_2^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix},$$

and the digit set $D = \left\{ d_1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right], d_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \right\}$, $F = F(\mathcal{T}, D)$ is a tile, called the Heighway dragon. $F = F(\mathcal{T}, D)$ is depicted in the next Figure. For this example, the vertex set is

$$\mathcal{V}_{19,\infty} = \mathcal{V}(F) = \{\overline{2211}, 221\overline{2211}, 21\overline{2211}, 211\overline{2211}, 2111\overline{2211}, \\ 21111\overline{2211}, 1111\overline{2211}, 111\overline{2211}, 11\overline{2211}, 1\overline{2211}\}.$$

Therefore, $\mathcal{C}(F)$ is a decagon. We use this decagon to approximate the Lebesgue measure. Thus the fourth column of the Table implies $0 < \mu(F) \leq 0.32872$. □

Examples

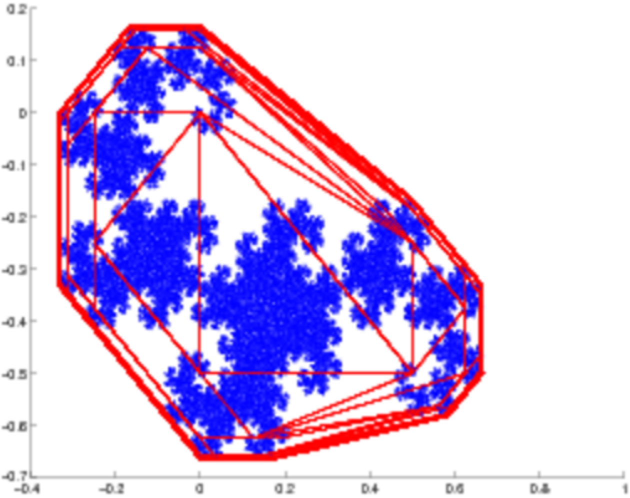


Figure: Heighway's Dragon

Example

[4] *A 3-digit tile.*
For the expansive matrices

$$T_1^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \qquad T_2^{-1} = T_1^{-1} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, \qquad T_3^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

and the digit set $D = \left\{ d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, d_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$, $F = F(\mathcal{T}, D)$ is a tile (see the next Figure). By Algorithm 2, the vertex set is obtained as

$$\mathcal{V}(F) = \{11\overline{1112}, 1\overline{1112}, \overline{1112}, 112\overline{1112}, 12\overline{1112}, 2\overline{1112}, 2112\overline{1112},$$
$$2\overline{3332}, 2332\overline{3332}, 33\overline{3332}, 3\overline{3332}, \overline{3332}, 332\overline{3332}, 32\overline{3332}\}.$$

Therefore, $\mathcal{C}(F)$ is a tetradecagon. For the Lebesgue measure, the fifth column of the Table yields $0 < \mu(F) \leq 0.34363$ so that $\mu(F)$ is not an integer. □

Examples

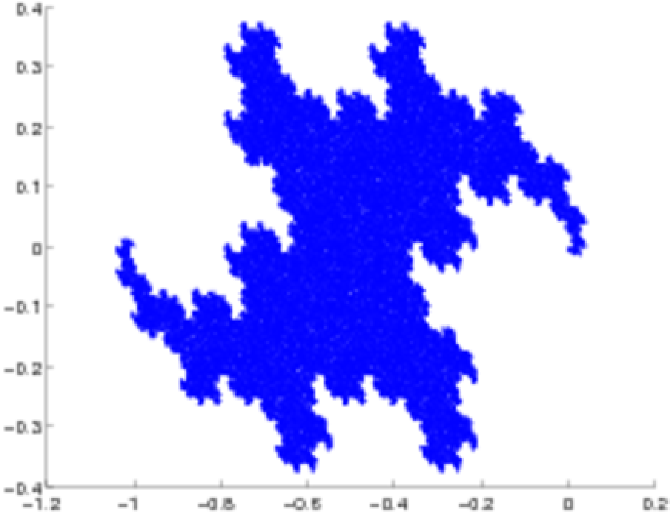


Figure: The 3-digit tile

Examples

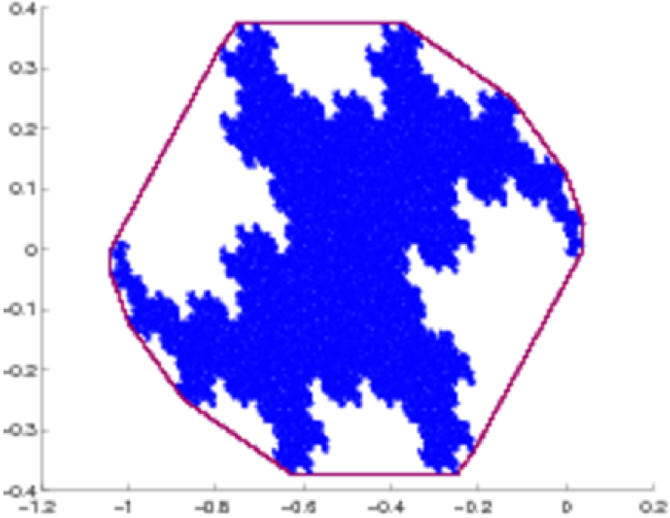


Figure: The 3-digit tile