Bernoulli systems and ONBs of exponentials I: Measure of the overlap

Keri Kornelson

University of Oklahoma - Norman

Fractals and Tilings
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Iterated Function Systems (IFSs)

[Hutchinson, 1981]

Definition

An Iterated Function System (IFS) is a finite collection $\{\tau_i\}_{i=1}^{k}$ of contractive maps on a complete metric space.

By the Banach Contraction Mapping Theorem, there exists a unique compact set X satisfying

$$\bigcup_{i=1}^{k} \tau_i(X) = X. \quad (1)$$

The set X is called the attractor of the IFS.

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Given any compact set $A_0 \subset \mathbb{R}^d$, successive iterations of our contraction

$$A_{n+1} = \bigcup_{i=1}^{k} \tau_i(A_n)$$

converge (in Hausdorff metric) to the attractor X.

IFS Measure

[Hutchinson, 1981]

Given an IFS $\{\tau_i\}_{i=1}^k$ and probability weights $\{p_i\}_{i=1}^k$ with $\sum_{i=1}^k p_i = 1$, there is a unique probability measure, supported on X, such that

$$\mu = \sum_{i=1}^{k} p_i(\mu \circ \tau_i^{-1}), \tag{2}$$

This measure μ is often called an equilibrium or Hutchinson measure.

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Given any probability measure μ_0, iterations of the measure transformation

$$\mu_{n+1} = \sum_{i=1}^{k} p_i(\mu_n \circ \tau_i^{-1})$$

converge to the equilibrium measure μ

Bernoulli IFS

Definition

A Bernoulli IFS consists of two affine maps τ_0 and τ_1 on $\mathbb{R}$ of the form

$$\tau_0(x) = \lambda x \qquad \tau_1(x) = \lambda(x+1)$$

for $0 < \lambda < 1$.

We denote the attractor set by X_λ and the the equilibrium measure using equal weights $p_0 = p_1 = \frac{1}{2}$ by μ_λ.

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The measure μ_λ can be realized as the probability distribution of the random variable $\sum_i \omega_i \lambda^i$, where each ω_i is a random variable taking on the values 0 or 1 with equal probability.

Bernoulli IFS attractors

- $\lambda < \frac{1}{2}$: X_λ is a Cantor set with non-integer Hausdorff dimension.
- $\lambda = \frac{1}{2}$: $X_{\frac{1}{2}}$ is the interval $[0, 1]$ and the measure $\mu_{\frac{1}{2}}$ is Lebesgue measure.
- $\lambda > \frac{1}{2}$: X_λ is the interval $\left[0, \frac{\lambda}{1-\lambda}\right]$. These measures have overlap, i.e.

$$\tau_0(X_\lambda) \cap \tau_1(X_\lambda)$$

has nonzero Lebesgue measure.

Some known results about μ_λ

- [Erdös, 1939] If $\lambda > \frac{1}{2}$ is the inverse of a Pisot number, the measure μ_λ is singular with respect to Lebesgue measure.
- [Kakutani, 1947] If $\lambda_1 < \lambda_2 < \frac{1}{2}$ then μ_{λ_1} and μ_{λ_2} are mutually singular.
- [Garsia, 1962] There is a countable family of μ_λ with $\lambda \in (\frac{1}{2}, 1)$ which are absolutely continuous.
- [Solomyak, 1995] For almost every $\lambda \in (\frac{1}{2}, 1)$, μ_λ is absolutely continuous.
- [Jorgensen & Pedersen, Strichartz, 1998] $\mu_{\frac{1}{4}}$ is a spectral measure, but $\mu_{\frac{1}{3}}$ is not. More generally, for $\lambda = \frac{1}{n}$, if n is even, there is an ONB of exponentials but when n is odd, there is not.

The ONB question

Question

For which values of λ is μ_λ does $L^2(\mu_\lambda)$ have an ONB of exponential functions?

Theorem (Jorgensen, K, Shuman 2008; Hu, Lau 2008)

Let $\lambda = \frac{a}{b}$ in reduced form. If b is odd, then any orthonormal collection of exponentials in $L^2(X_\lambda, \mu_\lambda)$ must be finite. If b is even, then there exists a countably infinite collection of orthonormal exponentials in $L^2(X_\lambda, \mu_\lambda)$.

If $\lambda = \frac{a}{b} > \frac{1}{2}$, then the question arises of whether the (essential) overlap influences whether an ONB of exponentials exists.

Special Case: $\lambda = \frac{3}{4}$

Let

$$\begin{aligned}\Gamma &= \left\{ \sum_{j=0}^{p} a_j 4^j \;:\; a_j \in \{0,1\}, p \text{ finite} \right\} \\ &= \{0,1,4,5,16,17,20,21,\ldots\}. \qquad (3)\end{aligned}$$

The collection $\{e_\gamma : \gamma \in \Gamma\}$ is an ONB for $L^2(\mu_{\frac{1}{4}})$ (JoPe98).

It is also an orthonormal set in $L^2(\mu_{\frac{3}{4}})$, but is it an ONB?

Test for ONB

Using the Parseval identity for orthonormal bases and Stone-Weierstrass, we see that $\{e_\gamma : \gamma \in \Gamma\}$ is an ONB if and only if

$$\begin{aligned} \|e_t\|^2_{\mu_{\frac{3}{4}}} &= \sum_{\gamma\in\Gamma} |\langle e_t, e_\gamma\rangle|^2 \\ &= \sum_{\gamma\in\Gamma} |\widehat{\mu}_{\frac{3}{4}}(t-\gamma)|^2 \end{aligned}$$

Our test for an ONB is whether, for every value of t, this last line is equal to 1.

$$\sum_{\gamma\in\Gamma}[\widehat{\mu}_{\frac{3}{4}}(t-\gamma)]^2 = \sum_{\gamma\in\Gamma}\prod_{k=1}^{\infty}\cos^2\left(2\pi\left(\frac{3}{4}\right)^k(t-\gamma)\right) \equiv 1.$$

Evidence for non-ONB

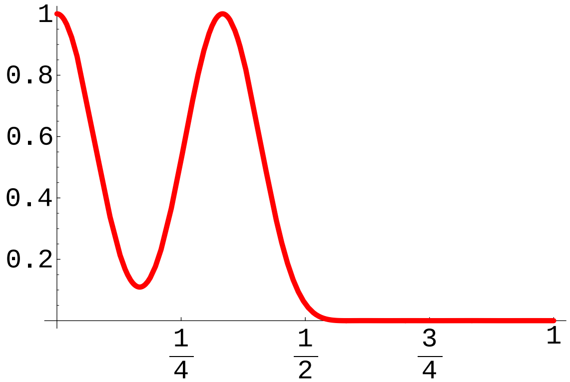

Figure: 40 terms in both the product and sum in the approximation of the function $\sum_{\gamma \in \Gamma} |\widehat{\mu}_{\frac{3}{4}}(t-\gamma)|^2$. The function does not appear to be identically 1.

Overlap and spectral measure

Theorem (Dutkay, Han, Jorgensen 2009)

If μ_λ is spectral, $\mu_\lambda(overlap) = 0$.

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This motivates a look at what we can say about the measure of the overlap set.

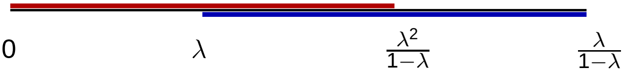
The overlap interval

Recall that for $\lambda > \frac{1}{2}$, the attractor set is

$$X_\lambda = \left[0, \frac{\lambda}{1-\lambda}\right].$$

Denote the right endpoint of the attractor set $b_\lambda = \frac{\lambda}{1-\lambda}$.

The overlap interval is $[\lambda, \frac{\lambda^2}{1-\lambda}]$, which clearly has nonzero Lebesgue measure when $\lambda > \frac{1}{2}$. We need to examine the measure with respect to μ_λ.



Symmetry Theorem

Theorem

For any $\alpha \in X_\lambda$*,*

$$\mu_\lambda([0,\alpha]) = \mu_\lambda([b_\lambda - \alpha, b_\lambda]).$$

In particular, $\mu_\lambda(\tau_0(X_\lambda)) = \mu_\lambda(\tau_1(X_\lambda))$*.*

Proof of Symmetry Theorem

Proof.

Consider μ_λ as the probability distribution for $\sum_{i=1}^{\infty} \omega_i \lambda^i$, where each ω_i is a random variable taking the values $\{0, 1\}$ with equal probability. Then $\mu_\lambda([0, \alpha])$ is the probability that

$$\sum_{i=1}^{\infty} \omega_i \lambda^i < \alpha.$$

Similarly, $\mu_\lambda([b_\lambda - \alpha, b_\lambda])$ is the probability that $\sum_{i=1}^{\infty} \omega_i \lambda^i > b_\lambda - \alpha$. This condition is equivalent to

$$\sum_{i=1}^{\infty} (1 - \omega_i) \lambda^i < \alpha.$$

But each $(1 - \omega_i)$ also takes on values in $\{0, 1\}$ with equal probability, so the probabilities must be the same. $\square$

Computing μ_λ measures

Use the convergence of measures from Hutchinson's theory, letting $\mu_0 = \delta_0$, a Dirac point mass measure.

$$\begin{aligned}
\mu_0 &= \delta_0 \\
\mu_1 &= \frac{1}{2}(\delta_0 \circ \tau_0^{-1} + \delta_0 \circ \tau_1^{-1}) \\
\vdots &\qquad \vdots \\
\mu_{n+1} &= \frac{1}{2}(\mu_n \circ \tau_0^{-1} + \mu_n \circ \tau_1^{-1}) \\
&= \frac{1}{2^n} \sum_{\omega_1\omega_2\cdots\omega_n} \delta_0 \circ \tau_{\omega_n}^{-1} \circ \tau_{\omega_{n-1}}^{-1} \circ \cdots \circ \tau_{\omega_1}^{-1} \qquad (\omega_i \in \{0,1\}) \\
&= \frac{1}{2^n} \sum_{\omega_1\omega_2\cdots\omega_n} \delta_{\omega_1\lambda + \omega_2\lambda^2 + \cdots \omega_n\lambda^n} \qquad \to \mu_\lambda
\end{aligned}$$

Therefore, for a set E,

$$\mu_\lambda(E) = \lim_{n\to\infty} \frac{1}{2^n} \# \left\{ \omega_1\omega_2\cdots\omega_n \ : \ \sum_{i=1}^{n} \omega_i \lambda^i \in E \right\}$$

Overlap for the golden ratio

Sidorov, Vershik 1998; Jorgensen, K, Shuman 2007

$$\lambda = \phi^{-1} = \frac{\sqrt{5}-1}{2}, \quad \lambda^2 + \lambda = 1$$

Theorem

When $\lambda = \frac{\sqrt{5}-1}{2}$, the μ_λ-measure of the overlap is $\frac{1}{3}$.

Proof

Compute $\mu_\lambda(\tau_1(X_\lambda)) = \mu_\lambda([\lambda, b_\lambda]) = \mu_\lambda\left(\left[\frac{\sqrt{5}-1}{2}, \frac{2}{\sqrt{5}-1}\right]\right)$:

$$\mu_\lambda([\lambda, b_\lambda]) \quad = \quad \lim_{n\to\infty} \frac{1}{2^n} \# \left\{ \omega_1\omega_2\cdots\omega_n \; : \; \sum_{i=1}^{n} \omega_i \lambda^i \in [\lambda, b_\lambda] \right\}$$

We count the number of length-n words $\omega = \omega_1\omega_2\cdots\omega_n$, $\omega_i \in \{0, 1\}$, such that $\sum_{i=1}^{n} \omega_i \lambda^i > \lambda$.

proof continued...

$1*$ $\quad \lambda + \sum_{i=2}^{n} \omega_i \lambda^i \geq \lambda$

$011*$ $\quad \lambda^2 + \lambda^3 + \sum_{i=4}^{n} \omega_i \lambda^i \geq \lambda$

$01011*$ $\quad \lambda^2 + \lambda^4 + \lambda^5 + \sum_{i=6}^{n} \omega_i \lambda^i \geq \lambda$

Lemma 1. For $\lambda = \frac{\sqrt{5}-1}{2}$,

$$\lambda^2 + \lambda^4 + \cdots + \lambda^{2k} + \lambda^{2k+1} = \lambda \qquad \forall k \geq 1.$$

Lemma 2. The *only* finite sums $\sum_{i=1}^{n} \omega_i \lambda^i \geq \lambda$ start as in Lemma 1.

proof continued ...

For fixed odd n, we count the number of words $\omega_1\omega_2\cdots\omega_n$ with $\sum_{i=1}^n \omega_i\lambda^i \geq \lambda$. Notice the cases are disjoint!

$1*$	$011*$	$01011*$	$\cdots$	$0101\cdots011$
2^{n-1}	2^{n-3}	2^{n-5}	$\cdots$	1

Then, we have

$$\frac{1}{2^n}\#\left\{\omega_1\omega_2\cdots\omega_n : \sum_{i=1}^n \omega_i\lambda^i \geq \lambda\right\} = \frac{1}{2}+\frac{1}{8}+\cdots+\frac{1}{2^n}.$$

This value is the same for odd n and the next even $n+1$. Taking the limit gives:

$$\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^n}\right)=\frac{2}{3}.$$

proof continued ...

We have shown that

$$\mu_\lambda(\tau_1(X_\lambda)) = \frac{2}{3}.$$

By the symmetry theorem, we also have

$$\mu_\lambda(\tau_0(X_\lambda)) = \frac{2}{3}.$$

The measure of the overlap can then be computed

$$\begin{aligned} \mu_\lambda\left(\left[\lambda, \frac{\lambda^2}{1-\lambda}\right]\right) &= \mu_\lambda(\tau_0(X_\lambda)) + \mu_\lambda(\tau_1(X_\lambda)) - \mu_\lambda(X_\lambda) \\ &= \frac{2}{3} + \frac{2}{3} - 1 = \frac{1}{3}. \end{aligned}$$

□

Overlap when $\lambda > \phi^{-1}$

Theorem (Jorgensen, K, Shuman 2007)

If $\lambda > \phi^{-1}$, the measure of the overlap is greater than $\frac{1}{3}$.

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Proof.

The same cases counted for $\lambda = \phi^{-1}$ all yield finite sums $\sum_{i=1}^{n} \omega_i \lambda^i \geq \lambda$. For example $\lambda^2 + \lambda^3 > \lambda$ so all words starting $011*$ yield finite sums greater than λ. But now, Lemma 2 no longer holds, so there may be other finite sums bigger than λ.

Example. $\lambda = \frac{3}{4}$, $\quad \lambda^3 + \lambda^4 + \lambda^5 > \lambda$.

Thus, $\mu_\lambda(\tau_1(X_\lambda)) > \frac{2}{3}$, so $\mu_\lambda(\tau_0(X_\lambda) \cap \tau_1(X_\lambda)) > \frac{1}{3}$. $\square$

Overlap when $\frac{1}{2} < \lambda < \phi^{-1}$

Theorem (Jorgensen, K, Shuman 2007)

If $\frac{1}{2} < \lambda < \phi^{-1}$, the measure of the overlap is greater than or equal to $\frac{1}{2^m - 1}$ for some m.

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Sketch of proof.

Since $\lambda > \frac{1}{2}$, $\exists m$ such that $\lambda + \lambda^2 + \cdots + \lambda^m > 1$.

We count finite words (as before) such that $\sum_{i=1}^{n} \omega_i \lambda^i \geq \lambda$. Let $w = 011\cdots1$ ($m-1$ ones.)

$1*$	$w1*$	$ww1*$	$\cdots$	$w\cdots w1$
2^{n-1}	2^{n-m-1}	2^{n-2m-1}	$\cdots$	1

Taking the limit yields $\mu_\lambda(\tau_1(X_\lambda)) \geq \frac{2^{m-1}}{2^m - 1}$, hence

$$\mu_\lambda\Big(\tau_0(X_\lambda) \cap \tau_1(X_\lambda)\Big) \geq \frac{1}{2^m - 1}. \qquad \square$$

Stay tuned...

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- In the next talk, Karen will discuss some results finding more exact measurements of overlaps. This is work done by one of her students last summer.
- Conclusion All Bernoulli measures with Lebesgue overlap also have essential, i.e. nonzero μ_λ-overlap. This means that Dorin's new result proves that no Bernoulli measures for $\lambda > \frac{1}{2}$ can be spectral.